JOURNAL OF COMMUTATIVE ALGEBRA
Volume 1, Number 2, Summer 2009

THE DILWORTH LATTICE OF ARTINIAN RINGS

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ABSTRACT. After Freese [**3**] we define the Dilworth lattice of an Artinian local ring to be the family of ideals with the largest number of generators. We prove that it is indeed a lattice. Moreover we prove that over certain Gorenstein algebras the maximum and the minimum of the family are powers of the maximal ideal.

1. Introduction. In his paper [**8**] the second author defined the Dilworth number of Artinian rings and obtained some elementary results. It was followed by Ikeda [**4**] and [**5**]. These were analogs of some results in "combinatorial order theory" of finite sets as are found in e.g. [**1**, Chapter VIII].

The present paper was inspired by [**3**], where Freese showed that the family of antichains in a finite poset forms a lattice, hence it has the maximum and the minimum. The considerations made in the papers [**4**] and [**8**] suggest that an analogous result should be true for Artinian rings. An antichain in a poset may be interpreted in terms of commutative rings, as a minimal generating set of an ideal. Thus one may conceive that the family of ideals with the largest number of generators forms a lattice. This indeed is true and we prove it in Theorem 3. This was the starting point of this paper.

Easy examples show that these are basically infinite families even if we restrict the ideals to homogeneous ones. However, this leads us to considering the maximum and the minimum members of the lattice. It seems to be a natural question to ask under what conditions they are powers of the maximal ideal. A general result obtained in this paper on this question is Lemma 9. Under a certain condition of a Gorenstein algebra, we deduce that the Dilworth lattice has powers of the maximal ideal as the maximum and minimum. The condition

Keywords and phrases. antichain, Dilworth lattice, finite chain product, Sperner property
Received by the editors on February 14, 2008, and in revised form on January 20, 2009.

DOI:10.1216/JCA-2009-1-2-315

of Lemma 9 is rather strong; nonetheless it has applications in two extremal cases. (Corollary 10 and Theorem 11.)

One other case where the same result can be proved is the monomial complete intersection (Theorem 13). We prove it using a result of the theory of finite sets.

Finally we treat the monomial complete intersections in two variables. In this case the homogeneous Dilworth family is a finite lattice and is isomorphic to the lattice of ideals in the product of two chains.

Throughout, $\mathcal{F}(A)$ denotes the family of ideals with the largest number of generators in an Artinian local ring A. We call it the Dilworth lattice of A. Analogously $\mathcal{G}(A)$ denotes the family of ideals with the largest "C-M type" of an ideal. This is used to prove that the lattice $\mathcal{F}(A)$ is symmetric over a Gorenstein Artinian ring A.

The authors would like to express their thanks to Professor Griggs very much for helpful suggestions for the finite chain product.

2. Definition and some properties of the Dilworth Family. Let (A, m, k) be a Noetherian local ring and let I be an ideal of A. We denote by $\mu(I)$ the number of elements in a minimal basis for the ideal I in A. Recall that we have

$$\mu(I) = \text{length } (I/mI).$$

$\mu(I)$ is usually called the (minimal) number of generators of I. If (A, m, k) is Artinian, we define the Dilworth number of A by

$$d(A) = \text{Max}\{\mu(I) | I \subset A\}.$$

Recall that $d(A) \leq \text{length}(A/yA)$ for any element $y \in m$; in particular, $d(A)$ is finite. (See [**8**, Theorem 2.3], [**6**, Theorem 1.1].)

Definition 1. *Let (A, m, k) be an Artinian local ring. Define the family $\mathcal{F}(A)$ of ideals by*

$$\mathcal{F}(A) = \{\textit{ideal } I \subset A \mid \mu(I) = d(A)\}.$$

If A is a graded Artinian ring, then define $\mathcal{F}_H(A)$ by

$$\mathcal{F}_H(A) = \{\textit{homogeneous ideal } I \subset A \mid \mu(I) = d(A)\}.$$

Remark 2. If $A = \oplus_{i=0}^{c} A_i$ is a graded ring, A_0 a field, then $\mathcal{F}(A)$ contains a homogeneous ideal. (See [**8**, Lemma 2.4].) Thus $\mathcal{F}_H(A)$ is a non-empty sublattice of $\mathcal{F}(A)$.

