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EXTENDED MODULES

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Introduction. Suppose R and S are local rings and $(R,\mathbf{m}) \to (S,\mathbf{n})$ is a flat local homomorphism. Given a finitely generated S-module N, we say N is *extended* (from R) provided there is an R-module M such that $S \otimes_R M$ is isomorphic to N as an S-module. If such a module M exists, it is unique up to isomorphism (cf. [**9**, (2.5.8)]), and it is necessarily finitely generated.

The $\mathbf{m}$-adic completion $R \to \widehat{R}$ and the Henselization $R \to R^{\mathrm{h}}$ are particularly important examples. One reason is that the Krull-Remak-Schmidt uniqueness theorem holds for direct-sum decompositions of finitely generated modules over a Henselian local ring. Indeed, failure of uniqueness for general local rings stems directly from the fact that some modules over the Henselization (or completion) are not extended. Understanding which R^{h}-modules are extended is the key to unraveling the direct-sum behavior of R-modules.

Throughout, we assume that $(R,\mathbf{m})$ and $(S,\mathbf{n})$ are Noetherian local rings and that $R \to S$ is a flat local homomorphism. Many of our results generalize easily to a mildly non-commutative setting. Moreover, it is not always necessary to assume that our rings are local. Thus, we assume that A is a commutative ring, that B is a faithfully flat commutative A-algebra, and that Λ is a module-finite A-algebra.

Given a finitely generated left $B \otimes_A \Lambda$-module N, we say that N is *extended* (from Λ) provided there is a finitely generated left Λ-module M such that $B \otimes_A M$ is isomorphic to N as a $B \otimes_A \Lambda$-module.

In Sections 1 and 2 of the paper, we examine how the extended modules sit inside the family of all finitely generated modules. In Sections 3 – 4 we consider rings of dimension 2 and 1, respectively, find

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criteria for a module to be extended, and show how the extendedness problem for one-dimensional rings reduces to the Artinian case. In Section 5, we find situations where every finitely generated B-module is a direct summand of an extended module, and in Section 6 we make a few observations about the Artinian case.

1. Two out of three: direct sums. Our goal in this section is to prove the following theorem, which generalizes Proposition 3.1 of [**12**]:

Theorem 1.1. *Let $A \to B$ be a faithfully flat homomorphism of commutative, Noetherian, semilocal rings. Let Λ be a module-finite A-algebra, and let N, N' and N'' be finitely generated left $B \otimes_A \Lambda$-modules, with $N \cong N' \oplus N''$. If two of the modules N, N', N'' are extended, so is the third.*

The first step in the proof is to observe that finitely generated modules over $\Gamma := B \otimes_A \Lambda$ satisfy direct-sum cancellation. This is essentially contained in E. G. Evans's paper [**8**], but we need a little argument to deal with the non-commutative setting.

Lemma 1.2. *Let Γ be a module-finite algebra over a commutative Noetherian semilocal ring B, and let U, V and W be finitely generated left Γ-modules. If $U \oplus W \cong V \oplus W$, then $U \cong V$.*

Proof. Let E be the endomorphism ring $\mathrm{End}_\Gamma(W)$, and let $\mathrm{J}(-)$ denote the Jacobson radical. Then $E/\,\mathrm{J}(E)$ is a module-finite $B/\,\mathrm{J}(B)$-algebra and therefore is Artinian. (Thus E is "semilocal" in the non-commutative sense.) Therefore E has 1 in the stable range (cf. [**10**, Theorem 4.4]), and by [**8**, Theorem 2] we have $U \cong V$. □

If, in Theorem 1.1, N' and N'' are extended, then N is obviously extended. The other two implications are symmetric, and so we want to prove that if N and N' are extended, then so is N''.

It is convenient to use the notation "$X \mid_\Lambda Y$", for Λ-modules X and Y, to mean that there is a Λ-module Z such that $X \oplus Z \cong Y$. When the ring Λ is understood, we will write "$X \mid Y$".

Assume, now, that $N \cong B \otimes_A M$ and $N' \cong B \otimes_A M'$, where M and M' are left Λ-modules. Then $(B \otimes_A M') \oplus N'' \cong B \otimes_A M$, so $B \otimes_A M' \mid_{B\otimes_A\Lambda} B \otimes_A M$. Suppose we can conclude that $M' \mid_\Lambda M$, say, $M' \oplus M'' \cong M$. Then we will have $(B \otimes_A M') \oplus (B \otimes_A M'') \cong B \otimes_A M$, and, by Lemma 1.2, $N'' \cong B \otimes_A M''$. Therefore Theorem 1.1 is a consequence of the following result (where we no longer need to assume that B is semilocal):

Theorem 1.3. *Let $A \to B$ be a faithfully flat homomorphism of commutative Noetherian rings, with A semilocal. Let Λ be a module-finite A-algebra, and let M and M' be finitely generated left Λ-modules. If $B \otimes_A M' \mid_{B\otimes_A\Lambda} B \otimes_A M$, then $M' \mid_\Lambda M$.*

Note that this includes the well-known result on faithfully flat descent of isomorphism [**9**, (2.5.8)]: If $B \otimes_A M' \cong_{B\otimes_A\Lambda} B \otimes_A M$, Theorem 1.3 implies that $M' \mid M$ and $M \mid M'$, and it follows easily that $M' \cong M$. Our proof of the theorem is based on the following beautiful result due to R. Guralnick:

Theorem 1.4. [**13**, Theorem A] *Let $(R, \mathbf{m}, k)$ be a local ring and Λ a module-finite R-algebra. Given finitely generated left Λ-modules U and V, there is an integer $e = e(U, V)$, depending only on U and V, with the following property: For each positive integer f and each Λ-homomorphism $\xi : U/\mathbf{m}^{e+f}U \to V/\mathbf{m}^{e+f}V$, there exists $\sigma \in \operatorname{Hom}_\Lambda(U, V)$ such that σ and ξ induce the same homomorphism $U/\mathbf{m}^f U \to V/\mathbf{m}^f V$. (Thus the outer and bottom squares in the diagram below both commute, though the top square may not.)*

$$\begin{array}{ccc}
U & \xrightarrow{\quad\sigma\quad} & V \\
\downarrow & & \downarrow \\
U/\mathbf{m}^{e+f}U & \xrightarrow{\ \xi\ } & V/\mathbf{m}^{e+f}V \\
\downarrow & & \downarrow \\
U/\mathbf{m}^{f}U & \xrightarrow{\overline{\xi} = \overline{\sigma}} & V/\mathbf{m}^{f}V
\end{array}$$

Proof. Choose exact sequences of left Λ-modules

$$\Lambda^{(n)} \xrightarrow{\alpha} \Lambda^{(m)} \to U \to 0,$$
$$\Lambda^{(n)} \xrightarrow{\beta} \Lambda^{(m)} \to V \to 0.$$

Define a Λ-homomorphism

$$T : \mathrm{End}_\Lambda(\Lambda^{(m)}) \times \mathrm{End}_\Lambda(\Lambda^{(n)}) \to \mathrm{Hom}_\Lambda(\Lambda^{(n)}, \Lambda^{(m)})$$

by $T(\mu,\nu) = \mu\alpha - \beta\nu$. Applying the Artin-Rees Lemma to the submodule $\mathrm{Im}(T)$ of $\mathrm{Hom}_\Lambda(\Lambda^{(n)}, \Lambda^{(m)})$, we get a positive integer $e = e(U,V)$ such that

(1) $\quad \mathrm{Im}(T) \cap \mathbf{m}^{e+f} \mathrm{Hom}_\Lambda(\Lambda^{(n)}, \Lambda^{(m)}) \subseteq \mathbf{m}^f \mathrm{Im}(T)$ for each $f > 0$.

