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COMPUTING GORENSTEIN COLENGTH

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ABSTRACT. Given an Artinian local ring R, we define (in [**1**]) its Gorenstein colength $g(R)$ to measure how closely we can approximate R by a Gorenstein Artin local ring. In this paper, we show that $R = T/\mathfrak{b}$ satisfies the inequality $g(R) \leq \lambda(R/\mathrm{soc}(R))$ in the following two cases: (a) T is a power series ring over a field of characteristic zero and $\mathfrak{b}$ an ideal that is the power of a system of parameters or (b) T is a 2-dimensional regular local ring with infinite residue field and $\mathfrak{b}$ is primary to the maximal ideal of T.

In the first case, we compute $g(R)$ by constructing a Gorenstein Artin local ring mapping onto R. We further use this construction to show that an ideal that is the nth power of a system of parameters is directly linked to the $(n-1)$st power via Gorenstein ideals. A similar method shows that such ideals are also directly linked to themselves via Gorenstein ideals.

1. Introduction. Let us first recall the definition of Gorenstein colength and review some of its basic properties from [**1**] in this section.

Definition 1.1. Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring. Define the Gorenstein colength of R, denoted $g(R)$ as: $g(R) = \min\{\lambda(S) - \lambda(R) : S$ is a Gorenstein Artin local ring mapping onto $R\}$, where $\lambda(\_)$ denotes length.

The main questions one would like to answer are the following:

Question 1.2.

a) How does one intrinsically compute $g(R)$?

b) How does one construct a Gorenstein Artin local ring S mapping onto R such that $\lambda(S) - \lambda(R) = g(R)$?

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In order to answer Question 1.2(a), we prove the following inequalities in [**1**], which give bounds on $g(R)$.

$$\lambda(R/(\omega^*(\omega))) \leq \quad \min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \text{ is an ideal in } R, \mathfrak{a} \simeq \mathfrak{a}^\vee\} \quad \leq \quad g(R) \quad \leq \quad \lambda(R),$$

Fundamental Inequalities

where ω is the canonical module of R, $\omega^*(\omega) = \langle f(\omega) : f \in \operatorname{Hom}_R(\omega, R)\rangle$ is the trace ideal of ω in R and $\mathfrak{a}^\vee = \operatorname{Hom}_R(\mathfrak{a}, \omega)$.

A natural question one can ask in this context is Question 3.10 in [**1**], which is the following:

Question 1.3. Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring and $\mathfrak{a}$ an ideal in R such that $\mathfrak{a} \simeq \mathfrak{a}^\vee$. Does there exist a Gorenstein Artinian local ring S mapping onto R, such that $\lambda(S) - \lambda(R) = \lambda(R/\mathfrak{a})$?

The socle of R, soc(R), is a direct sum of finitely many copies of k, hence it is isomorphic to $\operatorname{soc}(R)^\vee$. Hence a particular case of the above question is the following:

Question 1.4. Is there a Gorenstein Artin local ring S mapping onto R such that $\lambda(S) - \lambda(R) = \lambda(R/\operatorname{soc}(R))$?

A weaker question one can ask is the following:

Question 1.5. Is $g(R) \leq \lambda(R/\operatorname{soc}(R))$?

We answer Question 1.5 in two cases in this paper. In section 3, we show that if T is a power series ring over a field and $\mathfrak{d} = (f_1, \ldots, f_d)$ is an ideal generated by a system of parameters, then $g(T/\mathfrak{d}^n) \geq \lambda(T/\mathfrak{d}^{n-1})$. Further, if the residue field of T has characteristic zero, we construct a Gorenstein Artin local ring S mapping onto $T/\mathfrak{d}^n$ such that $\lambda(S) - \lambda(T/\mathfrak{d}^n) = \lambda(T/\mathfrak{d}^{n-1})$ using a theorem of L. Ried, L. Roberts and M. Roitman proved in [**7**]. This shows that $g(T/\mathfrak{d}^n) = \lambda(T/\mathfrak{d}^{n-1})$. In particular, this proves that $R = T/\mathfrak{d}^n$ satisfies the inequality in Question 1.5.

In [**5**], Kleppe, Migliore, Miro-Roig, Nagel and Peterson show that $\mathfrak{d}^n$ can be linked to $\mathfrak{d}^{n-1}$ via Gorenstein ideals in 2 steps and hence to $\mathfrak{d}$ in $2(n-1)$ steps. In section 4, we use the ideal corresponding to the Gorenstein ring constructed in section 3, to show that $\mathfrak{d}^n$ can be directly liked to $\mathfrak{d}^{n-1}$ and hence to $\mathfrak{d}$ in $(n-1)$ steps.

When R is an Artinian quotient of a two-dimensional regular local ring with an infinite residue field, we use a formula due to Hoskin and Deligne (Theorem 5.6) in order to answer Question 1.5 in section 5.

2. Computing $\omega^*(\omega)$**.** Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring with canonical module ω. As noted in [**1**], maps from ω to R play an important role in the study of Gorenstein colength. In this section, we prove a lemma which helps us compute the trace ideal $\omega^*(\omega)$ of ω in R. We use the following notation in this section.

Notation. Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring mapping onto R. Let

$$0 \to T^{b_d} \xrightarrow{\phi} T^{b_{d-1}} \to \cdots \to T \to R \to 0 \qquad (\sharp)$$

be a minimal resolution of R over T. Then a resolution of the canonical module ω of R over T is given by taking the dual of the above resolution, i.e., by applying $\operatorname{Hom}_T(\_, T)$ to the above resolution. Hence a presentation of ω is $T^{b_{d-1}} \xrightarrow{\phi^*} T^{b_d} \to \omega \to 0$. Tensor with R and apply $\operatorname{Hom}_R(-, R)$ to get an exact sequence $0 \longrightarrow \omega^* \longrightarrow R^{b_d} \xrightarrow{\phi \otimes R} R^{b_{d-1}}$. Let ω^* be generated minimally by b_{d+1} elements. Thus we have an exact sequence $R^{b_{d+1}} \xrightarrow{\psi} R^{b_d} \xrightarrow{\phi \otimes R} R^{b_{d-1}}$, where $\omega^* = \operatorname{im}(\psi)$.

