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MINIMAL INTERSECTIONS AND VANISHING (CO)HOMOLOGY

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ABSTRACT. We introduce a class of local Noetherian rings, which we call *minimal intersections*, and show that over such rings there exist classes of modules for which the derived functors Ext and Tor vanish non-trivially. This generalizes a well-known phenomenon of non-trivial vanishing of Ext and Tor for modules over complete intersections of codimension at least two.

1. Introduction. Let R be a commutative local Noetherian ring, and M and N finitely generated R-modules. In many cases the vanishing of all higher Ext and Tor can only occur in a trivial way. For instance, in [**9, 11, 12, 20**] it is shown that over hypersurfaces (which are codimension one complete intersections), Golod rings and Gorenstein rings of low codimension, the vanishing of all higher $\operatorname{Tor}_i^R(M,N)$ or $\operatorname{Ext}^i_R(M,N)$ implies that either M has finite projective dimension, or N has finite projective dimension (or finite injective dimension for Ext vanishing if R is not Gorenstein). This raises a question of the rarity of non-trivial vanishing of all higher homology and cohomology over local rings.

The most well-known class of local rings over which the vanishing of all higher Ext and Tor occurs non-trivially is that of complete intersections of codimension at least two (see, for example, [**14**, Theorem 3.1], and [**4**]). In this paper we isolate a property of complete intersections which enables non-trivial vanishing, and consider, more generally, local Noetherian rings having this property:

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Definition. Let $R = Q/I$ with Q a regular local ring and I an ideal in the square of the maximal ideal of Q. We say that R is a *minimal intersection (with respect to Q)* if I is the sum of two non-zero ideals I_1 and I_2 of Q such that $I_1 \cap I_2 = I_1 I_2$.

We prove that there exist classes of modules over a minimal intersection demonstrating non-trivial vanishing of all higher homology and cohomology. That is, if R is a minimal intersection then there exist classes of finitely generated R-modules M and N of infinite projective dimension over R, and (not necessarily finitely generated) R-modules L of infinite injective dimension over R, such that $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \gg 0$, and $\operatorname{Ext}_R^i(M, L) = 0$ for all $i \gg 0$. Outside of the special case of both Q/I_1 and Q/I_2 having only finitely many non-isomorphic indecomposable syzygy (or cosyzygy) modules, these classes of modules exhibiting non-trivial vanishing are quite large. If we further assume that R is Cohen-Macaulay, then they consist of finitely generated maximal Cohen-Macaulay modules.

Minimal intersections are a generalization of complete intersections of codimension two or greater. For if I is generated by a regular sequence $f_1, \ldots, f_c$ with $c \geq 2$, then for $1 \leq r \leq c$ we have $(f_1, \ldots, f_r) \cap (f_{r+1}, \ldots, f_c) = (f_1, \ldots, f_r)(f_{r+1}, \ldots, f_c)$.

In Section 2 we give general results for Ext and Tor that are needed in subsequent sections. We prove in Section 3 properties of minimal intersections which are also needed in subsequent sections. For instance, we show that the Cohen-Macaulay and Gorenstein properties are preserved after minimal intersection. Section 4 is the main part of the paper. We discuss how non-trivial vanishing of Ext and Tor can occur over complete intersections of codimension at least two, and then prove the same result assuming only that R is a minimal intersection. We also show that such non-trivial vanishing over Cohen-Macaulay and Gorenstein minimal intersections (respectively) behaves similarly to that over complete intersections. The final Section 5 gives several examples, and a sufficient condition for detecting modules in the classes exhibiting non-trivial vanishing. This sufficient condition looks at the form of the free resolution of the module, and is aptly implemented on the computer. We do this using the computer algebra package *Macaulay 2*.

2. General Results on Ext and Tor. In this section we give some preliminary results involving Ext and Tor.

We make major use of the following standard result (see, for example, [**19**, 11.51]).

2.1. Suppose that A is a commutative ring, J an ideal of A, and set $B = A/J$.

(1) If X is an A-module such that $\mathrm{Tor}_i^A(X,B) = 0$ for all $i \geq 1$, then for any B-module Y we have

$$\mathrm{Tor}_i^A(X,Y) \cong \mathrm{Tor}_i^B(X \otimes_A B, Y) \quad \text{for all } i,$$

and

$$\mathrm{Ext}^i_A(X,Y) \cong \mathrm{Ext}^i_B(X \otimes_A B, Y) \quad \text{for all } i.$$

(2) If Y is an A-module such that $\mathrm{Ext}^i_A(B,Y) = 0$ for all $i \geq 1$, then for any B-module X we have

$$\mathrm{Ext}^i_A(X,Y) \cong \mathrm{Ext}^i_B(X, \mathrm{Hom}_A(B,Y)) \quad \text{for all } i.$$

The following formula from [**12**, 2.2] is instrumental to the proofs in the subsequent sections.

2.2. Let X and Y be finitely generated modules over a local Noetherian ring A with $\mathrm{pd}_A X < \infty$. Then

$$\sup\{i \mid \mathrm{Tor}_i^A(X,Y) \neq 0\} = \sup\{\mathrm{depth}_{A_p} A_p - \mathrm{depth}_{A_p} X_p - \mathrm{depth}_{A_p} Y_p\},$$

where the second sup is taken over all $p \in \mathrm{Spec} A$.

Proposition 2.3. *Let A be a Gorenstein local ring, $B = A/J$ such that $\mathrm{pd}_A B < \infty$, and X be a maximal Cohen-Macaulay A-module. Then we have*

$$\mathrm{Hom}_A(X,A) \otimes_A B \cong \mathrm{Hom}_B(X \otimes_A B, B)$$

Proof. Hom-tensor adjointness gives $\mathrm{Hom}_B(X \otimes_A B, B) \cong \mathrm{Hom}_A(X,B)$, therefore it suffices to exhibit an isomorphism $\mathrm{Hom}_A(X,A) \otimes_A B \cong \mathrm{Hom}_A(X,B)$. This isomorphism is easily seen when X is a free A-module. In general, let $G \to F \to X \to 0$ be an A-free presentation of X. On the one hand we apply $\mathrm{Hom}_A(-,B)$, and on the other hand we apply $\mathrm{Hom}_A(-,A)$ first then $-\otimes_A B$. The result is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(X,B) \arrow[r] & \mathrm{Hom}_A(F,B) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_A(G,B) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_A(X,A)\otimes_A B \arrow[r] & \mathrm{Hom}_A(F,A)\otimes_A B \arrow[r] & \mathrm{Hom}_A(G,A)\otimes_A B
\end{tikzcd}$$

We just need to know that the bottom row is exact to establish the proposition. For this, consider the short exact sequences $0 \to \Omega \to F \to X \to 0$, and $0 \to \Omega' \to G \to \Omega \to 0$. Applying $\mathrm{Hom}_A(-,A)$, and using the fact that $\mathrm{Ext}^1_A(X,A) = \mathrm{Ext}^1_A(\Omega,A) = 0$ (since X and Ω are maximal Cohen-Macaulay A-modules), we get the short exact sequences $0 \to \mathrm{Hom}_A(X,A) \to \mathrm{Hom}_A(F,A) \to \mathrm{Hom}_A(\Omega,A) \to 0$ and $0 \to \mathrm{Hom}_A(\Omega,A) \to \mathrm{Hom}_A(G,A) \to \mathrm{Hom}_A(\Omega',A) \to 0$. Now applying $-\otimes_A B$ we obtain

$$\begin{aligned}
&\mathrm{Tor}_1^A(\mathrm{Hom}_A(\Omega,A),B) \to \mathrm{Hom}_A(X,A)\otimes_A B \to \\
&\qquad \mathrm{Hom}_A(F,A)\otimes_A B \to \mathrm{Hom}_A(\Omega,A)\otimes_A B \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Tor}_1^A(\mathrm{Hom}_A(\Omega',A),B) \to \mathrm{Hom}_A(\Omega,A)\otimes_A B \to \\
&\qquad \mathrm{Hom}_A(G,A)\otimes_A B \to \mathrm{Hom}_A(\Omega',A)\otimes_A B \to 0.
\end{aligned}$$