Theorem 3. *Let A be an Artinian local ring. Then $\mathcal{F}(A)$ is a lattice with sum and intersection as the "join" and the "meet".*

Proof. Let $I, J \in \mathcal{F}(A)$. We have to show that $I \cap J \in \mathcal{F}(A)$ and $I + J \in \mathcal{F}(A)$. Consider the exact sequence:

$$0 \to I \cap J \to I \oplus J \to I + J \to 0,$$

where the third map is defined by $(i, j) \mapsto i + j$. This gives rise to the exact sequence

$$(I \cap J) \otimes A/m \to (I \oplus J) \otimes A/m \to (I + J) \otimes A/m \to 0.$$

So we have

$$\mu(I) + \mu(J) \leq \mu(I \cap J) + \mu(I + J),$$

but the last sum does not exceed $2d(A)$. Hence

$$\mu(I \cap J) = \mu(I + J) = d(A),$$

as desired. $\square$

Corollary 4. *Let A be an Artinian local ring. Then $\mathcal{F}(A)$ has the maximum and the minimum members.*

Proof. Since A is Artinian, an infinite sum of ideals is in fact a finite sum. Hence $\mathcal{F}(A)$ has the maximum member. Similarly, an infinite intersection of ideals is in fact finite. Thus $\mathcal{F}(A)$ has the minimum member. $\square$

Let I be an ideal of an Artinian local ring (A, m, k). Recall that the type of I is defined by

$$\tau(I) = \text{length}((I : m)/I).$$

It was proved in [**8**, Theorem 2.6] that the number

$$\mathrm{Max}\{\tau(I) \mid I \text{ is an ideal of } A\}$$

is equal to $d(A)$. Define the family $\mathcal{G}(A)$ of ideals by

$$\mathcal{G}(A) = \{\text{ideal } I \subset A \mid \tau(I) = d(A)\}.$$

Proposition 5. *Let (A, m, k) be an Artinian local ring. Then $\mathcal{G}(A)$ is a lattice with sum and intersection as the "join" and the "meet."*

Proof. Let $I, J \in \mathcal{G}(A)$. We show that $I \cap J \in \mathcal{G}(A)$ and $I + J \in \mathcal{G}(A)$. Consider the exact sequence:

$$0 \to A/(I \cap J) \to A/I \oplus A/J \to A/(I+J) \to 0,$$

where the third map is defined by

$$(x \bmod I, y \bmod J) \mapsto x + y \bmod (I + J).$$

This gives rise to the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_A(k, A/(I \cap J)) \\ &\to \mathrm{Hom}_A(k, A/I) \oplus \mathrm{Hom}_A(k, A/J) \\ &\to \mathrm{Hom}_A(k, A/(I+J)). \end{aligned}$$

So we have

$$\tau(I) + \tau(J) \leq \tau(I \cap J) + \tau(I + J),$$

but the last sum does not exceed $2d(A)$. Hence

$$\mu(I \cap J) = \mu(I + J) = d(A),$$

as desired. $\square$

Theorem 6. *Let (A, m, k) be an Artinian local ring. Then the correspondence*

$$\begin{cases} \mathcal{F}(A) \ni I \mapsto mI \quad \in \mathcal{G}(A) \\ \mathcal{G}(A) \ni J \mapsto J : m \in \mathcal{F}(A) \end{cases}$$

is a lattice isomorphism between $\mathcal{F}(A)$ and $\mathcal{G}(A)$.

Proof. We divide the proof into four steps.

Step 1. $I \in \mathcal{F}(A) \Rightarrow mI \in \mathcal{G}(A)$. Suppose that $I \in \mathcal{F}(A)$. Then we have

$$\tau(mI) = \text{length}((mI : m)/mI) \geq \text{length}(I/mI) = d(A).$$

Step 2. $J \in \mathcal{G}(A) \Rightarrow J : m \in \mathcal{F}(A)$. Suppose that $J \in \mathcal{G}(A)$. Then

$$\mu(J : m) = \text{length }((J : m)/m(J : m)) \geq \text{length }((J : m)/J) = d(A)$$

Step 3. $mI : m = I$ for $I \in \mathcal{F}(A)$ and $m(J : m) = J$ for $J \in \mathcal{G}(A)$. These are immediate from the proof of Steps 1 and 2.