Lemma 2.1. *With notation as above, let ψ be given by the matrix (a_{ij}). Then the trace ideal of ω, $\omega^*(\omega)$, is the ideal generated by the a_{ij}'s.*

The above lemma is a particular case of the following lemma.

Lemma 2.2. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring and M a finitely generated R-module. Let $R^n \xrightarrow{B} R^m \longrightarrow M \longrightarrow 0$ be a minimal presentation of M. Apply $\operatorname{Hom}_R(\_, R)$ to get an exact* *sequence* $0 \longrightarrow M^* \longrightarrow (R^*)^m \xrightarrow{B^*} (R^*)^n$. *Map a free* R*-module, say* R^k, *minimally onto* M^* *to get an exact sequence* $R^k \xrightarrow{A} (R^*)^m \xrightarrow{B^*} (R^*)^n$, *where* $M^* = \ker(B^*) = \operatorname{im}(A)$. *Then the trace ideal of* M, $M^*(M) = (a_{ij} : a_{ij}$ *are the entries of the matrix* $A)$.

Proof. Let $w_1, \dots, w_m$ be a minimal generating set of M, $e_1, \dots, e_m$ be a basis of R^m such that $e_i \mapsto w_i$, and $e_1^*, \dots, e_m^*$ be the corresponding dual basis of $(R^*)^m$.

Let $f \in M^*$. Write $f = \Sigma_{i=1}^m r_i e_i^* \in (R^*)^m$. Then f acts on M by sending w_j to r_j. Hence if $A = (a_{ij})$, then the generators of M^* are $f_j = \Sigma_{i=1}^m a_{ij} e_i^*$, $1 \le j \le k$. Thus $f_j(w_i) = a_{ij}$. Thus $M^*(M) = (a_{ij})$. $\square$

Corollary 2.3. *With notation as above, let* $(T', \mathfrak{m}_{T'}, \mathsf{k})$ *be a regular local ring which is a flat extension of* T *such that* $\mathfrak{m}_T T' \subseteq \mathfrak{m}_{T'}$ *and let* $R' = T' \otimes_T R$. *Then* $\omega_{R'}^*(\omega_{R'}) = \omega^*(\omega)T'$.

Proof. Since T' is flat over T, $R' = T' \otimes_T R$ and $\mathfrak{m}_T T' \subseteq \mathfrak{m}_{T'}$, a minimal resolution of R' over T' is obtained by tensoring ($\sharp$) by T' over T. Therefore $\omega_{R'}^*(\omega_{R'})$ is the ideal generated by the entries of the matrix $\psi \otimes_T T'$. Now, by Lemma 2.1, the ideal in R generated by the entries of ψ is $\omega^*(\omega)$. Therefore, $\omega_{R'}^*(\omega_{R'}) = \omega^*(\omega)T'$. $\square$

3. Powers of Ideals Generated by a System of Parameters. In this section, the main theorem we prove is the following:

Theorem 3.1. *Let* $T = \mathsf{k}[|X_1, \dots, X_d|]$ *be a power series ring over a field* k *of characteristic zero. Let* $f_1, \dots, f_d$ *be a system of parameters in* T *and* $R = T/(f_1, \dots, f_d)^n$. *Then* $g(R) = \lambda(T/(f_1, \dots, f_d)^{n-1})$.

In order to prove this, we first prove the theorem when $f_i = X_i, i = 1, \dots, d$, and then use the fact that T is flat over $T' = \mathsf{k}[|f_1, \dots, f_d|]$.

Theorem 3.2. *Let* $T = \mathsf{k}[|X_1, \dots, X_d|]$ *be a power series ring over a field* k *with maximal ideal* $\mathfrak{m}_T = (X_1, \dots, X_d)$. *Let* $R = T/\mathfrak{m}_T^n$ *and* ω *be the canonical module of* R. *Then* $\omega^*(\omega) = \operatorname{soc}(R) = \mathfrak{m}_T^{n-1}/\mathfrak{m}_T^n$.

Proof. In order to prove this, we show that if $\phi \in \mathrm{Hom}(\omega, R)$, then $\phi(\omega) \subseteq \mathrm{soc}(R)$. Since $\mathrm{soc}(R) \subseteq \omega^*(\omega)$, this will prove the theorem.

Note that we can consider R to be the quotient of the polynomial ring $\mathsf{k}[X_1, \dots, X_d]$ by $(X_1, \dots, X_d)^n$. Thus change notation so that $T = \mathsf{k}[X_1, \dots, X_d]$ and $\mathfrak{m}_T = (X_1, \dots, X_d)$ is its unique homogenous maximal ideal.

The injective hull of k over T, $E_T(\mathsf{k})$, is $\mathsf{k}[X_1^{-1}, \dots, X_d^{-1}]$, where the multiplication is defined by

$$(X_1^{a_1} \cdots X_d^{a_d}) \cdot (X_1^{-b_1} \cdots X_d^{-b_d}) = \begin{cases} X_1^{a_1-b_1} \cdots X_d^{a_d-b_d} & \text{if } a_i \leq b_i \text{ for all } i \\ 0 & \text{otherwise} \end{cases}$$

and extended linearly (e.g., see [**6**]).

Let $\mathfrak{b} = \mathfrak{m}_T^n$. The canonical module ω of R is isomorphic to the injective hull of the residue field of R. Hence $\omega \simeq \mathrm{Hom}_R(R, E_T(\mathsf{k})) \simeq (0 :_{\mathsf{k}[X_1^{-1}, \dots, X_d^{-1}]} \mathfrak{b})$. Note that $\mathfrak{b} \cdot (X_1^{-a_1} \cdots X_d^{-a_d}) = 0$ whenever $a_i \geq 0$ and $n > \sum a_i$. Since $\lambda(\omega) = \lambda(R)$, we conclude that

$$\omega \simeq \mathsf{k}\text{-span of } \left\{ X_1^{-a_1} \cdots X_d^{-a_d} : a_i \geq 0; n > \sum_{i=1}^{d} a_i \right\}.$$

Observe that ω is generated by $\{X_1^{-a_1} \cdots X_d^{-a_d} \colon \sum_{i=1}^d a_i = n-1\}$ as an R-module. Let $\phi \in \omega^*$. We will now show that $\phi(X_1^{-a_1} \cdots X_d^{-a_d}) \in \mathrm{soc}(R)$ by induction on a_1. Let $w = X_1^{-a_1} \cdots X_d^{-a_d}$, $\sum_{i=1}^d a_i = n-1$.