Suppose now that $\xi : U/\mathbf{m}^{e+f}U \to V/\mathbf{m}^{e+f}V$. We can lift ξ to homomorphisms $\overline{\mu_0}$ and $\overline{\nu_0}$ making the following diagram commute:

$$(2) \qquad \begin{array}{ccccccc} (\Lambda/\mathbf{m}^{e+f}\Lambda)^{(n)} & \xrightarrow{\overline{\alpha}} & (\Lambda/\mathbf{m}^{e+f}\Lambda)^{(m)} & \longrightarrow & U/\mathbf{m}^{e+f}U & \longrightarrow & 0 \\ \downarrow{\scriptstyle \overline{\nu_0}} & & \downarrow{\scriptstyle \overline{\mu_0}} & & \downarrow{\scriptstyle \xi} & & \\ (\Lambda/\mathbf{m}^{e+f}\Lambda)^{(n)} & \xrightarrow{\overline{\beta}} & (\Lambda/\mathbf{m}^{e+f}\Lambda)^{(m)} & \longrightarrow & V/\mathbf{m}^{e+f}V & \longrightarrow & 0 \end{array}$$

Lifting $\overline{\mu_0}$ and $\overline{\nu_0}$ to maps $\mu_0 \in \mathrm{End}_\Lambda(\Lambda^{(m)})$ and $\nu_0 \in \mathrm{End}_\Lambda(\Lambda^{(n)})$, we see, by commutativity of (2), that $T(\mu_0,\nu_0) \in \mathbf{m}^{e+f} \mathrm{Hom}_\Lambda(\Lambda^{(n)}, \Lambda^{(m)})$. By (1), $T(\mu_0,\nu_0) \in \mathbf{m}^f \mathrm{Im}(T)$. Therefore there is a pair $(\mu_1,\nu_1) \in \mathbf{m}^f(\mathrm{End}_\Lambda(\Lambda^{(m)}) \times \mathrm{End}_\Lambda(\Lambda^{(n)}))$ such that $T(\mu_1,\nu_1) = T(\mu_0,\nu_0)$. Set $(\mu,\nu) := (\mu_0,\nu_0) - (\mu_1,\nu_1)$. Then $T(\mu,\nu) = 0$, so μ induces a Λ-homomorphism $\sigma : U \to V$. Since μ and μ_0 agree modulo $\mathbf{m}^f$, σ and ξ induce the same map $U/\mathbf{m}^fU \to V/\mathbf{m}^fV$. $\square$

Corollary 1.5. [**13**, Corollary 2]. *Let $(R,\mathbf{m},k)$ be a local ring and Λ a module-finite R-algebra. Given finitely generated left Λ-modules U and V, put $\ell := \max\{e(U,V), e(V,U)\}$, where $e(-,-)$ is as in Theorem 1.4. If $U/\mathbf{m}^{\ell+1}U \mid V/\mathbf{m}^{\ell+1}V$, then $U \mid V$.*

Proof. Choose Λ-module homomorphisms $\xi : U/\mathbf{m}^{\ell+1}U \to V/\mathbf{m}^{\ell+1}V$ and $\eta : V/\mathbf{m}^{\ell+1}V \to U/\mathbf{m}^{\ell+1}U$ such that $\eta\xi = 1_{U/\mathbf{m}^{\ell+1}U}$. By Theorem

1.4 there exist Λ-homomorphisms $\sigma : U \to V$ and $\tau : V \to U$ such that σ agrees with ξ modulo $\mathbf{m}$ and τ agrees with η modulo $\mathbf{m}$. By Nakayama's lemma, $\tau\sigma : U \to U$ is surjective and therefore an automorphism. It follows that $U \mid V$. □

We need one more preliminary result before we can prove Theorem 1.3.

Lemma 1.6. [**22**, 1.2] *Let $A \to B$ be a faithfully flat homomorphism of commutative rings, let Λ be a module-finite A-algebra, and let U and V be finitely presented left Λ-modules. If $B \otimes_A U \mid_{B\otimes_A\Lambda} B \otimes_A V$, then there is a positive integer r such that $U \mid_\Lambda V^{(r)}$.*

Proof. Choose $B \otimes_A \Lambda$-homomorphisms $B \otimes_A U \xrightarrow{\alpha} B \otimes_A V$ and $B \otimes_A V \xrightarrow{\beta} B \otimes_A U$ such that $\beta\alpha = 1_{B\otimes_A U}$. Since ${}_\Lambda V$ is finitely presented, the natural map

$$B \otimes_A \operatorname{Hom}_\Lambda(V, U) \to \operatorname{Hom}_{B\otimes_A\Lambda}(B \otimes_A V, B \otimes_A U)$$

is an isomorphism. Therefore we can write $\beta = b_1 \otimes \sigma_1 + \cdots + b_r \otimes \sigma_r$, with $b_i \in B$ and $\sigma_i \in \operatorname{Hom}_\Lambda(V, U)$ for each i. Put $\sigma := [\sigma_1 \cdots \sigma_r] : V^{(r)} \to U$. Then

$$(1_B \otimes \sigma) \begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix} \alpha = 1_{B\otimes_A U},$$

so $1_B \otimes \sigma : B \otimes_A V^{(r)} \to B \otimes_A U$ is a split surjection. It follows that $1_B \otimes \sigma$ induces a surjective map $\operatorname{Hom}_{B\otimes_A\Lambda}(B \otimes_A U, B \otimes_A V^{(r)}) \to \operatorname{Hom}_{B\otimes_A\Lambda}(B \otimes_A U, B \otimes_A U)$. Since ${}_\Lambda U$ is finitely presented, this map is obtained by change of rings from the map $\operatorname{Hom}_\Lambda(U, V^{(r)}) \to \operatorname{Hom}_\Lambda(U, U)$ induced by σ. By faithful flatness, the map $\operatorname{Hom}_\Lambda(U, V^{(r)}) \to \operatorname{Hom}_\Lambda(U, U)$ is surjective, whence σ is a split surjection. □

Proof of Theorem 1.3. Suppose first that A is local with maximal ideal $\mathfrak{m}$. By choosing a maximal ideal $\mathfrak{n}$ of B with $\mathfrak{m}B \subseteq \mathfrak{n}$, we can replace $A \to B$ by the flat local homomorphism $A \to B_\mathfrak{n}$. Thus we may assume that B is local. By Corollary 1.5 we can replace $A \to B$ by $A/\mathfrak{m}^{\ell+1} \to B/\mathfrak{m}^{\ell+1}B$ and thereby assume that A is Artinian. Under these assumptions, we proceed by induction on the length of M' as a Λ-module. Let

$$(B \otimes_A M') \oplus U \cong B \otimes_A M$$

as $B \otimes_A \Lambda$-modules.

By Lemma 1.6, we have $M' \mid_\Lambda M^{(r)}$ for some positive integer r. Assuming $M' \neq 0$, write ${}_\Lambda M' = V \oplus X$, where V is an indecomposable left Λ-module. Then $V \mid_\Lambda M^{(r)}$. By the Krull-Remak-Schmidt Theorem for finite-length modules, $V \mid_\Lambda M$, say, ${}_\Lambda M \cong V \oplus Y$. Then

$$\begin{aligned}(B \otimes_A V) \oplus (B \otimes_A X) \oplus U &\cong (B \otimes_A M') \oplus U \cong B \otimes_A M \\ &\cong (B \otimes_A V) \oplus (B \otimes_A Y).\end{aligned}$$

By Lemma 1.2, $(B \otimes_A X) \oplus U \cong B \otimes_A Y$. By the inductive hypothesis, $X \mid_\Lambda Y$, and now $M' \mid_\Lambda M$ as desired. This completes the proof of Theorem 1.3 in the case of a local ring A.

For the general case, we use a typical "partition of unity". Let $\mathfrak{m}_1, \dots, \mathfrak{m}_t$ be the maximal ideals of A. We know, by the local case, that $M'_{\mathfrak{m}_i} \mid_{\Lambda_{\mathfrak{m}_i}} M_{\mathfrak{m}_i}$ for each i. Choose $\Lambda_{\mathfrak{m}_i}$-homomorphisms

$$M'_{\mathfrak{m}_i} \xrightarrow{\varphi_i} M_{\mathfrak{m}_i} \xrightarrow{\psi_i} M'_{\mathfrak{m}_i}$$

such that $\psi_i \varphi_i = 1_{M'_{\mathfrak{m}_i}}$ for $i = 1, \dots, t$. Since ${}_\Lambda M'$ is finitely presented, the natural map $A_{\mathfrak{m}_i} \otimes_A \operatorname{Hom}_\Lambda(M', M) \to \operatorname{Hom}_{\Lambda_{\mathfrak{m}_i}}(M'_{\mathfrak{m}_i}, M_{\mathfrak{m}_i})$ is an isomorphism for each i. Therefore there is a homomorphism $\sigma_i : M' \to M$, whose localization at $\mathfrak{m}_i$ agrees with φ_i up to multiplication by a unit. Similarly, there is a homomorphism $\tau_i : M \to M'$ that localizes to a unit multiple of ψ_i. For each i, the composition $\tau_i \sigma_i$ induces a surjective endomorphism of $M'_{\mathfrak{m}_i}$.