Step 4. For $I_1, I_2 \in \mathcal{F}(A)$, we have $m(I_1 + I_2) = mI_1 + mI_2$ and $m(I_1 \cap I_2) = mI_1 \cap mI_2$. The first assertion is obvious. The second follows from Step 3 which shows that the correspondence is one to one. $\square$

Example 7. If $A = k[X, Y]/(X^4, Y^3)$, then $d(A) = 3, |\mathcal{F}_H(A)| = 4$ and

$$\mathcal{F}_H(A) = \{(X, Y)^2, (XY, Y^2) + (X, Y)^3, (Y^2) + (X, Y)^3, (X, Y)^3\}.$$

If k is an infinite field, then $\mathcal{F}(A)$ is not a finite family. In fact,

$$I = (Y^2 + aX^3) + (X, Y)^3$$

are Dilworth ideals for any $a \in k$.

The following example shows that in most cases $\mathcal{F}(A)$ and even $\mathcal{F}_H(A)$ are not finite.

Example 8. Let k be a field and let $A = k[X, Y, Z]/(X^5, Y^3, Z^2)$. Then $d(A) = 6$. Put

$$I_{(a,b)} = (aXY^2 + bXYZ, Y^2Z) + (X, Y, Z)^4,$$

for $a, b \in k$. It is easy to see that $\mu(I_{(a,b)}) = 6$, but for different $b \in k$ the ideals $I_{(1,b)}$ are all different. Thus $\mathcal{F}_H(A)$ is infinite provided that k is infinite.

3. The maximum and the minimum of the Dilworth family. Throughout this section, $A = \oplus_{i=0}^{c} A_i$ denotes a graded Artinian ring with $k := A_0$ a field. We say that A has the weak Lefschetz property (WLP) if there is a linear form $l \in A_1$ such that the homomorphism $A_i \to A_{i+1}$ induced by multiplication by l is either injective or surjective for all $i = 0, 1, \cdots, c-1$. In this case, we call l a weak Lefschetz element of A. In [**8**, Proposition 3.2] it was proved that if A has the WLP, then $\mu(m^r) = d(A)$ for some integer r, where m is the maximal ideal of A.

Lemma 9. *Let A be a graded Artinian k-algebra such that $A = k[A_1]$. Assume that* $\dim_k A_1 > 1$ *and let p be the least integer such that $\mu(m^p) \geq \mu(m^{p+1})$. If there exists a weak Lefschetz element l of A such that the initial degree of the socle of A/lA is greater than or equal to p, then m^p is the maximum member of $\mathcal{F}(A)$.*

Proof. Let I be the maximum member of $\mathcal{F}(A)$. First we show that I is contained in m^p with the additional assumption that I is homogeneous. If $p = 1$, there is nothing to prove. So we may assume that $p > 1$. By way of contradiction assume that I is not contained in m^p. Let q be the initial degree of I. Then, since we are assuming that $I \not\subset m^p$, we have $0 < q < p$. Let $a_1, \cdots, a_n$ be a basis of I_q and put $J = (a_1, \cdots, a_n)A$. Then obviously we have $\mu(lJ) \leq \mu(mJ)$. Moreover we have $\mu(J) = \mu(lJ)$, since l is a weak Lefschetz element. We treat the two cases (1) $\mu(J) < \mu(mJ)$ and (2) $\mu(J) = \mu(mJ)$ separately and in either case we lead a contradiction.

First assume that $\mu(J) < \mu(mJ)$. Then we have

$$\mu(I) = \mu(J) + \mu(I \cap m^{q+1}) - \mu(mJ) < \mu(I \cap m^{q+1}).$$

This contradicts the fact that $\mu(I)$ is the largest.