If $a_1 = 0$, then $X_1 \cdot w = 0$. Hence $\phi(w) \in (0 :_R X_1) = \mathrm{soc}(R)$. If not, then $X_1 w = X_2(X_1^{-(a_1-1)} X_2^{-(a_2+1)} \cdots X_d^{-a_d})$. We have $\phi(X_1^{-(a_1-1)} X_2^{-(a_2+1)} \cdots X_d^{-a_d}) \in \mathrm{soc}(R)$ by induction. Thus $X_2 \phi(X_1^{-(a_1-1)} X_2^{-(a_2+1)} \cdots X_d^{-a_d}) = 0$ which yields $X_1 \phi(w) = 0$. But $(0 :_R X_1) = \mathrm{soc}(R)$, which proves that $\phi(\omega) \subseteq \mathrm{soc}(R)$. □

Since we know that $\lambda(R/(\omega^*(\omega)) \leq g(R)$ by the fundamental inequalities, we immediately get the following:

Corollary 3.3. *With notation as in Thoerem* 3.2*, $g(R) \geq \lambda(R/\mathrm{soc}(R))$.*

We prove the reverse inequality in Theorem 3.8 by constructing a Gorenstein Artin ring S mapping onto R such that $\lambda(S) - \lambda(R) =$

$\lambda(R/\mathrm{soc}(R))$. The following theorem of Ried, Roberts and Roitman is used in the construction.

Theorem 3.4. (Reid, Roberts, Roitman) *Let k be a field of characteristic zero, $S = \mathsf{k}[X_1, \dots, X_d]/(X_1^{n_1}, \dots, X_d^{n_d}) = \mathsf{k}[x_1, \dots, x_d]$. Let $m \geq 1$ and f be a nonzero homogeneous element in S such that $(x_1 + \cdots + x_d)^m f = 0$. Then $\deg(f) \geq (t - m + 1)/2$, where $t = \sum_{i=1}^{d}(n_i - 1)$.*

We use the following notation in this section.

Notation. Let k be a field. For any graded ring S (with $S_0 = \mathsf{k}$), by $h_S(i)$ we mean the k-dimension of the ith graded piece of the ring S and if S is Artinian, $\mathrm{Max}(S) := \max\{i : h_S(i) \neq 0\}$. All k-algebras in this section are standard graded, i.e., they are generated as a k-algebra by elements of degree 1.

We also need the following basic fact in order to prove Theorem 3.8.

Remark 3.5. Let $S = \mathsf{k}[X_1, \dots, X_d]/(X_1^{n_1}, \dots, X_d^{n_d})$ be a quotient of the polynomial ring over a field k and f be a non-zero homogeneous element in S of degree s. Then $S/(0 :_S f)$ is Gorenstein and $\mathrm{Max}(S/(0 :_S f)) = \mathrm{Max}(S) - s$.

Proposition 3.6. *Let $T = \mathsf{k}[X_1, \dots, X_d]$ be a polynomial ring over k and $\mathfrak{m}_T = (X_1, \dots, X_d)$ be its unique homogeneous maximal ideal. Let f be a homogeneous element and $\mathfrak{c} = (X_1^n, \dots, X_d^n) :_T f$ be such that $\mathfrak{c} \subseteq \mathfrak{m}^n$. Then the following are equivalent:*

i) $\lambda(\mathfrak{m}_T^n/\mathfrak{c}) = \lambda(T/\mathfrak{m}_T^{n-1})$.

ii) $\mathrm{Max}(T/\mathfrak{c}) = 2(n-1)$.

iii) $\deg(f) = (d-2)(n-1)$.

Proof. Since $\mathrm{Max}(T/(X_1^n, \dots, X_d^n)) = d(n-1)$, (ii) $\Leftrightarrow$ (iii) follows from Remark 3.5.

Let $R = T/\mathfrak{m}_T^n$ and $S = T/\mathfrak{c}$. Since $T/(X_1^n, \dots, X_d^n)$ is a Gorenstein Artin local ring, so is S. Note that $\mathrm{soc}(R) = \mathfrak{m}_T^{n-1}/\mathfrak{m}_T^n$ and $\lambda(S) - \lambda(R) = \lambda(\mathfrak{m}_T^n/\mathfrak{c})$.

The rings R and S are quotients of the polynomial ring $\mathsf{k}[X_1, \ldots, X_d]$ by homogeneous ideals. Thus, both R and S are graded under the standard grading. Since $\mathfrak{c} \subseteq \mathfrak{m}_T^n$,

$$h_S(i) = h_R(i) \text{ for } i < n. \qquad (*)$$

Since S is Gorenstein,

$$h_S(i) = h_S(\mathrm{Max}(S) - i). \qquad (**)$$

Using $(*)$ and $(**)$, we see that the Hilbert function of S is:

degree i	0	1	2	3	...	n-1
$h_R(i)$	1	d	$\binom{d+1}{2}$	$\binom{d+2}{3}$	...	$\binom{d+n-2}{n-1}$
$h_S(i)$	1	d	$\binom{d+1}{2}$	$\binom{d+2}{3}$	...	$\binom{d+n-2}{n-1}$

degree i	...	Max(S) - (n-1)	Max(S) - (n-2)	...	Max(S) - 1	Max(S)
$h_R(i)$	...	0	0	...	0	0
$h_S(i)$	...	$\binom{d+n-2}{n-1}$	$\binom{d+n-3}{n-2}$	...	d	1

Thus we have

$$\begin{aligned}
\lambda(T/\mathfrak{m}_T^{n-1}) &= h_R(n-2) + h_R(n-3) + \cdots + h_R(0) \\
&= h_S(n-2) + h_S(n-3) + \cdots + h_S(0) \\
&= h_S(\mathrm{Max}(S) - (n-2)) + h_S(\mathrm{Max}(S) - (n-3)) \\
&\quad + \cdots + h_S(\mathrm{Max}(S)) \\
&= \sum_{i \geq \mathrm{Max}(S)-(n-2)} h_S(i) \\
&\leq \lambda(S) - \lambda(R) = \lambda(\mathfrak{m}_T^n/\mathfrak{c}).
\end{aligned}$$

Moreover, from the above table, equality holds if and only if $\mathrm{Max}(S) - (n-1) = n-1$, proving (i) $\Leftrightarrow$ (ii). ◻

In the following corollary, we show that $f = (X_1 + \ldots + X_d)^{(d-2)(n-1)}$ satisifies the hypothesis of Proposition 3.6.

