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GENERATING IDEALS IN PULLBACKS

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ABSTRACT. Let T be a domain, M a maximal ideal of T, $\varphi : T \to k = T/M$ the canonical projection, D a subring of the field k, and $R = \varphi^{-1}(D)$. We prove that if $I \nsubseteq M$ is an ideal of R for which $\varphi(I)$ can be generated by n elements of D and IT can be generated by m elements of T, then I can be generated by $\max\{2, n, m\}$ elements of R.

Consider the following pullback diagram:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & k \end{array}$$

Here, T is a domain, M is a maximal ideal of T, $k = T/M$, $\varphi : T \to k$ is the canonical projection, D is a subring of k, and $R = \varphi^{-1}(D)$. Pullbacks of this type have frequently been used to provide important (counter-)examples for many years now. In [**3**] Gabelli and the present author discussed much of what is known about ideal theory in pullbacks. Although that work was primarily a survey, we did consider the problem of determining the number of generators of an ideal I of R from knowledge of the number of generators of $\varphi(I)$ in D and of IT in T. The purpose of this note is to give a complete solution to that problem. Our main result is

Theorem. *Let $I \nsubseteq M$ be an ideal of R such that $\varphi(I)$ is an n-generated ideal of D and IT is an m-generated ideal of T. Then I can be generated by* $\max\{2, n, m\}$ *elements of R.*

Since an r-generated ideal I of R both maps to an r-generated ideal of D and extends to an r-generated ideal of T, it is easy to see that this is

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the best possible result one could hope to obtain when $\max\{m,n\} \geq 2$. It is also best possible when $m = n = 1$; this is explained below. Perhaps a comment on the requirement $I \not\subseteq M$ is in order. If $I \subseteq M$, then $\varphi(I) = (0)$, and the number of generators of I in R cannot be determined from the number of generators of IT alone. For example, if F is a field, $k = F(X)$, $T = k[[Y]]$, $M = YT$, and $D = F$; then $R = F + M$, and the common ideal M of R and T is principal in T but not even finitely generated in R.

Notation. As in [**3**], for an element $t \in T \setminus M$, we use t' to denote an element of T for which $\varphi(t') = 1/\varphi(t)$. It is useful to observe that $tt' \in R$ (since $\varphi(tt') = 1 \in D$) and $1 - tt' \in M$.

We shall need the following result from [**3**].

Lemma 1. ([**3**, Lemma 2.25]) *If P is a maximal ideal of R with $P \supseteq M$ and $x \in R \setminus M$, then $M \subseteq xR_P$.*

Lemma 2. *If I, J are ideals of R which are not contained in M and are such that $\varphi(I) = \varphi(J)$ and $IT = JT$, then $I = J$. In particular, if a subset A of I satisfies $\varphi(I) = \varphi(A)D$ and $IT = AT$, then A generates I in R.*

Proof. First, let $x \in I$. Then $\varphi(x) \in \varphi(I) = \varphi(J)$, so that $x \in \varphi^{-1}(\varphi(J)) = J + M$. Hence $I \subseteq J + M$, and, by symmetry, $I + M = J + M$. We now establish the result locally. Thus let P be a maximal ideal of R. If $P \supseteq M$, then by Lemma 1, $M \subseteq IR_P$ and $M \subseteq JR_P$, so that

$$IR_P = (I + M)R_P = (J + M)R_P = JR_P.$$

If $P \not\supseteq M$, it is well known (see [**3**, Theorem 1.9]) that there is a unique prime Q of T with $Q \cap R = P$ and $R_P = T_Q$, and we have

$$IR_P = IT_Q = JT_Q = JR_P. \qquad \square$$

We now proceed to prove our Theorem in a series of steps. We begin by reproving the main theorem in [**3**] on the number of generators of I.

Lemma 3. ([**3**, Theorem 2.26]) *Let I be an ideal of R with $I \nsubseteq M$, suppose that $x_1, \dots, x_n$ are elements of I with $x_1 \notin M$ whose images under φ generate $\varphi(I)$ in D, and suppose that $y_1, \dots, y_m$ are elements of T which generate IT. Then I is generated in R by the following elements:*

$$x_1, \dots, x_n, (1 - x_1 x_1')y_1, \dots, (1 - x_1 x_1')y_m.$$

Proof. First note that $(1 - x_1 x_1')y_i \in MIT \subseteq I$. Now the images of the x_i generate $\varphi(I)$ by hypothesis, and the equation

$$y_i = x_1 x_1' y_i + (1 - x_1 x_1')y_i$$

shows that the generators y_i of IT are contained in the ideal

$$(x_1, \dots, x_n, (1 - x_1 x_1')y_1, \dots, (1 - x_1 x_1')y_m)T.$$

The result now follows from Lemma 2. □

The next result handles the case $m = n = 1$ in our Theorem.