Choose, for each i, an element

$$r_i \in (\mathfrak{m}_1 \cap \cdots \cap \widehat{\mathfrak{m}_i} \cap \cdots \cap \mathfrak{m}_t) - \mathfrak{m}_i\,,$$

and put $\varphi := \sum_{i=1}^t r_i \sigma_i$ and $\psi := \sum_{i=1}^t r_i \tau_i$. By Nakayama's lemma, the composition $M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'$ is surjective and therefore an isomorphism (since M' is a Noetherian Λ-module). It follows that φ is a split injection, with splitting map $(\psi\varphi)^{-1}\psi$.

This completes the proof of Theorem 1.3 and, consequently, of Theorem 1.1. □

In the special case where $(A, \mathfrak{m})$ is a Noetherian local ring and $B = \widehat{A}$, the $\mathfrak{m}$-adic completion, Reiner and Roggenkamp [**18**] gave a simpler proof of Theorem 1.3, which we sketch here. Their result was stated only for the case where A is a discrete valuation ring, but the same proof works in general (cf. also [**23**] and [**25**, Proposition 4.1]). With Λ, M and M' as in Theorem 1.3, we identify $B \otimes_A \Lambda, B \otimes_A M$ and $B \otimes_A M'$ with their $\mathfrak{m}$-adic completions $\widehat{\Lambda}, \widehat{M}$ and $\widehat{M'}$, respectively.

Assuming $\widehat{M'} \mid_{\widehat{\Lambda}} \widehat{M}$, we choose $\widehat{\Lambda}$-homomorphisms $\varphi : \widehat{M'} \to \widehat{M}$ and $\psi : \widehat{M} \to \widehat{M'}$ such that $\psi\varphi = 1_{\widehat{M'}}$. Since $H := \operatorname{Hom}_\Lambda(M', M)$ is a finitely generated A-module, it follows that $\widehat{H} = B \otimes_A H = \operatorname{Hom}_{\widehat{\Lambda}}(\widehat{M'}, \widehat{M})$. Therefore φ can be approximated to any order by an element of H. In fact, order 1 suffices: Choose $f \in \operatorname{Hom}_\Lambda(M', M)$ such that $\widehat{f} - \varphi \in \widehat{\mathfrak{m}}\widehat{H}$. Similarly, we can choose $g \in \operatorname{Hom}_\Lambda(M, M')$ with $\widehat{g} - \psi \in \widehat{\mathfrak{m}} \operatorname{Hom}_{\widehat{\Lambda}}(\widehat{M}, \widehat{M'})$. Then $\widehat{g}\widehat{f} - 1_{\widehat{M'}} = \widehat{g}\widehat{f} - \psi\varphi = \widehat{g}(\widehat{f} - \varphi) + (\widehat{g} - \psi)\varphi$, and it follows that the image of $\widehat{g}\widehat{f} - 1_{\widehat{M'}}$ is in $\widehat{\mathfrak{m}}\widehat{M'}$. Nakayama's lemma now implies that $\widehat{g}\widehat{f}$ is surjective, and therefore an isomorphism. It follows that $\widehat{g}$ is a split surjection (with splitting map $\widehat{f}(\widehat{g}\widehat{f})^{-1}$). By faithful flatness g is a split surjection. □

Examples 1.8. The assumption that A be semilocal cannot be omitted from the hypotheses of Theorem 1.3. To see this, let A be any Dedekind domain with a non-principal ideal I and suppose there is an integral domain B containing A such that B is faithfully flat as an A-module and IB is a principal ideal of B. Then $I \nmid_A A$, but $B \otimes_A I \mid_B B$. For a specific example, take A to be the ring of integers in an algebraic number field K with non-trivial class group, and let B the integral closure of A in the Hilbert class field of K.

Alternatively, one can take $A = \mathbb{R}[X, Y]/(X^2 + Y^2 - 1)$, the affine coordinate ring of the unit circle C. Let $I = (x - 1, y)A$ and $B = \mathbb{C}[X, Y]/(X^2 + Y^2 - 1)$. Then $I^2 = (x - 1)A$, so I is an invertible ideal. To see that I is not principal, suppose $I = Bg$, where $g \in A$. Then g vanishes at the point $(1, 0)$ but at no other point of the unit circle. By the intermediate value theorem, g cannot change signs on $C - \{(1, 0)\}$. We may assume that $g(x, y) > 0$ for $(x, y) \in C - \{(1, 0)\}$.

Writing $y = ag$ with $a \in A$, we see that the a must be positive on the top half of the circle and negative on the bottom half. It follows that $a(1,0) = 0$. But then y would vanish to order at least 2 at $(1,0)$, which is false. □

In these examples, faithfully flat descent of isomorphism fails too. As mentioned before, this condition is formally weaker than the conclusion of Theorem 1.3, but we have no example to show that it is actually weaker. Here we summarize the state of our ignorance regarding these issues.

Questions 1.9. Let $A \to B$ be a faithfully flat homomorphism of commutative Noetherian rings, and let Λ be a module-finite A-algebra. Consider the following three conditions:

(1.9.1) (descent of isomorphism) If M_1 and M_2 are finitely generated left Λ-modules and $B \otimes_A M_1 \cong_{B \otimes_A \Lambda} B \otimes_A M_2$ then $M_1 \cong_\Lambda M_2$.

(1.9.2) ("two out of three") If N' and N'' are finitely generated left $B \otimes_A \Lambda$-modules and N' and $N' \oplus N''$ are extended from Λ, then N'' is extended from Λ.

(1.9.3) (descent of direct summand relations) If M' and M are finitely generated left Λ-modules and $B \otimes_A M' \mid_{B \otimes_A \Lambda} B \otimes_A M$, then $M' \mid_\Lambda M$.

Clearly (1.9.3) implies (1.9.1). Are there any other implications among the three conditions? We note that (1.9.2) can fail, even when A is a field. Take, for example, the homomorphism $\mathbb{R} \to B$, where $B = \mathbb{R}[X,Y,Z]/(X^2 + Y^2 + Z^2 - 1)$, the coordinate ring of the real 2-sphere. The tangent bundle to the sphere provides a non-free projective module V such that $V \oplus B \cong B^{(3)}$. Obviously B and $B^{(3)}$ are extended, but V is not.

2. Two out of three: short exact sequences. In this section we restrict our attention to flat local homomorphisms $\varphi : (R, \mathbf{m}) \to (S, \mathbf{n})$ satisfying the property:

(†) (i) $\mathbf{m}S = \mathbf{n}$; and (ii) φ induces an isomorphism on residue fields.

This condition is equivalent to the following: The induced homomorphism $R/\mathbf{m} \to S/\mathbf{m}S$ is bijective. Familiar examples satisfying (†) include the completion $R \to \widehat{R}$ and the Henselization $R \to R^{\mathrm{h}}$. Homomorphisms satisfying (†) are studied in detail in [**12**]. Our main result in this section, Theorem 2.2, is a variant of [**12**, Proposition 3.2]. We begin with an easy lemma, which we will need only in the case $\Lambda = R$, and which is very familiar in the case of the completion:

Lemma 2.1. *Let $(R, \mathbf{m}) \to (S, \mathbf{n})$ be a flat local homomorphism satisfying (†), and let Λ be a module-finite R-algebra.*

(1) *If N is a left $S \otimes_R \Lambda$-module of finite length t, then N has length t as a Λ-module.*

(2) *If M is a left Λ-module of finite length, then the natural map $M \to S \otimes_R M$ is an isomorphism.*

Proof. Suppose first that N is simple as an $S \otimes_R \Lambda$-module. Since, by [**10**, Proposition 1.2], $\mathbf{n}(S \otimes_R \Lambda)$ is contained in the Jacobson radical of $S \otimes_R \Lambda$, $\mathbf{n}$ annihilates N. Therefore N is a simple module over $(S/\mathbf{n}) \otimes_R \Lambda = (R/\mathbf{m}) \otimes_R \Lambda = \Lambda/\mathbf{m}\Lambda$. This proves (1) in the case of a simple $S \otimes_R \Lambda$-module.