Corollary 3.7. *Let $T = \mathsf{k}[X_1, \dots, X_d]$ be a polynomial ring over k, a field of characteristic zero, and $\mathfrak{m}_T = (X_1, \dots, X_d)$ be its unique homogeneous maximal ideal. Let $\mathfrak{c}_n = (X_1^n, \dots, X_d^n) : l^{(d-2)(n-1)}$, where $l = X_1 + \cdots + X_d$. Then $\mathfrak{c}_n \subseteq \mathfrak{m}_T^n$.*

Moreover, $\lambda(\mathfrak{m}_T^n/\mathfrak{c}_n) = \lambda(T/\mathfrak{m}_T^{n-1})$.

Proof. By Theorem 3.4, if F is a homogeneous element in T such that $l^m F \in (X_1^n, \dots, X_d^n)$, then $\deg(F) \geq (d(n-1) - m + 1)/2$. Therefore, for $m = (d-2)(n-1)$, we see that $\deg(F) \geq n - 1/2$, i.e., $F \in \mathfrak{m}_T^n$. Thus $(X_1^n, \dots, X_d^n) : (X_1 + \cdots + X_d)^{(d-2)(n-1)} \subseteq \mathfrak{m}_T^n$.

Moreover, by Proposition 3.6, since $\deg(l^{(d-2)(n-1)}) = (d-2)(n-1)$, $\lambda(\mathfrak{m}_T^n/\mathfrak{c}_n) = \lambda(T/\mathfrak{m}_T^{n-1})$. $\square$

Theorem 3.8. *Let $T = \mathsf{k}[|X_1, \dots, X_d|]$ be a power series ring over a field k of characteristic zero, with unique maximal ideal $\mathfrak{m}_T = (X_1, \dots, X_d)$. Let $R := T/\mathfrak{m}_T^n$. Then $g(R) \leq \lambda(R/\mathrm{soc}(R)) = \lambda(T/\mathfrak{m}^{n-1})$.*

Proof. Let $\mathfrak{c}_n = (X_1^n, \dots, X_d^n) :_T l^{(d-2)(n-1)}$, where $l = X_1 + \cdots + X_d$. Let $S = T/\mathfrak{c}_n$. Then S is a Gorenstein Artin local ring mapping onto R. Note that $R \simeq \mathsf{k}[X_1, \dots, X_d]/(X_1, \dots, X_d)^n$ and $S \simeq \mathsf{k}[X_1, \dots, X_d]/((X_1^n, \dots, X_d^n) :_T l^{(d-2)(n-1)})$.

Hence, by Corollary 3.7, $\lambda(S) - \lambda(R) = \lambda(R/\mathrm{soc}(R)) = \lambda(T/\mathfrak{m}_T^{n-1})$. This shows that $g(R) \leq \lambda(R/\mathrm{soc}(R))$. $\square$

Remark 3.9. The ring S constructed in the proof of the theorem does not work when $\mathrm{char}(\mathsf{k}) = 2$. For example, when $d = 3$ and $n = 3$, we have $h_R(i) = 1, 3, 6$ and $h_S(i) = 1, 2, 5, 2, 1$.

Remark 3.10. Let S be a graded Gorenstein Artin quotient of $T = \mathsf{k}[X_1, \dots, X_d]$, where k is a field of characteristic zero. We say that S is a compressed Gorenstein algebra of socle degree $t = \mathrm{Max}(S)$, if for each i, $h_S(i)$ is the maximum possible given d and t, i.e., $h_S(i) = \min\{h_T(i), h_T(t-i)\}$ (e.g., see [**3**]). Note that the proofs of Proposition 3.6 and Corollary 3.7 show that $S = T/((X_1^n, \dots, X_d^n) :_T l^{(d-2)(n-1)})$ is a compressed Gorenstein Artin algebra of socle degree $2n-2$. A similar technique also shows that $S = T/((X_1^n, \dots, X_d^n) :_T l^{(d-2)(n-1)-1})$ is a compressed Gorenstein Artin algebra of socle degree $2n-1$.

In the following remark, we record some key observations which we will use to prove Theorem 3.1.

Remark 3.11. Let $T = \mathsf{k}[|X_1, \dots, X_d|]$ be a power series ring over a field k. Let $f_1, \dots, f_d$ be a system of parameters. Then $T' = \mathsf{k}[|f_1, \dots, f_d|]$ is a power series ring and T is free over T' of rank $e = \lambda(T/(f_1, \dots, f_d))$. Thus, if $\mathfrak{b}$ and $\mathfrak{c}$ are ideals in T', then $(\mathfrak{c} :_{T'} \mathfrak{b})T = (\mathfrak{c}T :_T \mathfrak{b}T)$ and $\lambda(T/\mathfrak{b}T) = e \cdot \lambda(T'/\mathfrak{b})$.