Since $\mathrm{Hom}_A(\Omega,A)$ and $\mathrm{Hom}_A(\Omega',A)$ are maximal Cohen-Macaulay A-modules and $\mathrm{pd}_A B < \infty$, 2.2 shows that $\mathrm{Tor}_i^A(\mathrm{Hom}_A(\Omega,A),B) = \mathrm{Tor}_i^A(\mathrm{Hom}_A(\Omega',A),B) = 0$ for all $i > 0$, and so we have short exact sequences

$$0 \to \mathrm{Hom}_A(X,A)\otimes_A B \to \mathrm{Hom}_A(F,A)\otimes_A B \to \mathrm{Hom}_A(\Omega,A)\otimes_A B \to 0$$

and

$$0 \to \mathrm{Hom}_A(\Omega,A)\otimes_A B \to \mathrm{Hom}_A(G,A)\otimes_A B \to \mathrm{Hom}_A(\Omega',A)\otimes_A B \to 0.$$

Splicing these together, we see that the bottom row of the diagram above is exact. ◻

Proposition 2.4. *Assume that A is a Cohen-Macaulay local ring with canonical module ω. Let $(-)^\vee$ denote the dual $\operatorname{Hom}_A(-,\omega)$. Let X and Y be finitely generated A-modules, with Y maximal Cohen-Macaulay. Then $\operatorname{Ext}^i_A(X,Y) = 0$ for all $i \gg 0$ if and only if $\operatorname{Tor}_i^A(X,Y^\vee) = 0$ for all $i \gg 0$.*

If X is moreover maximal Cohen-Macaulay, then the following are equivalent:

(1) $\operatorname{Ext}^i_A(X,Y) = 0$ *for all* $i \geq 1$*;*

(2) $\operatorname{Tor}_i^A(X,Y^\vee) = 0$ *for all* $i \geq 1$*, and* $X \otimes_A Y^\vee$ *is maximal Cohen-Macaulay.*

Proof. That (1) and (2) are equivalent is proven in [**16**, 2.7]. To prove the "if" direction of the first statement, choose a sufficiently high syzygy module $\Omega^n(X)$ of X such that $\operatorname{Tor}_i^A(\Omega^n(X),Y^\vee) = 0$ for all $i \geq 1$. This yields short exact sequences

$$0 \to \Omega^{n+i}(X) \otimes_A Y^\vee \to F_{n+i-1} \otimes_A Y^\vee \to \Omega^{n+i-1}(X) \otimes_A Y^\vee \to 0$$

for all $i \geq 1$, derived from a minimal free resolution

$$\cdots \to F_n \to \cdots \to F_1 \to F_0 \to X \to 0$$

of X. Since $F_{n+i-1} \otimes_A Y^\vee$ are maximal Cohen-Macaulay for all i, counting depths along these short exact sequences shows that $\Omega^{n+i}(X) \otimes_A Y^\vee$ are maximal Cohen-Macaulay for all $i \geq d = \dim A$. Thus we have $\operatorname{Tor}_i^A(\Omega^{n+d}(X),Y^\vee) = 0$ for all $i \geq 1$, and $\Omega^{n+d}(X) \otimes_A Y^\vee$ is maximal Cohen-Macaulay. By the second part of the theorem, $\operatorname{Ext}^i_A(\Omega^{n+d}(X),Y) = 0$ for all $i \geq 1$, and so $\operatorname{Ext}^i_A(X,Y) = 0$ for all $i \gg 0$.

The "only if" direction of the first statement also uses the second part of the theorem, and is easier. □

3. Minimal Intersections. Throughout this section we assume that Q is a regular local ring, and $R = Q/(I_1 + I_2)$ with I_1 and I_2 nonzero ideals contained in the square of the maximal ideal of Q, and we set $R_1 = Q/I_1$ and $R_2 = Q/I_2$. We start with some basic facts.

3.1. The ring R is a minimal intersection if and only if $\operatorname{Tor}_i^Q(R_1,R_2) = 0$ for all $i \geq 1$. Indeed, it is standard that $\operatorname{Tor}_1^Q(R_1,R_2) = 0$ if and only if $I_1 \cap I_2 = I_1 I_2$ (see for example [**19**]). The statement follows now from rigidity of Tor for regular local rings [**1, 18**].

Lemma 3.2. *Assume that R is a minimal intersection. Then*

(1) $\mathrm{pd}_Q R = \mathrm{pd}_Q R_1 + \mathrm{pd}_Q R_2$*;*

(2) $\mathrm{depth}_Q Q + \mathrm{depth}_Q R = \mathrm{depth}_Q R_1 + \mathrm{depth}_Q R_2$.

Proof. The first statement follows by noting that if $\mathbf{F}$ and $\mathbf{G}$ are minimal free resolutions of R_1 and R_2 over Q, then $\mathbf{F} \otimes_Q \mathbf{G}$ is a minimal free resolution of $R \cong R_1 \otimes_Q R_2$ over Q, by 3.1, and this resolution is of length $\mathrm{pd}_Q R_1 + \mathrm{pd}_Q R_2$. Statement (2) follows from Statement (1) and the Auslander-Buchsbaum formula:

$$\begin{aligned}
\mathrm{depth}_Q Q + \mathrm{depth}_Q R &= \mathrm{depth}_Q Q + (\mathrm{depth}_Q Q - \mathrm{pd}_Q R) \\
&= \mathrm{depth}_Q Q + (\mathrm{depth}_Q Q - (\mathrm{pd}_Q R_1 + \mathrm{pd}_Q R_2)) \\
&= (\mathrm{depth}_Q Q - \mathrm{pd}_Q R_1) + (\mathrm{depth}_Q Q - \mathrm{pd}_Q R_2) \\
&= \mathrm{depth}_Q R_1 + \mathrm{depth}_Q R_2. \qquad \square
\end{aligned}$$

The following result discusses Cohen-Macaulay and Gorenstein minimal intersections.

Proposition 3.3. *With the notation above, we have:*

(1) *R is Cohen-Macaulay if and only if both R_1 and R_2 are Cohen-Macaulay. When this is the case,* $\mathrm{height}(I_1 + I_2) = \mathrm{height} I_1 + \mathrm{height} I_2$.

(2) *R is Gorenstein if and only if both R_1 and R_2 are Gorenstein.*

(3) *R is a complete intersection if and only if both R_1 and R_2 are complete intersections.*

Proof. The "if" direction of (1) is given in [**8**, Lemma 1.10]. For the "only if" direction we use the the Intersection Theorem of Peskine-Szpiro and Roberts, which implies the inequality

$$\dim Q + \dim R \geq \dim R_1 + \mathrm{depth} R_2.$$

(See [**5**, Corollary 9.4.6].) Therefore by Lemma 3.2(2)

$$\begin{aligned}\operatorname{depth} R_1 + \operatorname{depth} R_2 &= \operatorname{depth}_Q Q + \operatorname{depth}_Q R \\ &= \dim Q + \dim R \\ &\geq \dim R_1 + \operatorname{depth} R_2.\end{aligned}$$

Thus $\operatorname{depth} R_1 \geq \dim R_1$ and so R_1 is Cohen-Macaulay. By symmetry, so is R_2.

To prove the second statement of (1), the statement of Lemma 3.2(2) gives

$$\begin{aligned}\operatorname{height} I &= \dim Q - \dim R \\ &= \operatorname{depth}_Q Q - \operatorname{depth}_Q R \\ &= \operatorname{depth}_Q Q - (\operatorname{depth}_Q R_1 + \operatorname{depth}_Q R_2 - \operatorname{depth}_Q Q) \\ &= (\operatorname{depth}_Q Q - \operatorname{depth}_Q R_1) + (\operatorname{depth}_Q Q - \operatorname{depth}_Q R_2) \\ &= (\dim Q - \dim R_1) + (\dim Q - \dim R_2) \\ &= \operatorname{height} I_1 + \operatorname{height} I_2.\end{aligned}$$