Next we prove (2) for a simple left Λ-module M. As above, $\mathbf{m}M = 0$. Also, $\mathbf{n}(S \otimes_R M) = \mathbf{m}(S \otimes_R M) = 0$. A glance at the following commutative diagram now confirms (2):

$$\begin{array}{ccc} M & \longrightarrow & S \otimes_R M \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ (R/\mathbf{m}) \otimes_R M & \xrightarrow{\cong} & (S/\mathfrak{n}) \otimes_R M \end{array}$$

We have now proved (1) and (2) for simple modules, and the general cases follows easily by induction on length.

Theorem 2.2. *Let $(R, \mathbf{m}) \to (S, \mathbf{n})$ be a flat local homomorphism satisfying (†), and let Λ be a module-finite R-algebra. Let*

$$(\xi) \qquad 0 \to N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \to 0$$

be an exact sequence of finitely generated left $S \otimes_R \Lambda$*-modules.*

(1) *Assume* N' *and* N'' *are extended. If* $\operatorname{Ext}^1_{S\otimes_R\Lambda}(N'', N')$ *has finite length as an* R*-module (e.g., if* N''_P *is* Λ_P*-free for each prime* $P \neq \mathbf{n}$*), then* N *is extended.*

(2) *Assume* N *and* N'' *are extended. If* $\operatorname{Hom}_{S\otimes_R\Lambda}(N, N'')$ *has finite length as an* R*-module (e.g., if* ${}_{S\otimes_R\Lambda}N''$ *has finite length), then* N' *is extended.*

(3) *Assume* N' *and* N *are extended. If* $\operatorname{Hom}_{S\otimes_R\Lambda}(N', N)$ *has finite length as an* R*-module (e.g., if* ${}_{S\otimes_R\Lambda}N'$ *has finite length), then* N'' *is extended.*

Proof. For (1), write $N' = S \otimes_R N'_0$ and $N'' = S \otimes_R N''_0$, where N'_0 and N''_0 are finitely generated left Λ-modules. The natural map $S \otimes_R \operatorname{Ext}^1_\Lambda(N''_0, N'_0) \to \operatorname{Ext}^1_{S\otimes_R\Lambda}(N', N'')$ is an isomorphism (because $R \to S$ is flat, N''_0 is finitely generated and Λ is Noetherian). By faithful flatness, $\operatorname{Ext}^1_\Lambda(N''_0, N'_0)$ has finite length as an R-module. Now (2) of Lemma 2.1 implies that the natural map $\operatorname{Ext}^1_\Lambda(N''_0, N'_0) \to S \otimes_R \operatorname{Ext}^1_\Lambda(N''_0, N'_0)$ is an isomorphism. Combining these two natural isomorphisms, we see that the given exact sequence (ξ), regarded as an element of $\operatorname{Ext}^1_{S\otimes_R\Lambda}(N'', N')$, comes from a short exact sequence $0 \to N'_0 \to N_0 \to N''_0 \to 0$. Clearly, then, $S \otimes_R N_0 \cong N$.

To prove (2), we write $N = S \otimes_R N_0$ and $N'' = S \otimes_R N''_0$, where N_0 and N''_0 are finitely generated left Λ-modules. As in the proof of (1) we see that the natural map $\operatorname{Hom}_\Lambda(N_0, N''_0) \to \operatorname{Hom}_{S\otimes_R\Lambda}(N, N'')$ is an isomorphism. Therefore the $S \otimes_R \Lambda$-homomorphism β comes from a homomorphism $\beta_0 \in \operatorname{Hom}_\Lambda(N_0, N''_0)$. Clearly, $N' \cong S \otimes_R \operatorname{Ker}(\beta_0)$. The proof of (3) is essentially the same: Write $N = S \otimes_R N_0$ and $N' = S \otimes_R N'_0$. Show that α comes from $\alpha_0 \in \operatorname{Hom}_\Lambda(N'_0, N_0)$, and deduce that $N'' \cong S \otimes_R \operatorname{Coker}(\alpha_0)$. □

Part (1) of the theorem was used by Christensen, Piepmeyer, Striuli and Takahashi [**6**] to characterize the local rings having only finitely many isomorphism classes of indecomposable totally reflexive modules. More recently it was used by Crabbe and Striuli [**5**] to build indecomposable modules that are free of large constant rank on the punctured spectrum. In both cases, special properties of the completion (e.g., the Krull-Remak-Schmidt Theorem) were exploited, in order to prove the desired results in the complete case. The general result then followed by descent, via the theorem above.

3. Normal domains of dimension two. The ideal class group provides a perfect obstruction in the extendeness problem for normal domains of dimension 2. We refer to [**2**, Chapter VII, §4.7] for basic stuff about the divisor class group $\mathrm{Cl}(A)$ of a Noetherian normal domain A. In particular, $\mathrm{Cl}(A)$ is the group of isomorphism classes of divisorial fractional ideals of A. To each finitely generated S-module M, one assigns a *divisor class* $[M] \in \mathrm{Cl}(A)$, in such a way that (a) $\mathrm{Cl}(-)$ is additive on short exact sequences of finitely generated A-modules, and (b) for each non-zero fractional ideal J of A, $[J]$ is the isomorphism class of $(J^{-1})^{-1}$.

Theorem 3.1. *Let $(R, \mathbf{m})$ and $(S, \mathbf{n})$ be normal local domains of dimension 2, and let $(R, \mathbf{m}) \to (S, \mathbf{n})$ be a flat local homomorphism satisfying (†) (that is, the induced map $R/\mathbf{m} \to S/\mathbf{m}S$ is bijective). Let N be a finitely generated torsion-free S-module. Then N is extended if and only if the divisor class $[N]$ is in the image of the natural map $\Phi : \mathrm{Cl}(A) \to \mathrm{Cl}(B)$.*

Proof. Suppose first that $N \cong S \otimes_R M$. By faithful flatness, M is finitely generated and torsion-free. By "Bourbaki's Theorem" [**2**, Chapter VII, §4.9, Theorem 6], there is an exact sequence $0 \to F \to M \to J \to 0$, where J is a non-zero ideal of R and F is a free R-module. One checks, using (a) and (b), that $[N] = \Phi([M])$.

For the converse, we apply Bourbaki's Theorem to N, getting a short exact sequence

$$(1) \qquad 0 \to G \to N \to L \to 0,$$

where G is a free S-module and L is a non-zero ideal of S. Then $[L] = [N]$, so by hypothesis there is a divisorial ideal I of R such that $[S \otimes_R I] = [L]$, that is, $S \otimes_R I \cong (L^{-1})^{-1}$. Next, we consider the short exact sequence

$$(2) \qquad 0 \to L \to (L^{-1})^{-1} \to \frac{(L^{-1})^{-1}}{L} \to 0.$$

For each non-zero prime ideal $P \neq \mathfrak{n}$, S_P is a discrete valuation ring, so $L_P = ((L^{-1})^{-1})_P$. The module $\frac{(L^{-1})^{-1}}{L}$ therefore has finite length and hence is extended, by Lemma 2.1. Since $(L^{-1})^{-1}$ is extended, part (2) of Theorem 2.2 implies that L is extended. For each prime ideal $P \neq \mathbf{n}$, we have $L_P \cong S_P$ and it follows that $\mathrm{Ext}^1_S(L, G)$ has finite length. Now (1) of Theorem 2.2 implies that N is extended. □

This theorem was used in [**19**] to show that for each positive integer r there is a Cohen-Macaulay local ring R having an indecomposable module G such that $\widehat{R} \otimes_R G$ is isomorphic to the direct sum of r copies of the canonical module $\omega_{\widehat{R}}$.