Firstly, we construct a Gorenstein Artin ring S mapping onto R such that $\lambda(S) - \lambda(R) = \lambda(T/(f_1, \dots, f_d)^{n-1})$ which proves $g(R) \leq \lambda(T/(f_1, \dots, f_d)^{n-1})$. We do this as follows:

Suppose that $\mathrm{char}(\mathsf{k}) = 0$. Let $\mathfrak{d} = (f_1, \dots, f_d)$, $\mathfrak{c} = (f_1^n, \dots, f_d^n) :_{T'} l^{(d-2)(n-1)}$, where $l = (f_1 + \cdots + f_d)$. We see that since $(f_1^n, \dots, f_d^n) :_{T'} l^{(d-2)(n-1)} \subseteq \mathfrak{d}^n$ in T' by Corollary 3.7, the same holds in T by using Remark 3.11. Moreover, since $\lambda(\mathfrak{d}^n T/\mathfrak{c}T) = e\lambda(\mathfrak{d}^n/\mathfrak{c})$ and $\lambda(T/\mathfrak{d}^{n-1}T) = e\lambda(T'/\mathfrak{d}^{n-1})$, the length condition in Corollary 3.7 gives $\lambda(\mathfrak{d}^n T/\mathfrak{c}T) = \lambda(T/\mathfrak{d}^{n-1}T)$.

This implies that if $R = T/\mathfrak{d}^n T$, then $S = T/\mathfrak{c}T$ is a Gorenstein Artin ring mapping onto R and $\lambda(S) - \lambda(R) = \lambda(\mathfrak{d}^n T/\mathfrak{c}T) = \lambda(T/\mathfrak{d}^{n-1}T)$. Therefore $g(R) \leq \lambda(T/\mathfrak{d}^{n-1}T)$. Thus as a consequence of Theorem 3.8, we have proved

Theorem 3.12. *Let $T = \mathsf{k}[|X_1, \dots, X_d|]$ be a power series ring over a field k of characteristic zero, $f_1, \dots, f_d$ be a system of parameters and $\mathfrak{d} = (f_1, \dots, f_d)$. Let $R = T/\mathfrak{d}^n$. Then $g(R) \leq \lambda(T/\mathfrak{d}^{n-1})$.*

In order to prove Theorem 3.1, we know need to show that $g(R) \geq \lambda(T/\mathfrak{d}^{n-1})$. We prove this by first computing the trace ideal $\omega^*(\omega)$ of the canonical module and use the fundamental inequalities. We use the lemmas concerning the computation of $\omega^*(\omega)$ proved in section 2.

Theorem 3.13. *Let $T = \mathsf{k}[|X_1, \dots, X_d|]$ be a power series ring over a field k. Let $f_1, \dots, f_d$ be a system of parameters and $R = T/(f_1, \dots, f_d)^n$. Then $\omega^*(\omega) = (f_1, \dots, f_d)^{n-1}/(f_1, \dots, f_d)^n$, where ω is the canonical module of R.*

Proof. Let $T' = \mathsf{k}[[f_1, \dots, f_d]]$, $\mathfrak{d} = (f_1, \dots, f_d)^n T'$ and $R' \simeq T'/\mathfrak{d}^n$. By Theorem 3.2, $\omega^*_{R'}(\omega_{R'}) = \mathfrak{d}^{n-1}/\mathfrak{d}^n$. Therefore, since T is free over T', by Corollary 2.3, $\omega^*(\omega) = \mathfrak{d}^{n-1}T/\mathfrak{d}^n T = (f_1, \dots, f_d)^{n-1}/(f_1, \dots, f_d)^n$. $\square$

Proof of Theorem 3.1. By Theorem 3.12, $g(R) \leq \lambda(T/(f_1, \dots, f_d^{n-1}))$. The other inequality follows from Theorem 3.13 which can be seen as follows:

Let ω be the canonical module of R. We know that $g(R) \geq \lambda(R/\omega^*(\omega))$ by the fundamental inequalities. This yields $g(R) \geq \lambda(T/(f_1, \dots, f_d)^{n-1})$ since $R = T/(f_1, \dots, f_d)^n$ and $\omega^*(\omega) = (f_1, \dots, f_d)^{n-1}/(f_1, \dots, f_d)^n$. This gives us the equality $g(R) = \lambda(T/(f_1, \dots, f_d)^{n-1})$ proving the theorem. $\square$

Corollary 3.14. *Let $T = \mathsf{k}[|X_1, \dots, X_d|]$ be a power series ring over a field k of characteristic zero. Let $f_1, \dots, f_d$ be a system of parameters and $R = T/(f_1, \dots, f_d)^n$. Then $g(R) \leq \lambda(R/\operatorname{soc}(R))$.*

Proof. We have $\lambda(R/\operatorname{soc}(R)) \geq \lambda(T/(f_1, \dots, f_d)^{n-1}) = g(R)$, since $(f_1, \dots, f_d)^n :_T (X_1, \dots, X_d) \subseteq (f_1, \dots, f_d)^n :_T (f_1, \dots, f_d) = (f_1, \dots, f_d)^{n-1}$. $\square$

Remark 3.15. Let $T = \mathsf{k}[[X, Y]]$, $R = T/(X, Y)^n$ and $S = T/(X^n, Y^n)$. Then S is a Gorenstein Artin local ring mapping onto R such that $\lambda(S) - \lambda(R) = \lambda(T/\mathfrak{m}_T^{n-1}) = \lambda(R/\operatorname{soc}(R))$. This, together with Corollary 3.3, shows that $g(R) = \lambda(R/\mathfrak{m}^{n-1})$ without any assumptions on the characteristic of k. Thus, when $d = 2$, using the technique described in Remark 3.11, we see that the conclusion of Theorem 3.1 is independent of the characteristic of k.

Remark 3.16. By taking $\mathfrak{d}$ to be the maximal ideal in Theorem 3.1, we get the following: Let k be a field of characteristic zero and $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring over k. Let $\mathfrak{m}_T = (X_1, \ldots, X_d)$ be the maximal ideal of T and $R := T/\mathfrak{m}_T^n$. Then

$$g(R) = \lambda(T/\mathfrak{m}_T^{n-1}) = \lambda(R/\mathrm{soc}(R)).$$

This also follows immediately from Theorems 3.2 and 3.8.