Proposition. *Let $I \nsubseteq M$ be an ideal of R such that $\varphi(I)$ is principal in D and IT is principal in T. Then:*

(1) *I can be generated by two elements in R, and*

(2) *I is principal in $R \Leftrightarrow$ there is an element $x \in I$ with $\varphi(x)D = \varphi(I)D$ and $xT = IT$.*

Proof. That I can be generated by two elements follows from Lemma 3, and statement (2) follows easily from Lemma 2. □

We note that it is possible for $\varphi(I)$ and IT to be principal while I is non-principal. The paper [**2**] by Fontana and Gabelli contains an explicit example. (Other examples may be found in [**3**].) It is useful to compare statement (2) of the Proposition with [**2**, Theorem 2.3].

Lemma 4. *If I is an ideal of R with $I \nsubseteq M$, then*

(1) $IT \cap R = I \Leftrightarrow \varphi(I) = D$, *and*

(2) $IT = T \Leftrightarrow I \supseteq M$.

Proof. (1) Suppose that $IT \cap R = I$. Pick $x \in I \setminus M$. Then $xx' \in IT \cap R = I$, and $\varphi(xx') = 1$. For the converse, note that $D = \varphi(I) \subseteq \varphi(IT \cap R) \subseteq D$. Thus $\varphi(I) = \varphi(IT \cap R)$. Since $(IT \cap R)T = IT$, the result follows from Lemma 2.

(2) If $IT = T$, then $M = MT = IMT \subseteq I$. For the converse, note that we have assumed $I \nsubseteq M$, so that IT properly contains the maximal ideal M of T. □

Lemma 5. *If I is an ideal of R with $I \nsubseteq M$ and is such that $\varphi(I)$ is principal in D and IT can be generated by $m > 1$ elements in T, then there are m elements $a_1, \dots, a_{m-1}, z \in I$ such that*

(1) $a_i \in M \cap I$ for $i = 1, \dots, m-1$,

(2) $\varphi(z)D = \varphi(I)$, *and*

(3) $I = (a_1, \dots, a_{m-1}, z)$.

Proof. We first deal with the case where $\varphi(I) = D$. Let $IT = (y_1, \dots, y_m)$. We may assume $y_1 \notin M$. By Lemma 4, $I = IT \cap R$. In particular, $y_1 y_1' \in I$. Now consider the m elements

$$(1 - y_1 y_1')y_1, \dots, (1 - y_1 y_1')y_{m-1}, y_1 y_1' + (1 - y_1 y_1')y_m.$$

Since $1 - y_1 y_1' \in M$, these elements satisfy condition (1), and the last one satisfies (2). With an eye toward using Lemma 2, we observe that, in particular, the images of these elements generate $\varphi(I) = D$. Moreover, the equation

$$y_1 = (1 - y_1 y_1')y_1(-y_m + 1) + [y_1 y_1' + (1 - y_1 y_1')y_m]y_1$$

shows that the ideal generated by these elements in T contains the element y_1. It then follows easily that the ideal also contains $y_2, \dots, y_m$. Lemma 2 now applies.

For the general case, pick $x \in I$ so that $\varphi(x)$ generates $\varphi(I)$ in D. Note that $x \notin M$ since $I \nsubseteq M$. Then

$$\varphi(x'I) = \varphi(x')\varphi(I) = \varphi(x')\varphi(x)D = D;$$

in particular, $x'I$ is an ideal of R. We have that $\varphi(x'I) = D$ and $x'IT$ is generated by $m > 1$ elements of T. By the first case, $x'I$ is generated by elements $a_1, \dots, a_{m-1}, z$ with $a_i \in M \cap x'I$ for $i = 1, \dots, m-1$ and $\varphi(z)D = D$. Thus I is generated by the elements $a_1/x', \dots, a_{m-1}/x', z/x'$. For $i = 1, \dots, m-1$, we have $a_i/x' \in M \cap I$ (since $x' \notin M$). Moreover,

$$\varphi(z/x')D = \varphi(x'x)\varphi(z/x')D = \varphi(x)\varphi(z)D = \varphi(x)D = \varphi(I),$$

and the proof is complete. □

Lemma 6. *Let $I \nsubseteq M$ be an ideal of R with $\varphi(I)$ n-generated in D and IT m-generated in T and such that $m \geq n > 1$. Then I can be generated by m elements.*

Proof. Let $x_1, \dots, x_n$ be elements of I whose images under φ generate $\varphi(I)$ in D. We may assume $x_1 \notin M$. Let IT be generated by $y_1, \dots, y_m$ in T. By Lemma 3, I can be generated by $x_1, \dots, x_n, (1 - x_1x_1')y_1, \dots, (1-x_1x_1')y_m$. Consider the ideal $J \subseteq I$ of R generated by $x_1, (1 - x_1x_1')y_1, \dots, (1 - x_1x_1')y_m$. Note that $\varphi(J)$ is principal in D. Moreover, since $y_i = x_1x_1'y_i + (1 - x_1x_1')y_i$, we have $y_i \in JT$, whence $JT = IT$. By Lemma 5, J has a generating set $a_1, \dots, a_{m-1}, z$ where $\varphi(z)D = \varphi(x_1)D$ and each $a_i \in M \cap J$. Since $JT = IT$, we see that IT is generated by $z, a_1, \dots, a_{m-1}$ in T. We shall use this below. Also, we have that I is generated by $z, x_2, \dots, x_n, a_1, \dots, a_{m-1}$.