Now assume that $\mu(J) = \mu(mJ)$. This implies that $lJ = mJ$, since $lJ \subset mJ$. Hence the image $\bar{J}$ of J in A/lA is contained in the socle. Since the initial degree of the socle is at least p and since $q < p$, we have $\bar{J} = 0$ and $J \subset lA$. Thus there exist homogeneous elements $b_1, \cdots, b_n \in A_{q-1}$ of degree $q-1$ such that $lb_i = a_i$. Assume for the moment that $q > 1$. Then if we let $J' = (b_1, \cdots, b_n)$, we have

$$\mu(l^2 J') = \mu(lJ) = \mu(mJ) = \mu(mlJ').$$

Since l annihilates no elements of A_q, this means that $\mu(lJ') = \mu(mJ')$. Thus we have $\mu(J' + I) = \mu(I)$ which contradicts the maximality of I. Finally assume $q = 1$. This means that $n = 1$ and that $l = a_1$ up to constant multiple. Thus J is a principal ideal generated by l. Since we assume that the map $\times l : A_1 \to A_2$ is injective and since $lJ = mJ$, $\mu(m) = 1$. This contradicts the assumption $\dim A_1 > 1$. Thus we have proved that the maximum member of $\mathcal{F}_H(A)$ is contained in m^p.

Now we prove that the maximum member of $\mathcal{F}(A)$ is contained in m^p. We introduce some notation. For $f \in A$, if we write $f = f_d + f_{d+1} + \cdots$ with $f_i \in A_i$, $f_d \neq 0$, we denote by f° the initial part f_d. Let I be an ideal of A. Then by I° we denote the ideal generated by the set

$$\{a^\circ \mid a \in I\}.$$

Now let I be the maximum member of $\mathcal{F}(A)$. It is easy to see that $\mu(I) \leq \mu(I^\circ)$. Hence I° is a member of $\mathcal{F}_H(A)$, and I° is contained in m^p. Hence I is contained in m^p. Since A has the weak Lefschetz property, $d(A) = \mu(m^r)$ for some r. (See [**8**, Proposition 3.2].) Thus we have proved that m^p is the maximum in $\mathcal{F}(A)$. $\square$

Corollary 10. *Let $A = k[X,Y]/(f_1, \cdots, f_n)$ be an Artinian local ring, where k is an infinite field and $f_1, \cdots, f_n$ are homogeneous polynomials in $k[X,Y]$. Then the maximum member of $\mathcal{F}(A)$ is a power of the maximal ideal of A.*

Proof. By [**5**, Theorem 4.2], there is a weak Lefschetz element l of A. Note that the H-vector of A is of the form

$$(1, 2, \cdots, (p+1) = h_p, h_{p+1}, \cdots),$$

where $h_p \geq h_r$ for $p < r$. It follows that A/lA is a Gorenstein ring with socle degree p. Hence, by Lemma 9, the proof is complete.

Theorem 11. *Let k be an infinite field and let $f_1, \cdots, f_n$ be homogeneous polynomials in $k[X_1, \cdots, X_n]$ with $\deg f_i = p_i$. We assume that $2 \leq p_1 \leq \cdots \leq p_n$. Suppose that $A = k[X_1, \cdots, X_n]/(f_1, \cdots, f_n)$ is a complete intersection. If $p_n > p_1 + \cdots + p_{n-1} - (n-1)$, then the maximum member and the minimum members of $\mathcal{F}(A)$ are powers of the maximal ideal of A.*

Proof. Put

$$S = k[X_1, \cdots, X_n]/(f_1, \cdots, f_{n-1})$$

and

$$p = p_1 + \cdots + p_{n-1} - (n-1).$$

Then S is a graded Cohen-Macaulay ring of dimension one. Let $(h_0, h_1, \cdots)$ be the H-vector of S. Then we have

$$h_0 < h_1 < \cdots < h_{p-1} < h_p = h_{p+1} = \cdots .$$

Let $l \in S$ be a linear form which is a non-zero-divisor of S. The homomorphism $S_i \to S_{i+1}$ induced by multiplication by l is injective for $i < p$ and bijective for $i \geq p$. Let M be the homogeneous maximal ideal of S. It follows that $lS \supset M^{p+1}$. Notice that $A/lA \cong S/(f_n, l)S \cong S/(l)$, since $p_n > p$. So A/lA is a complete intersection with the socle degree p. It is easy to see that A has the weak Lefschetz property and that l is a weak Lefschetz element of A. (Cf. [**9**, Main theorem].) Let m be the maximal ideal of A. Then we have $\mu(m^i) < \mu(m^p)$ for $i < p$ and $\mu(m^p) \geq \mu(m^i)$ for $p \leq i$. By Lemma 9 we see that m^p is the maximum member of $\mathcal{F}(A)$.