Remark 3.17. If $R = \mathsf{k}[X_1, \ldots, X_d]/(X_1, \cdots, X_d)^n$, where k is a field of characteristic zero, it follows from Theorem 3.1 and Theorem 3.8 that $g(R) = \lambda(R/\omega^*(\omega))$. Thus Question 3.9 in [**1**] has a positive answer, i.e., in this case,

$$\min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^\vee\} = g(R).$$

4. Applications to Gorenstein Linkage.

Proposition 4.1. *Let $(S, \mathfrak{m}, \mathsf{k})$ be a graded Gorenstein Artin local ring such that $\deg(\mathrm{soc}(S)) = t$. Let $f \in \mathfrak{m}$ be a homogeneous element in S of degree s and $\mathfrak{c} = (0 :_S f)$. Then $(\mathfrak{c} :_S \mathfrak{m}^n) = \mathfrak{m}^{(t+1)-(s+n)} + \mathfrak{c}$.*

Proof. Note that $\mathfrak{m}^{(t+1)-(s+n)} \cdot \mathfrak{m}^n \cdot f \subseteq \mathfrak{m}^{t+1} = 0$. Hence $\mathfrak{m}^{(t+1)-(s+n)} + \mathfrak{c} \subseteq \mathfrak{c}_n :_S \mathfrak{m}^n$. To prove the other inclusion, let g be a homogeneous form of degree less than $(t+1)-(s+n)$. Then $g \cdot f$ is a homogeneous form of degree $t-n$ or less. If $g \cdot f = 0$, then $g \in \mathfrak{c}$. If $g \cdot f \neq 0$, since S is Gorenstein, there is some element $h \in \mathfrak{m}^n$ such that $(gf) \cdot h$ generates soc(S) and hence is not zero. Thus $gf\mathfrak{m}^n \neq 0$ for $g \notin \mathfrak{m}^{(t+1)-(s+n)} + \mathfrak{c}$. Therefore $(\mathfrak{c} :_S \mathfrak{m}^n) \subseteq \mathfrak{m}^{(t+1)-(s+n)} + \mathfrak{c}$, proving the proposition. $\square$

Corollary 4.2. *Let k be a field of characteristic zero and $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring. Let $\mathfrak{m} = (X_1, \ldots, X_d)$ and $\mathfrak{c}_n = ((X_1^n, \ldots, X_d^n) :_T l^s)$, where $l = (X_1 + \cdots + X_d)$ and $s \geq (d-2)(n-1)-1$. Then $(\mathfrak{c}_n :_T \mathfrak{m}^n) = \mathfrak{m}^{(d-1)(n-1)-s}$.*

Proof. By taking $S = T/(X_1^n, \ldots, X_d^n)$, it follows from Proposition 4.1 that $(\mathfrak{c}_n :_T \mathfrak{m}^n) = \mathfrak{m}^{(d-1)(n-1)-s} + \mathfrak{c}_n$. It remains to prove that $\mathfrak{c}_n \subseteq \mathfrak{m}^{(d-1)(n-1)-s}$.

Let f be a homogeneous element of T such that $f \in \mathfrak{c}$, i.e., $f \cdot l^s \subseteq (X_1^n, \ldots, X_d^n)$. Hence by Theorem 3.4, $deg(f) \geq \frac{(d(n-1)-s+1)}{2} \geq (d-1)(n-1) - s$ by the hypothesis on s. This shows that $\mathfrak{c}_n \subseteq m^{(d-1)(n-1)\ -\ s}$. $\square$

Let $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring over a field k. Let $f_1, \ldots, f_d$ be a system of parameters. Let $T' = \mathsf{k}[|f_1, \ldots, f_d|]$, $\mathfrak{d} = (f_1, \ldots, f_d)^n T'$ and $\mathfrak{c}_n = (f_1^n, \ldots, f_d^n) :_{T'} l^s$, where $l = f_1 + \cdots + f_d$ and $s \geq (d-2)(n-1) - 1$. Since, by Corollary 4.2, $(\mathfrak{c}_n :_{T'} \mathfrak{d}^n) = \mathfrak{d}^{(d-1)(n-1)-s}$ in T', the same holds in T by Remark 3.11. Therefore $(\mathfrak{c}_n T :_T \mathfrak{d}^n T) = \mathfrak{d}^{(d-1)(n-1)-s} T$. Thus we see that

Proposition 4.3. *Let k be a field of characteristic zero and $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring. Let $\mathfrak{d} = (f_1, \ldots, f_d)$, where $f_1, \ldots, f_d$ form a system of parameters. Let $l = f_1 + \cdots + f_d$ and $s \geq (d-2)(n-1) - 1$. Then $\mathfrak{c}_n = ((f_1^n, \ldots, f_d^n) :_T l^s)$ is a Gorenstein ideal such that $(\mathfrak{c}_n :_T \mathfrak{d}^n) = \mathfrak{d}^{(d-1)(n-1)-s}$.*

Definition 4.4. Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring. An unmixed ideal $\mathfrak{b} \subseteq T$ is said to be in the *Gorenstein linkage class of a complete intersection (glicci)* if there is a sequence of ideals $\mathfrak{c}_n \subseteq \mathfrak{b}_n$, $\mathfrak{b}_0 = \mathfrak{b}$, satisfying

1) $T/\mathfrak{c}_n$ is Gorenstein for every n
2) $\mathfrak{b}_{n+1} = (\mathfrak{c}_n :_T \mathfrak{b}_n)$ and
3) $\mathfrak{b}_n$ is a complete intersection for some n.

We say that $\mathfrak{b}$ is *linked* to $\mathfrak{b}_n$ via Gorenstein ideals in n steps.

Remark 4.5.

1. Let k be a field of characteristic zero and $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring. Let $\mathfrak{d} = (f_1, \ldots, f_d)$, where $f_1, \ldots, f_d$ form a system of parameters. In [**5**], Kleppe, Migliore, Miro-Roig, Nagel and Peterson show that $\mathfrak{d}^n$ can be linked to $\mathfrak{d}^{n-1}$ via Gorenstein ideals in 2 steps and hence to $\mathfrak{d}$ in $2(n-1)$ steps. But in Proposition 4.3, by taking $s = (d-2)(n-1)$, we see that $\mathfrak{d}^n$ can be linked directly via the

Gorenstein ideal $(f_1^n, \dots, f_d^n) :_{T'} l^{(d-2)(n-1)}$ to $\mathfrak{d}^{n-1}$, and hence to $\mathfrak{d}$, a complete intersection, in $n-1$ steps.