We shall now show that I is generated by the following m elements:

$$z, a_1 + zz'x_2, a_2 + zz'x_3, \dots, a_{n-1} + zz'x_n, a_n, \dots, a_{m-1}.$$

(Of course, if $m = n$, the list stops with the element $a_{n-1} + zz'x_n$.) Recall that $\varphi(z)D = \varphi(x_1)D$. Also, $\varphi(a_i + zz'x_{i+1}) = \varphi(x_{i+1})$ for $i = 1, \dots, n-1$. Hence the images of these elements generate $\varphi(I)$. Moreover, the fact that $z, a_i + zz'x_{i+1}$ lie in the extension to T of the ideal generated by these elements immediately implies that a_i is also in this ideal. It now follows from Lemma 2 that the given elements generate I. □

We note that this completes the proof of our Theorem in the case $m \geq n$. The case $1 \leq m < n$ follows from this case by adding

$n - m$ "dummy" generators to an m-element generating set for IT. The following result yields an alternate approach.

Lemma 7. *If $I \nsubseteq M$ is an ideal of R such that $\varphi(I)$ is generated by $n > 1$ elements in D and IT is generated by $m < n$ elements in T, then I can be generated by n elements in R.*

Proof. Choose $x_1, \dots, x_n \in I$ such that their images generate $\varphi(I)$ in D. We may assume $x_1 \neq 0$. Let $IT = (y_1, \dots, y_m)T$. We claim that the following n elements of I generate I:

$$x_1, \dots, x_{n-m}, x_{n-m+1}x_1x_1' + (1 - x_1x_1')y_1, \dots, x_nx_1x_1' + (1 - x_1x_1')y_m.$$

Observe that the image in D of the element $x_{n-m+j}x_1x_1' + (1 - x_1x_1')y_j$ is $\varphi(x_{n-m+j})$, so that the images of these elements do generate $\varphi(I)$. Moreover, the equation

$$y_j = x_1x_1'(-x_{n-m+j} + y_j) + x_{n-m+j}x_1x_1' + (1 - x_1x_1')y_j$$

shows that the extension to T of the ideal generated by these elements is $(y_1, \dots, y_m)T$. Apply Lemma 2. □

We observe that by following the proofs of Lemmas 5 and 6, one can actually write down the generators for I. Then one can show that the generators work using only Lemma 2. Thus assuming that $x_1, \dots, x_n \in I$ are chosen so that their images generate $\varphi(I)$ (with $x_1 \notin M$) and $y_1, \dots, y_m \in T$ are chosen so that they generate IT (with $y_1 \notin M$), set $a = 1 - x_1x_1'y_1y_1'$ and $z = x_1y_1y_1' + ay_m$. We claim that the following elements generate I:

$$z, ay_1 + zz'x_2, \dots, ay_{n-1} + zz'x_n, ay_n, \dots, ay_{m-1}.$$

(If $m \leq n$, the list stops at $ay_{n-1} + zz'x_n$, and if $m < n - 1$, we set $y_i = 0$ for $i > m$.) To verify the claim, let J be the ideal of R generated by these elements. To show that $I = J$, note that the images of the first n elements listed are exactly $\varphi(x_1), \dots, \varphi(x_n)$. The fact that $z, ay_1 + zz'x_2$ are listed in the set of purported generators implies that ay_1 is in JT; the equation

$$y_1 = zy_1x_1' + ay_1(-y_mx_1' + 1)$$

then shows that y_1 is also in JT. It is now easy to see that $y_2, \ldots, y_m \in JT$. Thus $I = J$ by Lemma 2.

We close with a few remarks. The section on generators in [**3**] ended with two problems. The first was to determine whether the bound given on the number of generators in Lemma 3 is the best possible. Of course, our Theorem answers that question in the negative. Indeed, apart from the case $m = n = 1$, the situation is exactly as in the generalized $D + M$-construction (see [**3**, Theorem 2.28], which was inspired by [**1**, Theorem 10]. The second problem asked whether a pullback construction such as we have considered here could be used to produce examples of Prüfer domains with invertible ideals requiring more than two generators. Unfortunately, our Theorem again shows that the answer is negative.

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