In order to describe another application, we need some notation and terminology. Given a finitely generated module M over a local ring $(R, \mathbf{m})$, we let $\mathrm{add}(M)$ denote the additive semigroup consisting of isomorphism classes $[N]$ of finitely generated modules that are isomorphic to direct summands of direct sums of copies of M. Write $\widehat{M} = V_1^{(c_1)} \oplus \cdots \oplus V_t^{(c_t)}$, where $V_1, \dots, V_t$ are pairwise non-isomorphic indecomposable $\widehat{R}$-modules and the c_i are positive. If $[N] \in \mathrm{add}(M)$, then, by the Krull-Remak-Schmidt Theorem, $\widehat{N} \cong V_1^{(n_1)} \oplus \cdots \oplus V_t^{(n_t)}$, and the sequence $(n_1, \dots, n_t)$ is uniquely determined by $[N]$. The map $\nu : [N] \mapsto (n_1, \dots, n_t)$ embeds $\mathrm{add}(M)$ as a subsemigroup of the free semigroup $\mathbb{N}_0^{(t)}$. Moreover, Theorem 1.3 implies that if $x, y \in \mathrm{add}(M)$ and $\nu(x) \leq \nu(y)$ (in the coordinatewise partial order on $\mathbb{N}_0^{(t)}$), then there is an element $z \in \mathrm{add}(M)$ such that $x + z = y$. In other words, $\mathrm{add}(M)$ is isomorphic to a *full* subsemigroup [**1**, §6.1] of $\mathbb{N}_0^{(t)}$. Semigroups that have such an embedding are called *positive affine normal semigroups* in [**1**, §6.1] and *reduced, finitely generated Krull monoids* in the literature on abstract commutative monoids, e.g. [**4, 11**]. In other places, these gadgets are called *Diophantine monoids*, since they can be realized as the set of non-negative integer solutions to a homogeneous system of linear equations with integer coefficients (cf. Exercise 6.4.16 in [**1**]). The main theorem in [**24**], which makes use of Theorem 3.1, is that given *any* Diophantine monoid H there exist a local unique factorization domain and a finitely generated reflexive module M such that $\mathrm{add}(M) \cong H$. Thus every bit of pathology (and there is plenty!) in the behavior of Diophantine monoids can be realized as bad direct-sum behavior among modules over local rings.

4. One-dimensional rings. For one-dimensional rings, the extendedness problem quickly reduces, via Theorem 2.2, to the same problem over Artinian (zero-dimensional) rings. This phenomenon was discovered and exploited by Levy and Odenthal in [**17**]. Here is how it works: Given any commutative ring R, let $\mathrm{K}(R)$ be the ring of fractions obtained by inverting the elements outside the union of the minimal prime ideals of R. Then $\mathrm{K}(R)$ is a zero-dimensional semilocal ring. (Of course the canonical map $R \to \mathrm{K}(R)$ is not necessarily one-to-one, but that does not matter.)

We return to the context of Theorem 2.2, but assume in addition that R has dimension one.

Theorem 4.1. [**17**] *Let* $(R, \mathbf{m}) \to (S, \mathbf{n})$ *be a flat local homomorphism satisfying* (†) *of* §2, *and let* Λ *be a module-finite* R*-algebra. Assume* $\dim(R) = 1$. *Let* N *be a finitely generated left* $S \otimes_R \Lambda$*-module. Then* N *is extended from* Λ *if and only if* $\mathrm{K}(S) \otimes_S N$ *is extended from* $\mathrm{K}(R) \otimes_R \Lambda$.

Proof. To simplify notation, we let $K = \mathrm{K}(R)$ and $L = \mathrm{K}(S)$. We observe first that S is one-dimensional too, by [**1**, Theorem A.11]. Also, if Q is a minimal prime ideal of S, then $Q \cap R$ is a minimal prime ideal of R, since "going down" holds for flat extensions [**1**, Lemma A.9]. Therefore the inclusion $R \to S$ induces a homomorphism $K \to L$, and this homomorphism is faithfully flat, since the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective [**1**, Lemma A.10]. The "only if" direction is clear from the change-of-rings diagram:

$$(\dagger) \qquad \begin{array}{ccc} S & \longrightarrow & L \\ \uparrow & & \uparrow \\ R & \longrightarrow & K \end{array}$$

For the converse, let X be a finitely generated $K \otimes_R \Lambda$-module such that $L \otimes_K X \cong L \otimes_S N$. Since $K \otimes_R \Lambda$ is a localization of Λ, there is a finitely generated left Λ-module M such that $K \otimes_R M \cong X$. Since $L \otimes_S N \cong L \otimes_S (S \otimes_R M)$, there is a homomorphism $\alpha : N \to S \otimes_R M$ inducing an isomorphism from $L \otimes_S N$ to $L \otimes_S (S \otimes_R M)$. Thus the kernel U and cokernel V of α have finite length and therefore are extended, by Lemma 2.1. Now we break the exact sequence

$$0 \to U \to N \xrightarrow{\alpha} S \otimes_R M \to V \to 0$$

into two short exact sequences:

$$0 \to U \to N \to W \to 0$$
$$0 \to W \to S \otimes_R M \to V \to 0$$

Applying (2) of Theorem 2.2 to the second short exact sequence, we see that W is extended. Now we apply (1) of Theorem 2.2 to the first short exact sequence, to conclude that N is extended. □

Even in the case $\Lambda = R$, the question of which $\mathrm{K}(S)$-modules are extended from $\mathrm{K}(R)$ appears to be difficult. In the last section of the paper we will discuss this problem. If, however, S is reduced (i.e., has no non-zero nilpotent elements), then $\mathrm{K}(R)$ and $\mathrm{K}(S)$ are direct products of fields, and the extendedness problem comes down to a simple combinatorial problem of compatibility of vector-space dimensions (cf. Corollary 4.4 below). We leave the proof of the following observation to the reader.

Proposition 4.2. *Let $K = K_1 \times \cdots \times K_s$, where each K_i is an Artinian local ring. For each i, let $K_i \to L_{i1}, \ldots, K_i \to L_{it_i}$ be flat local homomorphisms, with each L_{ij} Artinian. Put $L_i = L_{i1} \times \cdots \times L_{it_i}$ for each i, and let $L = L_1 \times \cdots \times L_s$. Given a finitely generated projective L-module N, let r_{ij} be the rank of the free L_{ij}-module $L_{ij}N$, for $i = 1, \ldots, s$ and $j = 1, \ldots, t_i$. Then N is extended from K if and only if $r_{i1} = \cdots = r_{it_i}$ for $i = 1, \ldots, s$.* □

Suppose, now, that R and S are Noetherian local rings and that $(R, \mathbf{m}) \to (S, \mathbf{n})$ is a flat local homomorphism satisfying (†) of §2. (For example, S might be the completion or the Henselization of R.) Assume, further, that R is one-dimensional (and consequently S is one-dimensional as well, by [**1**, Theorem A.11]). Let N be a finitely generated S-module such that $K(S) \otimes_R N$ is $K(S)$-projective (where, as above, $K(S)$ is the localization of S outside the union of the minimal prime ideals of S). For each minimal prime ideal Q of S, we define the *rank of N at Q* to be the rank of the free S_Q-module N_Q.