To prove (2) it suffices by (1) to show only that

$$\operatorname{Ext}_Q^{\operatorname{pd}_Q R_1 + \operatorname{pd}_Q R_2}(R, Q) \cong R$$

if and only if both

$$\operatorname{Ext}_Q^{\operatorname{pd}_Q R_1}(R_1, Q) \cong R_1 \quad \text{and} \quad \operatorname{Ext}_Q^{\operatorname{pd}_Q R_2}(R_2, Q) \cong R_2,$$

assuming R is Cohen-Macaulay. Let $\mathbf{F}$ and $\mathbf{G}$ be (deleted) minimal Q-free resolutions of R_1 and R_2, respectively. By the vanishing of $\operatorname{Tor}_i^Q(R_1, R_2) = 0$ for all $i \geq 1$, a minimal Q-free resolution of R is given by $\mathbf{F} \otimes_Q \mathbf{G}$. Let $(-)^*$ denote the dual $\operatorname{Hom}_Q(-, Q)$. Since R_1 and R_2 are Cohen-Macaulay, both $\mathbf{F}^*$ and $\mathbf{G}^*$ are complexes with homology $\operatorname{Ext}_Q^{\operatorname{pd}_Q R_1}(R_1, Q)$ and $\operatorname{Ext}_Q^{\operatorname{pd}_Q R_2}(R_2, Q)$, respectively, and since R is Cohen-Macaulay, $(\mathbf{F} \otimes_Q \mathbf{G})^*$ is a complex with homology $\operatorname{Ext}_Q^{\operatorname{pd}_Q R}(R, Q)$. From the natural isomorphism of complexes $(\mathbf{F} \otimes_Q \mathbf{G})^* \cong \mathbf{F}^* \otimes_Q \mathbf{G}^*$ it follows that

$$\operatorname{Ext}_Q^{\operatorname{pd}_Q R_1 + \operatorname{pd}_Q R_2}(R, Q) \cong \operatorname{Ext}_Q^{\operatorname{pd}_Q R_1}(R_1, Q) \otimes_Q \operatorname{Ext}_Q^{\operatorname{pd}_Q R_2}(R_2, Q).$$

Now it is clear that R is Gorenstein if R_1 and R_2 are Gorenstein. For the converse, one concludes that if $\operatorname{Ext}_Q^{\operatorname{pd}_Q R_1+\operatorname{pd}_Q R_2}(R,Q) \cong R$, then $\operatorname{Ext}_Q^{\operatorname{pd}_Q R_1}(R_1,Q) \cong Q/I_1'$ and $\operatorname{Ext}_Q^{\operatorname{pd}_Q R_2}(R_2,Q) \cong Q/I_2'$ for ideals I_1' and I_2' of Q satisfying $I_1 \subseteq I_1'$ and $I_2 \subseteq I_2'$. Dualizing $\mathbf{F}^*$ and $\mathbf{G}^*$ back to $\mathbf{F}$ and $\mathbf{G}$ shows the reverse inclusions of ideals, yielding $I_1 = I_1'$ and $I_2 = I_2'$.

Statement (3) follows easily from (1) and the fact that $\mu_Q(I) = \mu_Q(I_1) + \mu_Q(I_2)$, where $\mu_Q(J)$ denotes the minimal number of generators of an ideal J of Q. $\square$

Theorem 3.4. *With the notations above, the following are equivalent:*

(1) *R is a minimal intersection.*

(2) *R_p is a minimal intersection for all prime ideals p of Q.*

(3) *For all primes p of Q,*

$$\operatorname{depth}_{Q_p} Q_p + \operatorname{depth}_{Q_p} R_p = \operatorname{depth}_{Q_p} (R_1)_p + \operatorname{depth}_{Q_p} (R_2)_p$$

If R_1 and R_1 are Cohen-Macaulay, then (1)–(3) are equivalent to

(4) *R_p is a proper intersection for all primes p of Q.*

Recall that R is a called a *proper intersection* if $\dim R = \dim R_1 + \dim R_2 - \dim Q$. Thus, in the Cohen-Macaulay case, part (4) of the theorem says that minimal intersections are proper intersections in a strong sense.

Proof. Suppose that R is a minimal intersection. We have $R = Q/(I_1 + I_2)$ with Q a regular local ring, and $I_1 I_2 = I_1 \cap I_2$. Let p be a prime ideal of Q. Then $R_p = Q_p/((I_1)_p + (I_2)_p)$ with Q_p a regular local ring. Thus R_p is a minimal intersection if and only if $(I_1)_p(I_2)_p = (I_1)_p \cap (I_2)_p$, but this follows easily from the fact that $I_1 I_2 = I_1 \cap I_2$.

That (2) implies (3) is simply Lemma 3.2(2).

To show that (3) implies (1) we use 2.2 and 3.1:

$$\begin{aligned}\sup\{i \mid \operatorname{Tor}_i^Q(R_1,R_2)\neq 0\} &= \sup\Big\{\operatorname{depth}_{Q_p} Q_p - \operatorname{depth}_{Q_p}(R_1)_p \\ &\qquad -\operatorname{depth}_{Q_p}(R_2)_p\Big\} \\ &= \sup\{-\operatorname{depth}_{Q_p} R_p\} = 0.\end{aligned}$$

The equivalence of (4) is clear, using Proposition 3.3. □

It is useful to have a criteria for when a local ring is a minimal intersection. Recall that if X is a module over a local ring A with residue field κ, then the *Poincaré series* of X over A is the formal power series $\mathrm{P}_X^A(t) = \sum_{i\geq 0} \dim_\kappa \operatorname{Tor}_i^A(X,\kappa)t^i$.

Proposition 3.5. *The local ring $R = Q/(I_1+I_2)$ is a minimal intersection (with respect to Q) only if* $\dfrac{d\mathrm{P}_R^Q}{dt}(-1) = 0$.

Proof. Since Q is a regular local ring, $\mathrm{P}_{R_1}^Q(t)$ and $\mathrm{P}_{R_2}^Q(t)$ are polynomials in t, and since R_1 and R_2 are Q-modules of rank zero, we have $\mathrm{P}_{R_1}^Q(-1) = 0$ and $\mathrm{P}_{R_2}^Q(-1) = 0$. Now 3.1 shows that $\mathrm{P}_R^Q(t) = \mathrm{P}_{R_1}^Q(t)\mathrm{P}_{R_2}^Q(t)$. Thus $\mathrm{P}_R^Q(t)$ has -1 as a double root. □

Remark. The converse of Proposition 3.5 does not hold. Indeed, the Poincaré series over $Q = k[[x,y]]$ of the local ring $R = k[[x,y]]/(x^2,xy)$ has -1 as a double root, yet R is not a minimal intersection with respect to Q.

4. Vanishing over Minimal Intersections. This section contains the main results on non-trivial vanishing of Ext and Tor for modules over minimal intersections. The phenomenon of non-trivial vanishing is patterned on what happens over complete intersections, so we first briefly describe how non-trivial vanishing can occur in this case.

Vanishing over Complete Intersections. We first recall the following remarkable theorem of Avramov and Buchweitz [**4**], which makes use of *support varieties*, and which are reviewed below:

4.1. ([**4**]) Let M and N be finitely generated modules over a complete intersection R. Then the following are equivalent.

(1) $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \gg 0$;

(2) $\mathrm{Ext}_R^i(M,N) = 0$ for all $i \gg 0$;

(3) $\mathrm{Ext}_R^i(N,M) = 0$ for all $i \gg 0$;

(4) $\mathrm{V}(M) \cap \mathrm{V}(N) = \{0\}$.

Thus non-trivial vanishing occurs over complete intersections precisely when M and N are finitely generated modules, both of infinite projective dimension, such that $\mathrm{V}(M) \cap \mathrm{V}(N) = \{0\}$. We now describe a situation in which this trivial intersection of support varieties holds:

Proposition 4.2. *Let Q be a regular local ring, and $R = Q/(f_1, \ldots, f_c)$ a compete intersection of codimension $c \geq 2$. For $1 \leq r \leq c$, let $R_1 = Q/(f_1, \ldots, f_r)$ and $R_2 = Q/(f_{r+1}, \ldots, f_c)$. Suppose that M' is a maximal Cohen-Macaulay module over R_1 and that N' is a maximal Cohen-Macaulay module over R_2. For $M = M' \otimes_{R_1} R$ and $N = N' \otimes_{R_2} R$ we have*

(1) $\mathrm{V}(M) \cap \mathrm{V}(N) = \{0\}$

(2) $\mathrm{pd}_{R_1} M' = \mathrm{pd}_R M$, *and* $\mathrm{pd}_{R_2} N' = \mathrm{pd}_R N$.

Thus non-trivial vanishing occurs whenever M' and N' are chosen to have infinite projective dimension over R_1 and R_2, respectively.

We briefly recall the definition of support variety (cf. [**2**]). Let R be a complete intersection. We can without loss of generality assume that the residue field k of R is algebraically closed. For any finitely generated R-module M, the sequence of Ext modules $\mathrm{Ext}_R(M,k)$ has the structure of finitely generated graded module over the polynomial ring $\mathcal{R} = k[\chi_1, \ldots, \chi_c]$ of *cohomology operators*. Thus $\mathrm{ann}_{\mathcal{R}}\mathrm{Ext}_R(M,k)$ is a homogeneous ideal of $\mathcal{R}$, and we define the *support variety* of M, $\mathrm{V}(M)$, to be the cone in c-dimensional affine space over k defined by $\mathrm{ann}_{\mathcal{R}}\mathrm{Ext}_R(M,k)$.