2. In a private conversation, Migliore asked if this technique will show that $\mathfrak{d}^n$ is self-linked. We see that this can be done by taking $s = (d-2)(n-1) - 1$ in Proposition 4.3. Thus $\mathfrak{d}^n$ is linked to itself via the Gorenstein ideal $(f_1^n, \dots, f_d^n) :_{T'} l^{(d-2)(n-1)-1}$.

A Possible Approach to the Glicci Problem.

The Glicci problem. Given any homogeneous ideal $\mathfrak{b} \subseteq T := \mathsf{k}[X_1, \dots, X_d]$, such that $R := T/\mathfrak{b}$ is Cohen-Macaulay, is it true that $\mathfrak{b}$ is glicci?

A possible approach to the glicci problem is the following: Choose $\mathfrak{c}_n \subseteq \mathfrak{b}_n$ to be the closest Gorenstein. The question is: Does this ensure that $\mathfrak{b}_n$ is a complete intersection for some n?

Example 4.6. Let $T = \mathsf{k}[|X_1, \dots, X_d|]$, where $\mathrm{char}(\mathsf{k}) = 0$. Let $\mathfrak{d} = (f_1, \dots, f_d)$ be an ideal generated minimally by a system of parameters. We know by Theorems 3.1 and 3.12 that the ideal $\mathfrak{c}_n = (f_1^n, \dots, f_d^n) :_T (f_1 + \cdots + f_d)^{(d-2)(n-1)}$ is a Gorenstein ideal closest to $\mathfrak{d}^n$. Now by taking $s = (d-2)(i-1)$ in Proposition 4.3, we see that $\mathfrak{c}_i :_T \mathfrak{d}^i = \mathfrak{d}^{i-1}$, $2 \leq i \leq n$. Thus $\mathfrak{d}^n$ can linked to $\mathfrak{d}$ by choosing a closest Gorenstein ideal at each step.

5. The Codimension Two Case. We begin this section by recalling the following result of Serre characterizing Gorenstein ideals of codimension two.

Remark 5.1. Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring of dimension two. Let $\mathfrak{c}$ be an $\mathfrak{m}_T$ primary ideal such that $S = T/\mathfrak{c}$ is a Gorenstein Artin local ring. Then S is a complete intersection ring, i.e., $\mathfrak{c}$ is generated by 2 elements.

Notation. For the rest of this section, we will use the following notation: Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring of dimension 2, where that k is infinite. By $\mu(\_)$, we denote the minimal number of generators of a module and by $e_0(\_)$, we denote the multiplicity of an $\mathfrak{m}_T$-primary ideal in T. For an ideal $\mathfrak{b}$ in T, by $\bar{\mathfrak{b}}$, we denote the integral closure of $\mathfrak{b}$ in T.

Remark 5.2. We state the basic facts needed in this section in this remark. Their proofs can be found in [**2**] (Chapter 14).

1. Let $\mathfrak{b}$ be an $\mathfrak{m}_T$-primary ideal. We define the order of $\mathfrak{b}$ as $\operatorname{ord}(\mathfrak{b}) = \max\{i : \mathfrak{b} \subseteq \mathfrak{m}_T^i\}$.

Since $\mathfrak{m}_T$ is integrally closed, $\operatorname{ord}(\mathfrak{b}) = \operatorname{ord}(\overline{\mathfrak{b}})$.

2. Let $\mathfrak{b}$ be an $\mathfrak{m}_T$-primary ideal. Since k is infinite, a minimal reduction of $\mathfrak{b}$ is generated by 2 elements.

Further, if $\mathfrak{c}$ is a minimal reduction of $\mathfrak{b}$, the multiplicity of $\mathfrak{b}$, $e_0(\mathfrak{b}) = \lambda(T/\mathfrak{c})$.

3. The product of integrally closed $\mathfrak{m}_T$-primary ideals is integrally closed. In particular, if $\mathfrak{b}$ is an integrally closed $\mathfrak{m}_T$-primary ideal, then so is $\mathfrak{b}^n$ for each $n \geq 2$.

4. For an $\mathfrak{m}_T$-primary ideal $\mathfrak{b}$, $\lambda((\mathfrak{b} : \mathfrak{m}_T)/\mathfrak{b}) = \mu(\mathfrak{b}) - 1 \leq \operatorname{ord}(\mathfrak{b})$. Further, if $\mathfrak{b}$ is integrally closed, $\mu(\mathfrak{b}) - 1 = \operatorname{ord}(\mathfrak{b})$.

In particular, this yields $\mu(\mathfrak{b}) \leq \mu(\overline{\mathfrak{b}})$.

Proposition 5.3. *Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring of dimension two and let $\mathfrak{b}$ be an $\mathfrak{m}_T$-primary ideal. The closest (in terms of length) Gorenstein ideals contained in $\mathfrak{b}$ are its minimal reductions.*

Proof. Let $\mathfrak{c} \subseteq \mathfrak{b}$ be any Gorenstein ideal (and hence a complete intersection by the above remark). It is easy to see that $\lambda(T/\mathfrak{c}) \geq \lambda(T/(f,g))$, where $(f,g) \subseteq \mathfrak{b}$ is a minimal reduction of $\mathfrak{b}$. The reason is that

$$\begin{aligned}\lambda(T/\mathfrak{c}) &= e_0(\mathfrak{c}) \\ &\geq e_0(\mathfrak{b}) && \text{since } \mathfrak{c} \subseteq \mathfrak{b} \\ &= \lambda(T/(f,g)).\end{aligned}$$

As a consequence,

$$\begin{gathered}\lambda(T/\mathfrak{c}) - \lambda(T/\mathfrak{b}) \geq \lambda(T/(f,g)) - \lambda(T/\mathfrak{b}), \\ \text{i.e., } \lambda(\mathfrak{b}/\mathfrak{c}) \geq \lambda(\mathfrak{b}/(f,g)).\end{gathered}$$

Thus the closest Gorenstein ideal contained in $\mathfrak{b}$ is a minimal reduction (f, g). $\square$

We now prove the following theorem which shows that $g(R) \leq \lambda(R/\mathrm{soc}(R))$ where R is the Artinian quotient of a 2-dimensional regular local ring.