Corollary 4.3. *Let $(R, \mathbf{m}) \to (S, \mathbf{n})$ be a flat local homomorphism satisfying (†) of §2. Assume R and S are Noetherian and one-dimensional. Let N be a finitely generated S-module, and assume that $\mathrm{K}(S) \otimes_R N$ is $\mathrm{K}(S)$-projective. Then N is extended from R if and only if the rank of N is constant on each fiber of the map $\mathrm{Spec}(S) - \{\mathbf{n}\} \to \mathrm{Spec}(R) - \{\mathbf{m}\}$.*

Proof. This is an immediate consequence of Theorem 4.1 and Proposition 4.2. □

Corollary 4.4 *Let $(R, \mathbf{m})$ be a local Noetherian ring and $(R, \mathbf{m}) \to (S, \mathbf{n})$ a flat local homomorphism satisfying (†) of §2. Assume, further, that R is one-dimensional and that S is reduced. Let N be a finitely generated S-module. Then N is extended from R if and only if the rank of N is constant on each fiber of the map $\mathrm{Spec}(S) - \{\mathbf{n}\} \to \mathrm{Spec}(R) - \{\mathbf{m}\}$.*

Proof. Since S is reduced, each S_Q is a field, and thus $\mathrm{K}(S) \otimes_R N$ is $\mathrm{K}(S)$-projective. Therefore Corollary 4.3 applies. □

Corollary 4.5. *Let $(R, \mathbf{m})$ and $(S, \mathbf{n})$ be one-dimensional Noetherian local rings, and let $(R, \mathbf{m}) \to (S, \mathbf{n})$ be a flat local homomorphism satisfying (†) of §2. These are equivalent:*

(1) *If N is a finitely generated S-module such that $\mathrm{K}(S) \otimes_R N$ is $\mathrm{K}(S)$-projective, then N is extended.*

(2) *The natural map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is bijective.*

Proof. By Corollary 4.3, (2) implies (1). For the converse, suppose there are two minimal primes Q_1 and Q_2 of S lying over the same minimal prime of R. The following lemma provides a finitely generated S-module N such that $\mathrm{K}(S) \otimes_S N$ is $\mathrm{K}(S)$-projective and the ranks of N at Q_1 and Q_2 are 1 and 0, respectively. By Corollary 4.3, N is not extended. □

Lemma 4.6. *Let $(S, \mathbf{n})$ be a one-dimensional Noetherian local ring with minimal primes $Q_1, \dots, Q_n$. For each i, there is a finitely generated S-module G_i such that*

(1) $\mathrm{K}(S) \otimes_S G_i$ *is* $\mathrm{K}(S)$*-projective, and*

(2) *the rank of G_i at Q_j is δ_{ij}.*

Proof. For each i, let F_i be the image of the natural map $S \to S_{Q_i}$. Then F_i is a finitely generated S-module, and $(F_i)_{Q_i} = S_{Q_i}$. Fix i, and choose an element $r_i \in Q_1 \cap \cdots \cap \widehat{Q_i} \cap \cdots \cap Q_n - Q_i$. Put $s_i := r_i^{q_i}$, where the exponent q_i is chosen so large that s_i maps to 0 in S_{Q_j} for $j \neq i$. The module $G_i = s_i F_i$ clearly does the job. □

4.7. *Much ado about something.* There are well-known examples of flat local homomorphisms where the map on spectra is not bijective. For an arithmetic example, take $R = \mathbb{Z}_{(5)}[5i]$. This is an integral domain, but the Henselization and the completion each have two minimal primes. Similarly, $\mathbb{C}[X, Y]_{(X,Y)}/(Y^2 - X^3 - X^2)$ is a domain, but the Henselization and the completion each have two minimal primes. (The completion is $\mathbb{C}[[X, Y]]/(Y^2 - X^3 - X^2)$, and $Y^2 - X^3 - X^2 = (Y + X\sqrt{X+1})(Y - X\sqrt{X+1})$.)

As a segue into the next section, we mention the following:

Corollary 4.8. *Let $(R, \mathbf{m}) \to (S, \mathbf{n})$ be a flat local homomorphism satisfying (†) of §2. Assume, further, that R is one-dimensional. Let N be a finitely generated S-module, and assume that $\mathrm{K}(S) \otimes_R N$ is $\mathrm{K}(S)$-projective. Then N is a direct summand of an extended module. In particular, if S is reduced, then every finitely generated S-module is a direct summand of an extended module.*

Proof. Let $Q_1, \dots, Q_n$ be the minimal prime ideals of S. For each i, let r_i be the rank of N at Q_i. Let $r = \max\{r_i : 1 \le i \le n\}$, and put $G := \sum_{i=1}^n G_i^{(r-r_i)}$, where the G_i are the modules given by Lemma 4.6. Then G is a finitely generated S-module, $\mathrm{K}(S) \otimes_S G$ is $\mathrm{K}(S)$-projective, and $N \oplus G$ has constant rank r at the minimal primes of S. By Corollary 4.3, $N \oplus G$ is extended. □

Question 4.9. Is Corollary 4.8 still true without the assumption that $\mathrm{K}(S) \otimes_R N$ be $\mathrm{K}(S)$-projective?

5. Direct summands of extended modules. Suppose $(R, \mathbf{m}) \to (S, \mathbf{n})$ is a flat local homomorphism of Noetherian local rings. To prove that finite representation type ascends from R to S, it is enough to know that every S-module is a direct summand of an extended module. In fact, flatness is not essential here.

Theorem 5.1. [**22**, Lemma 2.1] *Let $A \to B$ be a homomorphism of commutative, Noetherian rings, with B semilocal. Let $\mathcal{A}$, respectively, $\mathcal{B}$, be a class of finitely generated A-modules, respectively B-modules. Assume the following:*

(1) *$\mathcal{A}$ and $\mathcal{B}$ are closed under direct summands and under isomorphism.*

(2) *$\mathcal{A}$ contains only finitely many isomorphism classes of indecomposable modules.*

(3) *For each $N \in \mathcal{B}$, there exists $M \in \mathcal{A}$ such that $N \mid_B B \otimes_A M$.*

Then $\mathcal{B}$ contains only finitely many isomorphism classes of indecomposable B-modules.

Proof. Given a ring C and a C-module V, we let $\mathrm{add}_C(V)$ be the class of C-modules that are isomorphic to direct summands of direct sums of finitely many copies of V. Our goal is to find a finitely generated B-module V such that $\mathcal{B} \subseteq \mathrm{add}_B(V)$. Since B is Noetherian and semilocal, it is known [**22**, Theorem 2.1] that there are only finitely many isomorphism classes of indecomposable B-modules in $\mathrm{add}_B(V)$; so this will complete the proof.

Let $L_1, \dots, L_t$ be a complete list of representatives for the isomorphism classes of indecomposable modules in $\mathcal{A}$. Let $L := L_1 \oplus \cdots \oplus L_t$, and put $V = B \otimes_A L$. Given $N \in \mathcal{B}$, we want to show that $N \in \mathrm{add}_B(V)$. Choose, using assumption (3), a module $M \in \mathcal{A}$ such that $N \mid_B B \otimes_A M$. Since A is Noetherian, we can express M as a direct sum of finitely many indecomposable A-modules, and by (1) each of these is isomorphic to some L_i. Therefore $M \in \mathrm{add}_A(L)$, and it follows that $N \in \mathrm{add}_B(V)$. ◻

Recall [**16**] that an *étale neighborhood* (sometimes called a *pointed étale neighborhood*) of a Noetherian local ring $(R, \mathbf{m})$ is a flat local homomorphism $(R, \mathbf{m}) \to (S, \mathbf{n})$ such that

(1) condition (†) of §2 holds,

(2) the diagonal map $S \otimes_R S \to S$ splits as $S - S$-bimodules, and

(3) S is essentially of finite type over R (that is, S is a localization of a finitely generated R-algebra).

The isomorphism classes of étale neighborhoods of $(R, \mathbf{m})$ form a direct system, and the Henselization $(R, \mathbf{m}) \to (R^{\mathrm{h}}, \mathbf{m}R^{\mathrm{h}})$ is the direct limit of all of them.