Proof. The proof is really that of [**14**, 3.1]: by construction, M lifts to M', and the proof of [**14**, 3.1] gives $(\chi_{r+1},\dots,\chi_c) \subseteq \mathrm{ann}_{\mathcal{R}}\mathrm{Ext}_R(M,k)$. Thus we have $\mathrm{V}((\chi_{r+1},\dots,\chi_c)) \supseteq \mathrm{V}(M)$. Similarly, $\mathrm{V}((\chi_1,\dots,\chi_r)) \supseteq \mathrm{V}(N)$, and so

$$\mathrm{V}(M)\cap\mathrm{V}(N) \subseteq \mathrm{V}((\chi_1,\dots,\chi_r)) \cap \mathrm{V}((\chi_{r+1},\dots,\chi_c)) = \{0\}. \qquad \square$$

There are two other relevant properties of vanishing Ext and Tor which hold over complete intersections. Both are well-known, and the first is referred to as the *uniform Auslander condition* in [**17**]. See [**3**, 4.2] and [**13**, 2.2] for the proofs.

4.3. Let M and N be finitely generated modules over a complete intersection R. Then

(1) $\mathrm{Ext}^i_R(M,N)=0$ for all $i \gg 0$ if and only if $\mathrm{Ext}^i_R(M,N)=0$ for all $i > \min\{\mathrm{depth}R - \mathrm{depth}M, \mathrm{depth}R - \mathrm{depth}N\}$.

(2) $\mathrm{Tor}^R_i(M,N)=0$ for all $i \gg 0$ if and only if $\mathrm{Tor}^R_i(M,N)=0$ for all $i > \min\{\mathrm{depth}R - \mathrm{depth}M, \mathrm{depth}R - \mathrm{depth}N\}$.

The second relevant property is a special case of what is proved in [**4**, 5.6]. Below we let $(-)^*$ denote the dual $\mathrm{Hom}_R(-,R)$.

4.4. Let M and N be finitely generated modules over a complete intersection R. Then

(1) $\mathrm{Ext}^i_R(M,N)=0$ for all $i \gg 0$ if and only if $\mathrm{Ext}^i_R(M,N^*)=0$ for all $i \gg 0$.

(2) $\mathrm{Tor}^R_i(M,N)=0$ for all $i \gg 0$ if and only if $\mathrm{Tor}^R_i(M,N^*)=0$ for all $i \gg 0$.

We can generalize these aspects of non-trivial vanishing to minimal intersections. The trade-off to considering a class of rings much more general than the complete intersections is that we establish non-trivial vanishing for specific classes of modules — those consisting of modules as described in Proposition 4.2. We remark that Proposition 4.2 does not describe the only way in which non-trivial vanishing can occur over complete intersections. See Example 5.2 below.

Vanishing over Arbitrary Minimal Intersections. We assume now that $R = Q/(I_1 + I_2)$ is a minimal intersection with Q a regular local ring, and $R_1 = Q/I_1$ and $R_2 = Q/I_2$. The following theorem is the main result of the paper.

Theorem 4.5. *Let R be a minimal intersection, M' any sufficiently high syzygy module over R_1 of a finitely generated R_1-module, and N' any sufficiently high syzygy module over R_2 of a finitely generated R_2-module. Let L' any sufficiently high cosyzygy module over R_2 of a finitely generated R_2-module. Then for $M = M' \otimes_{R_1} R$, $N = N' \otimes_{R_2} R$, and $L = \operatorname{Hom}_Q(R_1, L')$ the following hold:*

(1) $\operatorname{Tor}_i^R(M, N) = 0$ *for all* $i > \dim Q$*;*

(2) $\operatorname{Ext}_R^i(M, L) = 0$ *for all* $i > \dim Q$*;*

(3) $\operatorname{pd}_R M = \operatorname{pd}_{R_1} M'$, $\operatorname{pd}_R N = \operatorname{pd}_{R_2} N'$, $\operatorname{id}_R L = \operatorname{id}_{R_2} L'$.

Proof. Let N'' be any finitely generated R_2-module. Since Q is a regular local ring we have $\operatorname{Tor}_i^Q(R_1, N'') = 0$ for all $i \gg 0$. Take an exact sequence $0 \to N' \to R_2^b \to N'' \to 0$. Then from the derived long exact sequence of Tor

$$\cdots \to \operatorname{Tor}_i^Q(R_1, N') \to \operatorname{Tor}_i^Q(R_1, R_2^b) \to \operatorname{Tor}_i^Q(R_1, N'') \to \cdots,$$

and the fact that $\operatorname{Tor}_i^Q(R_1, R_2) = 0$ for all $i \geq 1$, we have the isomorphisms $\operatorname{Tor}_{i+1}^Q(R_1, N'') \cong \operatorname{Tor}_i^Q(R_1, N')$ for all $i \geq 1$. Thus if N' is a sufficiently high syzygy over R_2, we may assume that

$$\operatorname{Tor}_i^Q(R_1, N') = 0 \quad \text{for all } i \geq 1. \tag{4.5.1}$$

Applying 2.1(1) we get the isomorphisms

$$\operatorname{Tor}_i^Q(M', N') \cong \operatorname{Tor}_i^{R_1}(M', N' \otimes_Q R_1) = \operatorname{Tor}_i^{R_1}(M', N)$$

for all i. Note that $\operatorname{Tor}_i^Q(R_1, R_2) = 0$ for all $i \geq 1$ implies that a minimal free resolution of R over R_1 is attained by tensoring a minimal free resolution of R_2 over Q with R_1. Thus R has finite projective dimension over R_1. By choosing a sufficiently high syzygy M' over R_1 we can assume that

$$\operatorname{Tor}_i^{R_1}(M', R) = 0 \quad \text{for all } i \geq 1. \tag{4.5.2}$$

By 2.1(1) we have the isomorphisms

$$\operatorname{Tor}_i^{R_1}(M', N) \cong \operatorname{Tor}_i^{R}(M' \otimes_{R_1} R, N) = \operatorname{Tor}_i^{R}(M, N)$$

for all i. Thus $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > \dim Q$, and this establishes the claim about the vanishing of homology.

To see the statements regarding the projective dimensions of M and N, note that by (4.5.2) a minimal free resolution of M over R is obtained by tensoring a minimal free resolution of M' over R_1 with R. Thus $\operatorname{pd}_R M = \operatorname{pd}_{R_1} M'$. By symmetry we have $\operatorname{pd}_R N = \operatorname{pd}_{R_2} N'$.

For (2), let L'' be an R_2-module. Since Q is a regular local ring, $\operatorname{Ext}_Q^i(R_1, L'') = 0$ for all $i \gg 0$. Let $0 \to L'' \to \mathcal{I} \to L' \to 0$ be an exact sequence of R_2-modules with $\mathcal{I}$ injective. Then $\mathcal{I}$ is a direct sum of injective hulls $E_{R_2}(R_2/p)$ of quotients R_2/p with p a prime ideal of R_2. If P is a prime ideal of Q which is a preimage of p, then $E_{R_2}(R_2/p) = \operatorname{Hom}_Q(R_2, E_Q(Q/P))$, where $E_Q(Q/P)$ is the injective hull of Q/P. Thus we can write $\mathcal{I} = \operatorname{Hom}_Q(R_2, \mathcal{J})$ where $\mathcal{J}$ is an injective Q-module. We have the isomorphisms $\operatorname{Ext}_Q^i(R_1, \operatorname{Hom}_Q(R_2, \mathcal{J})) \cong \operatorname{Hom}_Q(\operatorname{Tor}_i^Q(R_1, R_2), \mathcal{J})$ for all i (see, for example, [**19**, page 360]). Therefore we have $\operatorname{Ext}_Q^i(R_1, \mathcal{I}) = 0$ for all $i \geq 1$, and so from the long exact sequence of Ext

$$\cdots \to \operatorname{Ext}_Q^i(R_1, L'') \to \operatorname{Ext}_Q^i(R_1, \mathcal{I}) \to \operatorname{Ext}_Q^i(R_1, L') \to \cdots .$$

we get

$$\operatorname{Ext}_Q^i(R_1, L') \cong \operatorname{Ext}_Q^{i+1}(R_1, L'')$$

for all $i \geq 1$. Now it is clear that we can replace L'' by an R_2-module L' such that

$$\operatorname{Ext}_Q^i(R_1, L') = 0 \quad \text{for all} \quad i \geq 1. \tag{4.5.3}$$

By 2.1(2) we have the isomorphisms

$$\operatorname{Ext}_Q^i(M', L') \cong \operatorname{Ext}_{R_1}^i(M', \operatorname{Hom}_Q(R_1, L')) = \operatorname{Ext}_{R_1}^i(M', L)$$

for all i. As in the part of the proof for (1) above, we can choose a finitely generated R_1-module M' such that $\operatorname{Tor}_i^{R_1}(M', R) = 0$ for all $i \geq 1$, which by 2.1(1) gives

$$\operatorname{Ext}_{R_1}^i(M', L) \cong \operatorname{Ext}_R^i(M' \otimes_{R_1} R, L) = \operatorname{Ext}_R^i(M, L)$$

for all i. Therefore we have $\mathrm{Ext}^i_R(M, L) = 0$ for all $i > \dim Q$.