By Lemma 12 below, the minimum member of $\mathcal{F}(A)$ is $m^{p_n - 1}$. ◻

Lemma 12. *Let (A, m, k) be an Artinian Gorenstein local ring and let $\mathcal{F}(A)$ be the Dilworth lattice of A. Then the correspondence*

$$I \mapsto (0 : I) : m$$

gives an anti-isomorphism of the lattice $\mathcal{F}(A)$.

Proof. Let $I \in \mathcal{F}(A)$. Suppose that $I = (a_1, \cdots, a_d)$ with $d = \mu(I)$. Then $0 : I = \cap_{i=1}^{d}(0 : a_i)$ is an irredundant intersection of irreducible ideals. Hence $\tau(0 : I) = \mu(I)$ and $0 : I \in \mathcal{G}(A)$. By Theorem 6, we have $(0 : I) : m \in \mathcal{F}(A)$. It is easy to see that the correspondence

$$\mathcal{F}(A) \ni I \mapsto 0 : I \in \mathcal{G}(A)$$

is an anti-isomorphism of lattices. Thus by composing with $\mathcal{G}(A) \to \mathcal{F}(A)$ as given in Theorem 6, the assertion follows. ◻

Theorem 13. *Let $A = k[X_1, \cdots, X_n]/(X_1^{d_1}, \cdots, X_n^{d_n})$ be a monomial complete intersection. Then the maximum and the minimum of $\mathcal{F}(A)$ are powers of the maximal ideal.*

Proof. Let $C(d)$ denote a chain of length d, i.e., a totally ordered d-element set. Then the set of monomials in A may be identified with the finite chain product

$$P := C(d_1) \times \cdots \times C(d_n).$$

P is a poset with a rank function. It is well known that P has the Sperner property, that is, a level set is a maximum-sized antichain. (For this, there are many references. See, e.g., [**1**].) This in particular determines $d(P)$, and hence $d(A)$, as the maximum size of the level sets. Let $\mathcal{F}(P)$ denote the Dilworth lattice of P in the sense of Freese [**3**]. Namely $\mathcal{F}(P)$ is the set of maximum-sized antichains with the partial order defined by the containment of the order ideals the antichains generate.

It can be proved that the maximum member of $\mathcal{F}(P)$ is a level set. (For this we give a proof in the appendix, since we have been unable to find a suitable reference.)

Now let $I \in \mathcal{F}(A)$ be the maximum member. We would like to prove that $I = m^p$, where p is the least integer such that $\mu(m^p) \geq \mu(m^{p+1})$. Among the monomials of A introduce a graded monomial order, as is used in the theory of Groebner basis, and let $\mathrm{in}(I)$ be the ideal generated by $\{\mathrm{in}(f) \mid f \in I\}$, where $\mathrm{in}(f)$ is the initial monomial of f. Then $\mu(I) \leq \mu(\mathrm{in}(I))$. Hence, in fact, $\mu(\mathrm{in}(I)) = \mu(I) = d(A)$. Thus the minimal generating set of $\mathrm{in}(I)$, being a maximum sized antichain, is contained in the maximum of $\mathcal{F}(P)$. This means that the degree of any element of I is at least p. The assertion for the minimum of $\mathcal{F}(A)$ follows from Lemma 12. $\square$

Example 14. Let $A = k[X, Y, Z]/((Y, Z)Z + (Y^3, X^2Z) + (X, Y, Z)^4)$. The H-vector is $(1, 3, 4, 3)$. Note that X is a weak Lefschetz element and $d(A) = 4$. The maximum of $\mathcal{F}(A)$ is $(Z) + m^2$ and the minimum $(XZ) + m^3$. So the maximum and the minimum are not powers of the maximal ideal. $\square$

It seems conceivable that if A is a Gorenstein algebra with the weak Lefschetz property, then the maximum and the minimum of $\mathcal{F}(A)$ are powers of the maximal ideal. In particular, in Theorem 11 above, the restriction on the degrees of generators seems unnecessary.