The crucial condition, for our purposes, is (2), which is usually referred to as *separability* [**7**]. A local homomorphism essentially of finite type that satisfies (2) is said to be *unramified* [**16**]. The following result was proved in [**22**], although it was not stated explicitly there:

Theorem 5.2. [**12**, Theorem 3.4] *Let $\varphi : (R, \mathbf{m}) \to (S, \mathbf{n})$ be a flat local homomorphism of Noetherian local rings, and assume S is separable over R (that is, the diagonal map $S \otimes_R S \to S$ splits as $S \otimes_R S$-modules). Then every finitely generated S-module is a direct summand of a finitely generated extended module.*

Proof. Given a finitely generated S-module N, we apply $- \otimes_S N$ to the diagonal map, getting a split surjection of S-modules $\pi: S \otimes_R N \twoheadrightarrow N$, where the S-module structure on $S \otimes_R N$ comes from the S-action on S, not on N. Thus we have a split injection of S-modules $j: N \to S \otimes_R N$. Now write N as a direct union of finitely generated R-modules M_i. The flatness of φ implies that $S \otimes_R N$ is a direct union of the modules $S \otimes_R M_i$. The finitely generated S-module $j(N)$ must be contained in some $S \otimes_R M_i$. Since $j(N)$ is a direct summand of $S \otimes_R N$, it must be a direct summand of the smaller module $S \otimes_R M_i$. □

Corollary 5.3. *Let R be a Noetherian local ring. Then every finitely generated module over the Henselization R^{h} is a direct summand of an extended module.*

Proof. Let N be a finitely generated R^{h}-module. Since R^{h} is a direct limit of pointed étale neighborhoods of R, there exist a pointed étale neighborhood S of R and a finitely generated S-module M such that $N \cong_{R^{\mathrm{h}}} R^{\mathrm{h}} \otimes_S M$. By Theorem 5.2, there is a finitely generated R-module X such that $M \mid_S S \otimes_S X$. Then $N \mid_{R^{\mathrm{h}}} R^{\mathrm{h}} \otimes_R X$. □

In [**22**], these ideas were used to show that if $(R, \mathbf{m})$ is a Cohen-Macaulay local ring of finite Cohen-Macaulay type, then the Henselization R^{h} also has finite Cohen-Macaulay type. Of course, in order to use Theorem 5.1 to prove this, one needs a substantial improvement of Corollary 5.3: If N is a maximal Cohen-Macaulay S-module, then there is a *maximal Cohen-Macaulay* R-module X such that $N \mid_{R^{\mathrm{h}}} R^{\mathrm{h}} \otimes_R X$. This was proved in [**22**], under the additional assumption that R_P is Gorenstein for each prime ideal $P \neq \mathbf{m}$. Fortunately, the additional assumption is always satisfied in this situation [**14**]: If R has finite Cohen-Macaulay type then in fact R_P is a regular local ring for every $P \neq \mathbf{m}$.

Question 5.4. Let R be a Cohen-Macaulay Noetherian local ring, and let N be a maximal Cohen-Macaulay R^{h}-module. Is there necessarily a maximal Cohen-Macaulay R-module X such that $N \mid_{R^{\mathrm{h}}} R^{\mathrm{h}} \otimes_R X$?

The analogous question for the completion has a negative answer, even if we delete the second occurrence of "maximal Cohen-Macaulay". For example, let k be any countable field, and put $R = k[X, Y, Z]_{(X,Y,Z)}/(Z^2)$. Let $V_1, V_2, V_3, \ldots$ be a complete list of representatives for the finitely generated R-modules. Write each $\widehat{R} \otimes_R V_i$ as a direct sum of indecomposable $\widehat{R}$-modules, and let $\mathcal{S}$ be the (countable) set of indecomposable $\widehat{R}$-modules that occur in these decompositions. Since the Krull-Remak-Schmidt Theorem holds for complete rings, the modules in $\mathcal{S}$ are, up to isomorphism, the *only* indecomposable $\widehat{R}$-modules that are direct summands of extended modules. But by [**15**, Theorem 1.3] $\widehat{R}$ has uncountable Cohen-Macaulay type, because its singular locus is two-dimensional. Therefore there are uncountably many isomorphism classes of indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules that are not direct summands of extended modules.

If we delete *both* occurrences of "maximal Cohen-Macaulay" in Question 5.4, the answer is "no" for *every* countable Noetherian local ring $(R, \mathbf{m})$ of dimension at least two. (Compare with Corollary 4.8.) To see this, it suffices to show that the completion $\widehat{R}$ has uncountably many isomorphism classes of indecomposable finitely generated R-modules. The maximal ideal $\widehat{\mathbf{m}}$ is a union of height-one prime ideals, by Krull's Principal Ideal Theorem. On the other hand, $\widehat{R}$ has countable prime avoidance [**3, 20**]. Therefore $\widehat{R}$ must have uncountably many height-one prime ideals. The modules $\widehat{R}/P$, P ranging over the height-one prime ideals, are pairwise non-isomorphic indecomposable $\widehat{R}$-modules.

6. Étale extensions of Artinian local rings. This section, which is somewhat speculative and short on details, is motivated by the problem of determining which finitely generated modules over the Henselization of a one-dimensional Noetherian local ring are extended. If $(R, \mathbf{m})$ is a local ring, then every finitely generated module over the Henselization is extended from some étale neighborhood of R. Suppose, now, that we have an étale neighborhood $(R, \mathbf{m}) \to (S, \mathbf{n})$ of one-dimensional Noetherian local rings, and we seek criteria for a given finitely generated S-module N to be extended from R. A version of Zariski's Main Theorem [**16**, p. 64] implies that S is *essentially finite* over R, that is, it is a localization of a module-finite R-algebra. Let $K = \mathrm{K}(R)$ and $L = \mathrm{K}(S)$, the Artinian localizations of R and S (cf. §4.) By Theorem 4.1, N is extended from R if and only if $L \otimes_S N$ is extended from K. Therefore we seek criteria for a finitely generated L-module to be extended from K.

Of course K is a direct product of local rings, so we can work with one component at a time and assume that K is local. Now write $L = L_1 \times \cdots \times L_t$, where the L_j are local rings. Using [**16**, Chapter III, (1.2)–(1.4)], one can see that each extension $K \to L_j$ is étale. If we can solve the extendedness problem for each map $K \to L_j$, we will obtain the general answer by imposing compatibility requirements. That is, a finitely generated L-module N is extended if and only if each L_jN is extended, say, $L_jN \cong_{L_j} L_j \otimes_K M_j$, *and* the M_j are all K-isomorphic to each other.

Therefore, modulo elaborate bookkeeping, the extendedness problem for the Henselization of a one-dimensional local ring reduces to the extendedness problem for an étale extension $(K, \mathfrak{m}, \mathfrak{k}) \to (L, \mathfrak{n}, \mathfrak{l})$ of local Artinian rings. We remark that the extension $K \to L$ is finite (since it is essentially finite and since localization is surjective for Artinian rings). Because $K \to L$ is flat, L is a finitely generated free K-module.

We want to enlarge L to a Galois extension [**21**] of K. Since $\mathfrak{l}$ is a finite separable extension of $\mathfrak{k}$, we can pass to the Galois closure $\bar{\mathfrak{l}}/\mathfrak{k}$ of $\mathfrak{l}/\mathfrak{k}$. We have $\bar{\mathfrak{l}} = \mathfrak{l}[X]/(f)$, where f is a monic polynomial in $\mathfrak{l}[X]$. Let $F \in L[X]$ be a monic polynomial lifting f, and put $\bar{L} := L[X]/(F)$. Then $\bar{L}$ is a local ring with maximal ideal $\bar{\mathfrak{n}} := \mathfrak{n}\bar{L}$ and residue field $\bar{\mathfrak{l}}$. Moreover, $\bar{L}/K$ is a Galois extension whose Galois group is naturally isomorphic to the Galois group of $\bar{\mathfrak{l}}/\mathfrak{k}$.

Suppose, now, that N is a finitely generated L-module. If N is extended from K, then of course $\bar{L} \otimes_L N$ is extended from K. Conversely, if $\bar{L} \otimes_L N$ is extended from K, say, $\bar{L} \otimes_L N \cong_{\bar{L}} \bar{L} \otimes_K M$, then $N \cong_L L \otimes_K M$, by faithfully flat descent of isomorphism for the extension $L \to \bar{L}$. Therefore, modulo more bookkeeping, we may change notation and assume that L/K is a Galois extension of Artinian local rings.

For the rest of the paper, we let L/K be a Galois extension of Artinian local rings, and we let G be the Galois group. Thus G is a finite group of ring automorphisms of L and $K = L^G$ (the fixed ring). These two conditions, together with the following non-degeneracy condition, characterize Galois extensions [**7**, Chapter III, Proposition 1.2]:

- For each non-trivial $\sigma \in G$ there is an element $x \in L$ such that $\sigma(x) - x$ is a unit of L.