To finish the proof we just need to justify that $\mathrm{id}_R L = \mathrm{id}_{R_2} L'$. By 3.1, $\mathrm{Tor}^Q_i(R_1, R_2) = 0$ for all $i \geq 1$. Then 2.1(1) shows that $\mathrm{Ext}^i_Q(R_1, L') \cong \mathrm{Ext}^i_{R_2}(R, L')$ for all i, in particular, $L \cong \mathrm{Hom}_{R_2}(R, L')$. Then by (4.5.3) we have $\mathrm{Ext}^i_{R_2}(R, L') = 0$ for all $i \geq 1$. Thus a minimal injective resolution of L over R is obtained by applying $\mathrm{Hom}_{R_2}(R, -)$ to a minimal injective resolution of L' over R_2, and so $\mathrm{id}_R L = \mathrm{id}_{R_2} L'$. □

Over Cohen-Macaulay minimal intersections we can establish non-trivial vanishing of Tor for a larger class of modules, and non-trivial vanishing of Ext for pairs of finitely generated modules. Indeed, over a Cohen-Macaulay ring a higher syzygy module is maximal Cohen-Macaulay, but a maximal Cohen-Macaulay module need not be a higher syzygy module.

Note that for the classes of modules identified in the following corollary, Property 4.3 holds.

Recall from 3.3 that R is Cohen-Macaulay if and only if both R_1 and R_2 are Cohen-Macaulay. We let $(-)^\vee$ denote the dual $\mathrm{Hom}_R(-, \omega)$, were ω is the canonical module of R.

Corollary 4.6. *Let R be a Cohen-Macaulay minimal intersection. Suppose that M' is a maximal Cohen-Macaulay R_1-module, and N' is a maximal Cohen-Macaulay R_2-module. Then for $M = M' \otimes_{R_1} R$ and $N = N' \otimes_{R_2} R$ we have*

(1) *M, N, and $M \otimes_R N$ are maximal Cohen-Macaulay R-modules;*

(2) $\mathrm{Tor}^R_i(M, N) = 0$ *for all* $i \geq 1$*;*

(3) $\mathrm{Ext}^i_R(M, N^\vee) = 0$ *for all* $i \geq 1$*;*

(4) $\mathrm{pd}_R M = \infty$ *if and only if M' is not free;* $\mathrm{pd}_R N = \infty$ *if and only if N' is not free if and only if* $\mathrm{id}_R N^\vee = \infty$.

Proof. (1). From Theorem 3.4 we have that

$$\mathrm{depth}_{Q_p} Q_p - \mathrm{depth}_{Q_p} (R_1)_p - \mathrm{depth}_{Q_p} (R_2)_p = -\mathrm{depth}_{Q_p} R_p \leq 0$$

for all primes p of Q. Since M' is a maximal Cohen-Macaulay R_1-

module and N' is a maximal Cohen-Macaulay R_2-module,

$$\operatorname{depth}_{Q_p}(R_1)_p = \operatorname{depth}_{Q_p} M'_p \text{ and } \operatorname{depth}_{Q_p}(R_2)_p = \operatorname{depth}_{Q_p} N'_p$$

for all primes p of Q. Thus for all primes p of Q,

$$\operatorname{depth}_{Q_p} Q_p - \operatorname{depth}_{Q_p} M'_p - \operatorname{depth}_{Q_p} N'_p = \\ \operatorname{depth}_{Q_p} Q_p - \operatorname{depth}_{Q_p}(R_1)_p - \operatorname{depth}_{Q_p}(R_2)_p \leq 0$$

Thus by 2.2 we obtain

$$\operatorname{Tor}_i^Q(M', N') = 0 \quad \text{for all} \quad i \geq 1. \tag{4.6.1}$$

It follows that $\operatorname{pd}_Q(M' \otimes_Q N') = \operatorname{pd}_Q M' + \operatorname{pd}_Q N'$. Now the Auslander-Buchsbaum formula gives the equation

$$\operatorname{depth}_Q Q + \operatorname{depth}_Q(M' \otimes_Q N') = \operatorname{depth}_Q M' + \operatorname{depth}_Q N'$$

Using the fact that M' and N' are maximal Cohen-Macaulay, and comparing with $\operatorname{depth}_Q Q + \operatorname{depth}_Q R = \operatorname{depth}_Q R_1 + \operatorname{depth}_Q R_2$ from 3.2, we see that $\operatorname{depth}_Q R = \operatorname{depth}_Q(M \otimes_Q N)$, and this is the same as $\operatorname{depth}_R R = \operatorname{depth}_R(M \otimes_R N)$.

The same proof shows that M and N are both maximal Cohen-Macaulay, just by replacing N' by R_2, and M' by R_1, respectively.

(2). Following the proof of Theorem 4.5(1), and in light of (4.6.1), we just need to show that (4.5.1) and (4.5.2) hold. As in the argument for part (1), we have $\operatorname{Tor}_i^Q(R_1, N') = 0$ for all $i \geq 1$, which is (4.5.1). Similarly, $\operatorname{Tor}_i^Q(M', R_2) = 0$ for all $i \geq 1$, and since $\operatorname{Tor}_i^Q(R_1, R_2) = 0$ for all $i \geq 1$, 2.1(1) implies we also have $\operatorname{Tor}_i^{R_1}(M', R) = 0$ for all $i \geq 1$, which is (4.5.2).

Property (3) follows from (1), (2), and Proposition 2.4.

By Theorem 4.5(3,4) the only part of (4) we need to show is the last statement. We have N' is free over R_2 if and only if $\operatorname{Tor}_i^R(k, N) = 0$ for all $i \gg 0$ if and only if (by Proposition 2.4) $\operatorname{Ext}_R^i(k, N^\vee) = 0$ for all $i \gg 0$ if and only if $\operatorname{id}_R N^\vee < \infty$. □

Remark. The plentitude of modules involved in non-trivial vanishing according to Corollary 4.6 thus depends on the number of non-isomorphic indecomposable maximal Cohen-Macaulay modules over R_1

and R_2. Much work has been done on the classification of Cohen-Macaulay rings having only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules, the so-called rings of finite Cohen-Macaulay type. See [**21**] for a survey of the subject. In particular, a Cohen-Macaulay ring of finite Cohen-Macaulay type has at most an isolated singularity [**10**]. Outside of this case, the literature suggests that the number of non-isomorphic indecomposable maximal Cohen-Macaulay modules over a Cohen-Macaulay ring is quite large (see, for example, [**6**]).

A stronger analogy to vanishing over complete intersections is attained when we assume that R is Gorenstein: Properties (4) and (5) below mimic 4.4, and (6) that of 4.1(3).

Recall by Proposition 3.3 that R is Gorenstein if and only if both R_1 and R_2 are Gorenstein. Below we let $(-)^*$ denote the dual $\operatorname{Hom}_R(-, R)$.

Corollary 4.7. *Let R be a Gorenstein minimal intersection. Suppose that M' is a maximal Cohen-Macaulay R_1-module, and N' is a maximal Cohen-Macaulay R_2-module. Then for $M = M' \otimes_{R_1} R$ and $N = N' \otimes_{R_2} R$ we have*

(1) *M, N, and $M \otimes_R N$ are maximal Cohen-Macaulay R-modules;*

(2) $\operatorname{Tor}_i^R(M, N) = 0$ *for all $i \geq 1$;*

(3) $\operatorname{Ext}_R^i(M, N^*) = 0$ *for all $i \geq 1$;*

(4) $\operatorname{Tor}_i^R(M, N^*) = 0$ *for all $i \geq 1$;*

(5) $\operatorname{Ext}_R^i(M, N) = 0$ *for all $i \geq 1$;*

(6) $\operatorname{Ext}_R^i(N, M) = 0$ *for all $i \geq 1$;*

(7) $\operatorname{pd}_R M = \infty$ *if and only if M' is not free, and* $\operatorname{pd}_R N^* = \infty$ *if and only if N' is not free.*

Proof. Properties (1)-(3) and (7) are handled by Corollary 4.6. For (4), (5), and (6) it suffices to show that $N^* \cong \operatorname{Hom}_{R_2}(N', R_2) \otimes_{R_2} R$. But this is exactly the statement of Proposition 2.3. ◻

5. Examples and a Sufficient Condition. The following is an example illustrating that non-trivial vanishing can occur over rings which are not minimal intersections.