4. The Dilworth lattice of monomial complete intersections in two variables.

Theorem 15. *Let k be a field and let $A = k[X,Y]/(X^p, Y^q)$ where p, q are integers such that $p \geq q \geq 2$. Then $\mathcal{F}_H(A)$ is a distributive lattice isomorphic to the lattice of order ideals of the chain product $C(p-q) \times C(q)$. In particular $|\mathcal{F}_H(A)| = \binom{p}{q}$.*

Proof. Let $I \in \mathcal{F}_H(A)$. First note that $d(A) = q$, and $m^{q-1} \subset I \subset m^{p-1}$ by Lemmas 9 and 12. Fix the reverse lexicographic order of monomials in A with $X < Y$ and let $\mathrm{in}(f)$ denote the initial monomial in $f \in A$. Suppose that $I \in \mathcal{F}_H(A)$. Then $\mu(I) \leq \mu(\mathrm{in}(I))$. Hence $\mu(\mathrm{in}(I)) = q$. Suppose that $\{M_0, M_1, \cdots, M_{q-1}\}$ is the set of minimal monomials in $\mathrm{in}(I)$ put in the increasing order with respect to the Y-degree. Then it is easy to see that the only possibility for the Y-degree of M_i is i, and hence M_i take the form

$$M_i = X^{\alpha_i} Y^i, \qquad i = 0, 1, 2, \cdots, q-1, \tag{1}$$

where

$$p > \alpha_0 > \alpha_1 > \cdots > \alpha_{q-1} \geq 0.$$

Note that

$$\begin{cases} \deg M_0 \geq \deg M_1 \cdots \geq \deg M_{q-1}, \\ \deg M_i = \deg M_{i+1} \text{ if and only if } \alpha_i - \alpha_{i+1} = 1. \end{cases}$$

We may assume that I is minimally generated by homogeneous elements

$$f_0, f_1, \cdots, f_{q-1}$$

such that $\mathrm{in}(f_i) = M_i$, $i = 0, 1, \cdots, q-1$. Now suppose that a monomial M appears in f_i for some i. Notice that $M_i <_{(\text{rev lex})} M$. Thus M is either in the list (1) above or is divisible by a monomial in the list. In other words, every monomial that appears in f is contained in in(I). Thus we have $I \subset \text{in}(I)$. This show in fact that $I = \text{in}(I)$ and I can be generated by monomials. Put

$$\mathcal{I} = \{(\alpha_0, \alpha_1, \cdots, \alpha_{q-1}) \mid p > \alpha_0 > \cdots > \alpha_{q-1} \geq 0\}.$$

We have obtained a one-one correspondence

$$\mathcal{F}(A) \to \mathcal{I}.$$

In the obvious manner $\mathcal{I}$ is a lattice. The correspondence is actually a lattice isomorphism. It is easy to see that $\mathcal{I}$ is (isomorphic to) the lattice of order ideals in $C(p-q) \times C(q)$. (Cf. Remark 16 below.) $\square$

Remark 16. The lattice $\mathcal{I}$ above has many interpretations. See Stanley [**7**, pp. 28–31]. For example, it is the lattice that appears in the cellular decomposition of the Grassmann variety $G(q,p)$. Also $\mathcal{I}$ parametrizes the Young diagrams fit in a rectangle of size $r \times q$, where $r = p - q$. It also parametrizes the zigzag paths starting at $(0,0)$ and ending at (r,q) in a rectangle of size $r \times q$ divided into rq squares.

5. Appendix.

Proposition 17. *Let P be the finite chain product. Let $\mathcal{F}(P)$ be the Dilworth lattice. Then the maximum of $\mathcal{F}(P)$ is a level set.*

Proof. Let $P = \sqcup P_i$ be a level decomposition and let h_i be the level number: $h_i = |P_i|$. Let $I \subset P$ a maximum-sized antichain. Then

$$\sum_{a \in I} \frac{1}{h_{rk(a)}} \leq 1.$$

This is known as the LYM property of a poset and a finite chain product indeed has the LYM property. (See [**2**, Corollary 4.12].) Let I be the largest maximum-sized antichain of P. (This exists by Freese [**3**].) Let p be the least integer such that $h_p \geq h_{p+1}$. Then since P has the Sperner property, the order ideal which I generates contains P_p. If there is an element $a \in I \setminus P_p$, then it contradicts the LYM property. Thus P_p is the maximum member of $\mathcal{F}(P)$. $\square$

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