There is a natural action of G on isomorphism classes of finitely generated L-modules: Given a finitely generated L-module N and an element $\sigma \in G$, we let $\sigma(N)$ be the L-module whose underlying abelian group is $(N, +)$ and whose L-module structure is given by $\ell \cdot_\sigma x = \sigma(\ell)x$ for all $\ell \in L$ and $x \in N$. Alternatively if N is the cokernel of the matrix φ, then $\sigma(N)$ is the cokernel of the matrix obtained by applying σ to each entry of φ.

Theorem 6.1. [**21**, Proposition 2.5] *Let N be a finitely generated L-module. Then N is extended if and only if there is a linear action of G on $(N, +)$ satisfying $\sigma(\ell x) = \sigma(\ell)\sigma(x)$ for all $\ell \in L, x \in N$.* □

This gives, in some sense, a solution to the extendedness problem for the Henselization of a one-dimensional local ring. The following conjecture, if true, might provide a more workable solution:

Conjecture 6.2. *Let L/K be a Galois extension of Artinian local rings, with Galois group G, and let N be a finitely generated L-module. Then N is extended if and only if $N \cong_L \sigma(N)$ for each $\sigma \in G$.*

The "only if" part of the conjecture is clearly true: If $N \cong L \otimes_K M$ for some K-module M, write M as the cokernel of some matrix φ over K. The alternate description of the group action shows that, for each $\sigma \in G$, $\sigma(N)$ is the cokernel of the same matrix φ, regarded as a matrix over L.

Here is a proof of a special case of the conjecture.

Theorem 6.3. *Suppose, as above, that $(K, \mathfrak{m}, \mathfrak{k})$ is a module-finite $\mathbb{R}$-algebra with $\mathfrak{k} = \mathbb{R}$, and let $\mathbb{C} \otimes_{\mathbb{R}} K = (L, \mathfrak{n}, \mathfrak{l})$ be its complexification. Let σ be the non-trivial element of the Galois group G of L/K. The finitely generated L-module N is extended if and only if $N \cong_L \sigma(N)$.*

Proof. We have already observed that the "only if" implication is true. Therefore we assume that $N \cong_L \sigma(N)$, and we shall show that N is extended from a K-module. Without loss of generality we may suppose that N is indecomposable. To see this, we observe first of all that any L-module of the form $V \oplus \sigma(V)$ is extended, since the action $\sigma(x, y) = (y, x)$ satisfies the criterion of Theorem 6.1. Suppose, now, that $N = N_1 \oplus \cdots \oplus N_r$ is the decomposition into indecomposable L-modules N_i. Since $\sigma(N) \cong N$, there exists, by the Krull-Remak-Schmidt Theorem, a bijective map $\pi : \{1, \ldots, r\} \to \{1, \ldots, r\}$ such that $N_i \cong \sigma(N_{\pi(i)})$ for all $i \in \{1, \ldots, r\}$. Let $I = \{i \mid N_i \ncong \sigma(N_i)\}$. If $i \in I$, it follows from Krull-Remak-Schmidt that there exists $j(i) \in I - \{i\}$ such that $\sigma(N_i) \cong N_{j(i)}$. Then $N_i \oplus N_{j(i)}$ is of the form $V \oplus \sigma(V)$ and is therefore extended. It follows that $|I|$ is even and that $\bigoplus_{i \in I} N_i$ is extended. Therefore it is enough to show that N_i is extended for each $i \notin I$.

Thus we assume that ${}_L N$ is indecomposable and that $N \cong_L \sigma(N)$.

An easy argument shows that

$$(*) \qquad L \otimes_K N \cong_L N \oplus \sigma(N) \cong_L N \oplus N$$

Suppose we can show that N is decomposable as a K-module, say $N = M_1 \oplus M_2$, with non-zero K-modules M_i. It will then follow from $(*)$ and the Krull-Remak-Schmidt Theorem that $L \otimes_K M_1 \cong N$, and the proof will be complete.

Suppose to the contrary that N is indecomposable as an R-module. We seek a contradiction. Choose an isomorphism $\psi \in \mathrm{Hom}_L(N, \sigma(N))$, and put $A = \mathrm{End}_L(N)$ and $B = \mathrm{End}_K(N)$. Then $A \subseteq B$, ψ is a unit of B, and ψ^2 is a unit of A. By assumption, A and B are both local rings in the non-commutative sense, that is, $A/\mathrm{J}(A)$ and $B/\mathrm{J}(B)$ are both division rings. It follows that $\mathrm{J}(B)$ is exactly the set of nilpotent elements of B and that $\mathrm{J}(A) = \mathrm{J}(B) \cap A$. Since $\mathfrak{n}A$ is contained in the Jacobson radical $\mathrm{J}(A)$ of A, it follows that $D := A/\mathrm{J}(A)$ is a finite-dimensional division algebra over $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, D must be isomorphic to $\mathbb{C}$. Therefore $\psi^2 \equiv \kappa 1_N \pmod{\mathrm{J}(A)}$ for some complex number κ. Upon replacing ψ by $\frac{1}{\sqrt{\kappa}}\psi$, we may assume that $\psi^2 \equiv 1_N \pmod{\mathrm{J}(A)}$. Then $\psi^2 - 1 \equiv 0 \pmod{\mathrm{J}(B)}$, and, since $B/\mathrm{J}(B)$ is a division ring, $\psi \equiv \pm 1 \pmod{\mathrm{J}(B)}$. Upon replacing ψ by $-\psi$ if necessary, we may assume that $\psi \equiv 1 \pmod{\mathrm{J}(B)}$, that is, $\psi - 1$ is nilpotent.

Let $V := N/\mathfrak{m}N = N/\mathfrak{n}N$, a real vector space, and let $\psi' : V \to V$ be the $\mathbb{R}$-linear endomorphism induced by ψ. Since $\psi' - 1_V$ is nilpotent, there is a non-zero vector $v \in V$ such that $\psi'(v) = v$. Since ψ' is conjugate-linear with respect to the $\mathbb{C}$-module structure on V, we have $\psi'(iv) = -iv$. Choose an element $w \in N$ such that $w + \mathbf{n}N = iv$, and note that $(\psi - 1)w \equiv -2w \pmod{\mathbf{n}N}$. Since $w \notin \mathbf{n}N$, this clearly contradicts the fact that $\psi - 1$ is nilpotent. □

We close with an easy example that illustrates some aspects of Theorem 6.3.

Example 6.4. Let $R = \mathbb{R}[x, y] := \mathbb{R}[X, Y]/(X^2, XY, Y^2)$, and let $S = \mathbb{C} \otimes_{\mathbb{R}} R = \mathbb{C}[x, y]$. Given a complex number c, put $N_c := S/(x + cy)$. We claim that the following conditions are equivalent:

(1) N_c is extended.

(2) There exists an $n \geq 1$ such that $N_c^{(n)}$ is extended.

(3) $c \in \mathbb{R}$.

Clearly (3) $\implies$ (1) $\implies$ (2). Assuming (2), we will prove (3). Let σ be the non-trivial element of the Galois group of S over R. Note that $N_c^{(n)}$ is the cokernel of the diagonal matrix D with $x + cy$ on the diagonal, and $\sigma(N_c^{(n)})$ is the cokernel of the diagonal matrix $\bar{D}$ with $x + \bar{c}y$ on the diagonal (where $\bar{c}$ is the complex conjugate of c). By Theorem 6.3 (or Theorem 6.1), $N_c^{(n)} \cong \sigma(N_c^{(n)})$, and thus the matrices D and $\bar{D}$ are equivalent over S. The entries of the two matrices therefore generate the same ideal of S, that is, $S(x + cy) = S(x + \bar{c}y)$. Write $x + \bar{c}y = s(x + cy)$ with $s \in S$, and write $s = a + b$ with $a \in \mathbb{C}$ and $b \in Sx + Sy$. Then $x + \bar{c}y = a(x + cy)$, whence $a = 1$ and $\bar{c} = c$. This proves (3).

For $c \in \mathbb{C} - \mathbb{R}$, the module $N_c \oplus N_{\bar{c}}$ is extended, say, $N_c \oplus N_{\bar{c}} = S \otimes_R M$. As in the proof of Theorem 6.3, the Krull-Remak-Schmidt theorem implies that M is indecomposable. This shows that an indecomposable R-module can decompose upon extension to S.

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