Example 5.1. Let $Q = k[[W, X, Y, Z]]$ where k is a field, and

$$R = Q/(W^2, X^2, Z^2, XY, WX + XZ, WY + YZ, Y^2 - WZ).$$

Then one may check that R is a zero-dimensional local ring with $\mathrm{P}_R^Q(t) = 1+7t+13t^2+10t^3+3t^4$. According to Proposition 3.5, R is not a minimal intersection with respect to Q. Let w denote the image of W in R, etc. Consider the finitely generated R-modules $M = \operatorname{coker}\varphi$, where φ is represented with respect to the standard basis of R^2 by the matrix $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$, and $N = R$ Then we have $\operatorname{Ext}_R^i(M, N) = 0$ for all $i > 0$. Moreover, $\operatorname{pd}_R M = \infty$, and $\operatorname{id}_R N = \infty$.

The following example shows that Proposition 4.2 does not describe the only way non-trivial vanishing occurs over complete intersections. The details of the example are proven in [**15**].

Example 5.2. Let $Q = k[[V, W, X, Y, Z]]$, with k a field, and $R = Q/(VW, XY)$. Then R is a codimension two complete intersection. Let v denote the image of V in R, etc., and M be the cokernel of the map $\varphi : R^8 \to R^8$ represented with respect to the standard basis of R^8 by the matrix

$$\begin{pmatrix} -v & 0 & 0 & -z & 0 & 0 & 0 & y \\ -w & 0 & -z & 0 & 0 & 0 & y & 0 \\ 0 & 0 & v & -w & 0 & y & 0 & 0 \\ 0 & 0 & 0 & w & y & 0 & 0 & 0 \\ 0 & -w & 0 & x & 0 & 0 & 0 & 0 \\ 0 & -w & x & 0 & 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 & 0 & 0 & 0 \\ x & z & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For $N = R/(v)$, we have $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. Moreover, M is not of the form described in Proposition 4.2. That is, for no minimal generator f of (VW, XY) is there a maximal Cohen-Macaulay $R_1 = Q/(f)$-module M' such that $M \cong M' \otimes_{R_1} R$.

A Sufficient Condition. Let $R = Q/(I_1 + I_2)$ be a minimal intersection with Q a regular local ring, and $R_1 = Q/I_1$, $R_2 = Q/I_2$. In this section we discuss a sufficient condition for determining whether a finitely generated R-module M has a syzygy over R of the form $M' \otimes_{R_1} R$ for some R_1-module M' of infinite projective dimension over R_1 satisfying $\operatorname{Tor}_i^{R_1}(M', R) = 0$ for all $i \geq 1$, and hence of the form identified in Theorem 4.5.

Let

$$\mathbf{F} : \cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to M \to 0$$

be a minimal R-free resolution of M. Choose a sequence of free Q-modules $\widetilde{F}_i$ and maps $\widetilde{\partial}_i$ between them

$$\widetilde{\mathbf{F}} : \cdots \to \widetilde{F}_2 \xrightarrow{\widetilde{\partial}_2} \widetilde{F}_1 \xrightarrow{\widetilde{\partial}_1} \widetilde{F}_0 \to 0$$

such that $\mathbf{F}$ and $\widetilde{\mathbf{F}} \otimes_Q R$ are isomorphic complexes. It is useful to think of the maps ∂_i as being given by matrices over R (with respect to some fixed bases of the F_i), in which case the maps $\widetilde{\partial}_i$ may be thought of as matrices of preimages in Q of the entries of the matrices representing the ∂_i. Since $\mathbf{F}$ is a complex of R-modules we have $\widetilde{\partial}_{i-1}\widetilde{\partial}_i \equiv 0$ modulo $I_1 + I_2$, in other words $(\widetilde{\partial}_{i-1} \otimes_Q R)(\widetilde{\partial}_i \otimes_Q R) = 0$. For the sufficient condition given below we will be considering the sequences of maps

$$\widetilde{F}_i \otimes_Q R_j \xrightarrow{\widetilde{\partial}_i \otimes R_j} \widetilde{F}_{i-1} \otimes_Q R_j \xrightarrow{\widetilde{\partial}_{i-1} \otimes R_j} \widetilde{F}_{i-2} \otimes_Q R_j \tag{5.2.1}$$

For $j = 1, 2$ and $i \geq 2$.

Proposition 5.3. *Let M be a finitely generated R-module of infinite projective dimension over R, and suppose $(\widetilde{\mathbf{F}}, \widetilde{\partial})$ is some lifting to Q of a minimal R-free resolution $(\mathbf{F}, \partial)$ of M. If the sequence of maps* (5.2.1) *forms an exact sequence for some $i \geq 2$, then M has a syzygy over R of the form $M' \otimes_Q R_j$ where M' is an R_j module satisfying $\operatorname{Tor}_l^{R_j}(M', R) = 0$ for all $l \geq 1$, and hence M participates in a non-trivial vanishing of all higher Tor.*

Proof. Without loss of generality assume that $j = 1$, and that (5.2.1) forms an exact sequence for fixed $i \geq 2$. Let $M'_{i-2} := \operatorname{coker}(\widetilde{\partial}_{i-1} \otimes R_1)$. Then

$$\widetilde{F}_i \otimes_Q R_1 \xrightarrow{\widetilde{\partial}_i \otimes R_1} \widetilde{F}_{i-1} \otimes_Q R_1 \xrightarrow{\widetilde{\partial}_{i-1} \otimes R_1} \widetilde{F}_{i-2} \otimes_Q R_1 \to M'_{i-2} \to 0$$

is the beginning of an R_1-free resolution of M'_{i-2}. Tensoring this complex with R we get $F_i \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} F_{i-2}$, which is exact. This means that $\operatorname{Tor}_1^{R_1}(M'_{i-2}, R) = 0$. Since $\operatorname{Tor}_i^Q(R_1, R_2) = 0$ for all $i \geq 1$, by 2.1(1) we have $\operatorname{Tor}_l^{R_1}(M'_{i-2}, R) \cong \operatorname{Tor}_l^Q(M'_{i-2}, R_2)$ for all $l \geq 1$. Therefore, by rigidity of Tor for regular local rings, $\operatorname{Tor}_l^{R_1}(M'_{i-2}, R) = 0$ for all $l \geq 1$. This finishes the proof, since $M'_{i-2} \otimes_{R_1} R \cong \operatorname{coker}\partial_{i-1}$ is a syzygy of M over R. $\square$

Remark. Suppose $j = 1$. If I_2 happens to be generated by a Q-regular sequence, then we *a priori* only need to know that the sequence of maps in (5.2.1) with $j = 1$ forms a *complex* in order to invoke the conclusion of Proposition 5.3. For if (5.2.1) forms a complex with $j = 1$, and I_2 is generated by a Q-regular sequence, then this sequence is also regular on R_1, and by working our way inductively from R up to R_1, Nakayama's lemma yields that (5.2.1) is exact.

Next we give examples using *Macaulay* 2 which illustrate Proposition 5.3. We first discuss a few details of the liftings $(\widetilde{\mathbf{F}}, \widetilde{\partial})$, and define special maps based on the notion of Eisenbud operators, which were developed in [**7**] for finitely generated modules over a complete intersection.

Fix a minimal generating set $f_1, \ldots, f_c$ of $I_1 + I_2$ such that I_1 is generated by $f_1, \ldots, f_r$ and I_2 generated by $f_{r+1}, \ldots, f_c$. (By our assumption that I_1 and I_2 are non-zero, we have $1 \leq r \leq c-1$.) Since the products $\widetilde{\partial}_{i-1}\widetilde{\partial}_i$ are zero modulo $I_1 + I_2$, we may express them in terms of the f_j: write

$$\widetilde{\partial}_{i-1}\widetilde{\partial}_i = \sum_{j=1}^{c} f_j \widetilde{t}_{i,j}, \tag{5.3.1}$$

where the $\widetilde{t}_{i,j}$ are maps $\widetilde{t}_{i,j} : \widetilde{F}_i \to \widetilde{F}_{i-2}$. Note that these maps are not uniquely defined. They depend first on the resolution $\mathbf{F}$, then on the lifting $(\widetilde{\mathbf{F}}, \widetilde{\partial})$, and then on the choice of the expression in (5.3.1).