Theorem 5.4. *Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring of dimension 2, with infinite residue field k. Set $R = T/\mathfrak{b}$ where $\mathfrak{b}$ is an $\mathfrak{m}_T$-primary ideal. Then $g(R) \leq \lambda(R/\mathrm{soc}(R))$, i.e., there is a Gorenstein ring S mapping onto R such that $\lambda(S) - \lambda(R) \leq \lambda(R/\mathrm{soc}(R))$.*

In order to prove Theorem 5.4, we use a couple of formulae for $e_0(\mathfrak{b})$ and $\lambda(R)$ (which can be found, for example, in [**4**]). We need the following notation.

Let $(T, \mathfrak{m})$ and $(T', \mathfrak{n})$ be two-dimensional regular local rings. We say that T' birationally dominates T if $T \subseteq T'$, $\mathfrak{n} \cap T = \mathfrak{m}$ and T and T' have the same quotient field. We denote this by $T \leq T'$. Let $[T' : T]$ denote the degree of the field extension $T/\mathfrak{m} \subseteq T'/\mathfrak{n}$.

Further if $\mathfrak{b}$ is an $\mathfrak{m}$-primary ideal in T, let $\mathfrak{b}^{T'}$ be the ideal in T' obtained from $\mathfrak{b}$ by factoring $\mathfrak{b}T' = x\mathfrak{b}^{T'}$, where x is the greatest common divisor of the generators of $\mathfrak{b}T'$. The following theorem ([**4**], Theorem 3.7) gives a formula for $e_0(\mathfrak{b})$.

Theorem 5.5. *Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a two-dimensional regular local ring and $\mathfrak{b}$ be an $\mathfrak{m}_T$-primary ideal. Then*

$$e_0(\mathfrak{b}) = \sum_{T \leq T'} [T' : T] \mathrm{ord}(\mathfrak{b}^{T'})^2.$$

The following formula ([**4**], Theorem 3.10) is attributed to Hoskin and Deligne.

Theorem 5.6. (Hoskin-Deligne Formula) *Let T, $\mathfrak{b}$ and R be as in Theorem 5.4. Further assume that $\mathfrak{b}$ is an integrally closed ideal. Then,*

$$\lambda(R) = \sum_{T \leq T'} \binom{\mathrm{ord}(\mathfrak{b}^{T'}) + 1}{2} [T' : T].$$

Corollary 5.7. *Let T, $\mathfrak{b}$ and R be as in the Hoskin-Deligne formula. Then we have the inequality*

$$e_0(\mathfrak{b}) + \mathrm{ord}(\mathfrak{b}) \leq 2\lambda(R).$$

Proof. By Theorem 5.5, we have $e_0(\mathfrak{b}) = \sum_{T \leq T'} \mathrm{ord}(\mathfrak{b}^{T'})^2 [T' : T]$. Using the Hoskin-Deligne formula, we see that

$$\lambda(R) = \sum_{T \leq T'} \frac{\mathrm{ord}(\mathfrak{b}^{T'})^2 + \mathrm{ord}(\mathfrak{b}^{T'})}{2} [T' : T]$$

giving us

$$2\lambda(R) = e_0(\mathfrak{b}) + \sum_{T \leq T'} \mathrm{ord}(\mathfrak{b}^{T'})[T' : T].$$

Since $T \leq T$ and $\mathfrak{b}^T = \mathfrak{b}$, we get the required inequality. □

Corollary 5.8. *Let T, R and $\mathfrak{b}$ be as in Theorem 5.4. Then*

$$e_0(\mathfrak{b}) + \mu(\mathfrak{b}) - 1 \leq 2\lambda(T/\mathfrak{b}).$$

Proof. Let $\overline{\mathfrak{b}}$ be the integral closure of $\mathfrak{b}$. By the previous corollary, we have $e_0(\overline{\mathfrak{b}}) + \mathrm{ord}(\overline{\mathfrak{b}}) \leq 2\lambda(T/\overline{\mathfrak{b}})$. Since $\overline{\mathfrak{b}}$ is integrally closed, $\mathrm{ord}(\overline{\mathfrak{b}}) = \mu(\overline{\mathfrak{b}}) - 1$. Thus we get $e_0(\overline{\mathfrak{b}}) + \mu(\overline{\mathfrak{b}}) - 1 \leq 2\lambda(T/\overline{\mathfrak{b}})$. Now $e_0(\mathfrak{b}) = e_0(\overline{\mathfrak{b}})$, $\mu(\mathfrak{b}) \leq \mu(\overline{\mathfrak{b}})$ and $\lambda(T/\overline{\mathfrak{b}}) \leq \lambda(T/\mathfrak{b})$, giving the required inequailty. □

Proof of Theorem 5.4. For any ideal $\mathfrak{b}$ in T, we have $\mu(\mathfrak{b}) - 1 = \lambda((\mathfrak{b} : \mathfrak{m})/\mathfrak{m})$. But $(\mathfrak{b} : \mathfrak{m})/\mathfrak{b} \simeq \mathrm{soc}(R)$. Thus by the previous corollary, we have

$$e_0(\mathfrak{b}) + \lambda(\mathrm{soc}(R)) \leq 2\lambda(R). \qquad (\sharp\sharp)$$

Let (f, g) be a minimal reduction of $\mathfrak{b}$. Then $S := T/(f, g)$ is a complete intersection ring (and hence Gorenstein) mapping onto R. Moreover $\lambda(S) = e_0(\mathfrak{b})$. Thus $(\sharp\sharp)$ can be read as $\lambda(S) + \lambda(\mathrm{soc}(R)) \leq 2\lambda(R)$. Rearranging, we get $\lambda(S) - \lambda(R) \leq \lambda(R) - \lambda(\mathrm{soc}(R))$. This proves that $g(R) \leq \lambda(R/\mathrm{soc}(R))$. □

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