In investigating when the sequence (5.2.1)

$$\widetilde{F}_i \otimes_Q R_j \xrightarrow{\widetilde{\partial}_i \otimes R_j} \widetilde{F}_{i-1} \otimes_Q R_j \xrightarrow{\widetilde{\partial}_{i-1} \otimes R_j} \widetilde{F}_{i-2} \otimes_Q R_j$$

is exact, we proceed in two steps. First we need to know when it forms a complex. For $j = 1$ this is implied by the condition

$$\widetilde{t}_{i,r+1} \otimes R_1 = \cdots = \widetilde{t}_{i,c} \otimes R_1 = 0, \tag{5.3.2}$$

and for $j = 2$ the condition

$$\widetilde{t}_{i,1} \otimes R_2 = \cdots = \widetilde{t}_{i,r} \otimes R_2 = 0. \tag{5.3.3}$$

Once we know conditions (5.3.2) or (5.3.3) hold, we compute the homology of the corresponding complex (5.2.1) to see that it is zero.

In the following examples, we perform both steps using *Macaulay* 2. For the first step we use a special script, which can be obtained from the authors, called `getEisoplist` which computes the maps $\widetilde{t}_{ij}$ and stores them as a list of lists called `Eisoplist`. Because the internal indexing used by *Macaulay* 2 starts at 0, the element `Eisoplist#i#j` actually represents the map $\widetilde{t}_{i+2,j+1}$. The code may also compute the Eisenbud operators, which are the maps $t_{i,j} = \widetilde{t}_{i,j} \otimes_Q R$ defined in [**7**] in the case where R is a complete intersection.

The input for this script is a chain complex and an integer. Presumably, the chain complex is a free resolution $(\mathbf{F}, \partial)$ over R of the module M, and this may be obtained simply by using the `res` command in *Macaulay* 2. The integer tells the script up to what degree i the maps $\widetilde{t}_{i,j}$ should be computed. The lifting $(\widetilde{\mathbf{F}}, \widetilde{\partial})$ of the given resolution $(\mathbf{F}, \partial)$ is done in the script using the *Macaulay* 2 command `lift`. Finally, the choice of the $\widetilde{t}_{i,j}$ defined by expression (5.3.1) is decided in the script using the `//Igb` command in *Macaulay 2*, where `Igb` is a Gröbner basis of the ideal $(f_1, \ldots, f_c)$.

Example 5.4. Let $Q = \mathbb{Q}[x, y, z]$ and $R = Q/I$, where

$$I := (x^2 - yz, xz - y^2, z^2 - xy, x^2 + yz).$$

Then R is a zero-dimensional minimal intersection, and the R-module $M = R/(x + y + z)$ participates in non-trivial vanishing of all higher Tor.

We first load the script `getEisoplist`, then show that R is in fact an minimal intersection by testing $\operatorname{Tor}_1^Q(Q/I_1, Q/I_2) = 0$, then exhibit a minimal resolution of M, showing that it has infinite projective dimension over R.

```
i1 : load"getEisoplist.m2"
--loaded getEisoplist.m2

i2 : Q = QQ[x,y,z];

i3 : I = ideal(x^2-y*z,x*z-y^2,z^2-x*y,x^2+y*z);

o3 : Ideal of Q

i4 : Tor_1(coker matrix{{x^2-y*z,x*z-y^2,z^2-x*y}},coker
matrix{{x^2+y*z}}) == 0

o4 = true

i5 : R = Q/I

o5 = R

o5 : QuotientRing

i6 : M = coker matrix{{x+y+z}}

o6 = cokernel | x+y+z |

                             1
o6 : R-module, quotient of R

i7 : Mres = res(M,LengthLimit=>6)

      1      1      2      4      8      16      32
o7 = R  <-- R  <-- R  <-- R  <-- R  <-- R   <-- R

     0      1      2      3      4      5       6

o7 : ChainComplex
```

Now we compute the maps $\widetilde{t}_{i,j}$. What is shown is $\{\widetilde{t}_{2,1}, \widetilde{t}_{2,2}, \widetilde{t}_{2,3}, \widetilde{t}_{2,4}\}$. Notice that $\widetilde{t}_{2,4} = 0$, and so it is also zero modulo I_1.

```
i8 : MEisoplist = getEisoplist(Mres,2)

o8 = {{{2} | 0 1 |, {2} | -1 0 |, {2} | -1 -1 |, 0}}

o8 : List
```

The next step is check that the homology of the complex

$$\widetilde{F}_2 \otimes_Q R_1 \xrightarrow{\widetilde{\partial}_2 \otimes R_1} \widetilde{F}_1 \otimes_Q R_1 \xrightarrow{\widetilde{\partial}_1 \otimes R_1} \widetilde{F}_0 \otimes_Q R_1$$

is zero (although we do not really need this step, by the remark following Proposition 5.3, since $I_2 = (x^2 + yz)$ is generated by a regular element). First we need to define the ring R_1.

```
i9 : use Q

o9 = Q

o9 : PolynomialRing

i10 : R1 = Q/ideal(x^2-y*z,x*z-y^2,z^2-x*y)

o10 = R1

o10 : QuotientRing

i11 : homology(lift(Mres.dd_1,Q) ** R1,lift(Mres.dd_2,Q) ** R1) ==
0

o11 = true
```

Therefore, by Proposition 5.3, M participates in non-trivial vanishing.

We can build a companion module N for M as per Theorem 4.5, which yield non-trivial vanishing of all higher $\mathrm{Tor}_i^R(M,N)$. The steps below are: define R_2, resolve the residue field over this ring, take an appropriate syzygy, and tensor this syzygy down to the ring R.

```
i12 : use Q;

i13 : R2 = Q/ideal(x^2+y*z)

o13 = R2

o13 : QuotientRing
```

```
i14 : Ntres = res coker vars R2

        1       3       4       4       4
o14 = R2  <-- R2  <-- R2  <-- R2  <-- R2

      0       1       2       3       4

o14 : ChainComplex

i15 : N = (coker lift(Ntres.dd_3,Q)) ** R

o15 = cokernel {2} | x  z 0 x  |
               {2} | -y x 0 -y |
               {2} | z  0 x 0  |
               {2} | 0  z y x  |

                              4
o15 : R-module, quotient of R
```

The beginning of a minimal resolution of N over R is given to show that $\mathrm{pd}_R N = \infty$. Afterwards we compute $\{\widetilde{t}_{2,1}, \widetilde{t}_{2,2}, \widetilde{t}_{2,3}, \widetilde{t}_{2,4}\}$ for N. Note that $\widetilde{t}_{2,1} = \widetilde{t}_{2,2} = \widetilde{t}_{2,3} = 0$

```
i16 : Nres = res(N,LengthLimit=>6)

       4      4      4      4      4      4      4
o16 = R  <-- R  <-- R  <-- R  <-- R  <-- R  <-- R

      0      1      2      3      4      5      6

o16 : ChainComplex

i17 : NEisoplist = getEisoplist(Nres,2)

o17 = {{0, 0, 0, {2} | 1 0 0 1 |}}
                 {2} | 0 1 0 0 |
                 {2} | 0 0 1 0 |
                 {2} | 0 0 0 1 |

o17 : List
```

Finally, we compute the homology of the corresponding complex to show that it is zero. We also show that indeed the first several $\mathrm{Tor}_i^R(M,N)$ are zero.

```
i18 : homology(lift(Nres.dd_1,Q) ** R2,lift(Nres.dd_2,Q) ** R2) ==
0

o18 = true
```

```
i19 : Tor_1(M,N)==0,Tor_2(M,N)==0,Tor_3(M,N)==0

o19 = (true, true, true)

o19 : Sequence
```

Example 5.5. Let $Q = \mathbb{Q}[u, v, w, x, y, z]$ and $R = Q/I$, where

$$\begin{aligned} I = (&uv - vx, uw - uz - wx + xz, vw - vz, u^2 - v^2 - 2ux + x^2,\\ &v^2 - w^2 + 2wz - z^2, xy - vx, xz, yz - vz,\\ &x^2 - y^2 + 2vy - v^2, v^2 + y^2 - 2vy - z^2). \end{aligned}$$

Then R is a zero-dimensional Gorenstein minimal intersection.

```
i20 : Q = QQ[u,v,w,x,y,z];

i21 : I = ideal(u*v-v*x,u*w-u*z-w*x+x*z,v*w-v*z,u^2-v^2-2*u*x
+x^2,v^2-w^2+2*w*z-z^2,x*y-v*x,x*z,y*z-v*z,x^2-y^2+2*v*y-v^2,v^2
+y^2-2*v*y-z^2);

o21 : Ideal of Q
```

If we let I_1 be generated by the first five generators of I and I_2 generated by the second five, then we exhibit that R is an minimal intersection.

```
i22 : Tor_1(coker matrix{{ u*v-v*x,u*w-u*z-w*x+x*z,v*w-v*z,
u^2-v^2-2*u*x+x^2,v^2-w^2 +2*w*z-z^2}}, coker matrix{{x*y-v*x,x*z,
y*z-v*z,x^2-y^2+2*v*y-v^2,v^2+y^2-2*v*y-z^2}}) == 0

o22 = true
```

The last map in the following resolution of I over Q shows that in fact R is Gorenstein.

```
i23 : C=res I

       1      10       35       52       35       10       1
o23 = Q  <-- Q   <-- Q   <-- Q   <-- Q   <-- Q   <-- Q  <-- 0

      0      1        2        3        4        5        6      7

o23 : ChainComplex

i24 : ideal transpose C.dd_6 == I

o24 = true
```

Next we identify a module M which participates in non-trivial vanishing. We compute the $\widetilde{t}_{i,j}$ for M and show that $\widetilde{t}_{2,6} = \widetilde{t}_{2,7} = \widetilde{t}_{2,8} = \widetilde{t}_{2,9} = \widetilde{t}_{2,10} = 0$. Then we show that the corresponding complex

$$\widetilde{F}_2 \otimes_Q R_1 \xrightarrow{\widetilde{\partial}_2 \otimes R_1} \widetilde{F}_1 \otimes_Q R_1 \xrightarrow{\widetilde{\partial}_1 \otimes R_1} \widetilde{F}_0 \otimes_Q R_1$$

has zero homology.

```
i25 : R = Q/I;

i26 : M = coker matrix{{u-x,v,w-z}};

i27 : Mres = res(M,LengthLimit=>6)

       1      3      8      21      55      144      377
o27 = R  <-- R  <-- R  <-- R   <-- R   <-- R    <-- R

      0      1      2      3       4       5        6

o26 : ChainComplex

i28 : tM = getEisoplist(Mres,2);

i29 : tM#0#5,tM#0#6,tM#0#7,tM#0#8,tM#0#9

o29 = (0, 0, 0, 0, 0)


o29 : Sequence

i30 : use Q;

i31 : R1 = Q/ideal(u*v-v*x,u*w-u*z-w*x+x*z,v*w-v*z,
                 u^2-v^2-2*u*x+x^2,v^2-w^2+2*w*z-z^2);

i32 : homology(lift(Mres.dd_1,Q) ** R1,lift(Mres.dd_2,Q) ** R1) ==
0

o32 = true
```

Now we identify a companion module N for M such that the pair has non-trivial vanishing of all higher Tor. We compute the $\widetilde{t}_{i,j}$ for N and show that $\widetilde{t}_{2,1} = \widetilde{t}_{2,2} = \widetilde{t}_{2,3} = \widetilde{t}_{2,4} = \widetilde{t}_{2,5} = 0$. Then we show that the corresponding complex has zero homology.

```
i33 : use R;

i34 : N = coker matrix{{x,y-v,z}};

i35 : Nres = res(N,LengthLimit=>6)

       1      3      8      21       55       144       377
o35 = R  <-- R  <-- R  <-- R   <-- R   <-- R    <-- R

      0      1      2      3       4        5         6

o35 : ChainComplex

i36 : tN = getEisoplist(Nres,2);

i37 : tN#0#0,tN#0#1,tN#0#2,tN#0#3,tN#0#4

o37 = (0, 0, 0, 0, 0)

o37 : Sequence

i38 : use Q;

i39 : R2 =
Q/ideal(x*y-v*x,x*z,y*z-v*z,x^2-y^2+2*v*y-v^2,v^2+y^2-2*v*y-z^2);

i40 : homology(lift(Mres.dd_1,Q) ** R2,lift(Mres.dd_2,Q) ** R2) ==
0

o40 = true
```

Finally, we compute the first few Tors, and then following Theorem 4.7, we show that the first few $\mathrm{Ext}^i_R(M, \mathrm{Hom}_R(N,R))$ vanish.

```
i41 : Tor_1(M,N) == 0,Tor_2(M,N) == 0,Tor_3(M,N) == 0

o41 = (true, true, true)

o41 : Sequence

i42 : Hom(N,R)

o42 = image | z2 |

                                 1
o42 : R-module, submodule of R

i43 : Ext^1(M,Hom(N,R)) == 0,Ext^2(M,Hom(N,R)) ==
0,Ext^3(M,Hom(N,R)) == 0
```

```
o43 = (true, true, true)

o43 : Sequence
```

Appendix. Here is the *Macaulay 2* script `getEisoplist.m2`. The script can also be used to compute the Eisenbud operators in the case where R is a complete intersection by simply adding the line(s)

```
--- tensor t_(ij) with R ( = ring N)
tempCoef = substitute(tempCoef, ring N);
```

immediately following the line

```
l := 0;
```

near the end.

```
getEisoplist=method() getEisoplist(ChainComplex, ZZ):=(Nres,n)->(
    ---Input : A free resolution of a module over a graded quotient
    ---        ring and an integer indicating how far the
    ---        Eisenbud operators should be constructed.
    ---Output: A List of Lists of Matrices, which contain the generalized
    ---        Eisenbud operators t_(ij).
    --- get the module that the ChainComplex is a resolution of
    N := cokernel Nres.dd_1;
    --- get the ring that the module is over
    R := ring N;
    --- get the ring that R is a quotient of
    Q := ambient R;
    --- n is the highest degree for which the generalized Eisenbud
    --- operators are constructed
    firstMatrList := {};
    i := 1;
    --- the below loop builds a list of matrices which have entries
    --- in R and are the liftings of differentials of the resolution
    while (i <= n) do
    (
        firstMatrList = firstMatrList | {lift(Nres.dd_i, Q)};
        i = i + 1;
    );
    secondMatrList := {};
    i = 0;
    --- the below loop takes the entries of firstMatrList and composes
    --- the maps pairwise.  Using the n above, this creates a list of
    --- length n-1.
```

```
while (i < n-1) do
(
    secondMatrList =
    secondMatrList | {firstMatrList#i * firstMatrList#(i+1)};
    i = i + 1;
);
--- create a groebner basis for the ideal that we are modding out
--- in the ring of our module N
I := ideal R;
--- must use ChangeMatrix => true for division to be used
Igb := gb(I, ChangeMatrix => true);
i = 0;
Eisoplist := {};
--- the below loop builds a list of lists of matrices.  Each
--- entry in Eisoplist is a list of matrices whose length is
--- the number of generators of the ideal we are modding out by.
--- Since each product matrix is congruent to zero mod the f's,
--- we are able to represent the entries in terms of the generators
--- of the ideal.  We decompose each matrix to the form
--- (matrix_1)f1 * ... * (matrix_c)fc, where c is the number of
--- generators of the ideal we are modding out by.  Thus, when you
--- see Eisoplist#i#j below, it represents t_(ij), the f_jth component
--- of the ith matrix in the above secondMatrList
while (i < n-1) do
(
    j := 0;
    --- had to be a Mutable List to access and change elements
    tempMatrList := new MutableList from {};
    while (j < numgens target secondMatrList#i) do
    (
        tempRow := secondMatrList#i^{j};
        --- the below command inputs a row of a matrix and
        --- outputs a matrix whose columns are the components
        --- of the elements of the input matrix in the original
        --- basis of I.
        tempCoef := tempRow // Igb;
        --- if we are on the first row, we need to insert the
        --- matrix in the list, otherwise, just append to the
        --- one that we're on.
        l := 0;
        while (l < numgens I) do
        (
            if (j == 0) then
            (
                tempMatrList = append(tempMatrList, tempCoef^{l});
            )
            else
            (
                tempMatrList#l = tempMatrList#l || tempCoef^{l};
                );
```

```
                l = l + 1;
            );
            j = j + 1;
        );
        i = i + 1;
        Eisoplist = Eisoplist | {toList(tempMatrList)};
    );
    Eisoplist
)
```

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