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ON CHARACTERIZATIONS OF INTEGRALITY INVOLVING THE LYING-OVER AND INCOMPARABLE PROPERTIES

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ABSTRACT. The fact that residually algebraic pairs are the same as INC-pairs is generalized from the context of integral domains to that of arbitrary (commutative) rings. It is also shown that if $A \subseteq B$ are rings with D the integral closure of A in B, then B is integral over A if and only if (A, B) is an INC-pair for which the extension $D \subseteq B$ satisfies LO. However, a Noetherian local one-dimensional domain A is Henselian if and only if B is integral over A whenever B is a domain containing A such that (A, B) is an INC-pair for which the extension $A \subseteq B$ satisfies LO.

1. Introduction. All rings considered in this note are commutative with identity, and all subrings are unital. Following [**10**, page 28] we let LO, INC and GU denote the lying-over, incomparable and going-up properties for ring extensions. If $\mathcal{P}$ is a property of (some) ring extensions and $A \subseteq B$ are rings, we say that (A, B) is a $\mathcal{P}$-*pair* in case $D \subseteq E$ satisfies $\mathcal{P}$ for all rings $A \subseteq D \subseteq E \subseteq B$. The case of LO-pairs was introduced in [**5**], studied sporadically in the literature (cf. [**12**]), and recently given a new characterization in [**3**, Theorem 2.2]. It was shown in [**5**, Corollary 3.2] that GU-pairs are the same as LO-pairs. As for INC-pairs, they were introduced and characterized (without the terminology) in [**4**, Corollary 4] and studied further, but only in the context of extensions of (commutative integral) domains, in [**1**]. In particular, [**1**, Theorem 2.3] established that for extensions of domains, INC-pairs are the same as residually algebraic pairs. This domain-theoretic formulation has persisted in the summary of [**1**] given in the monograph [**8**], and several subsequent papers have also continued to study INC-pairs and residually algebraic pairs only for extensions of domains. Accordingly, our first order of business here is to generalize [**1**, Theorem 2.3] by showing that, for arbitrary ring extensions, the concepts of INC-pairs and residually algebraic pairs are equivalent.

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This is done in Proposition 2.1 by using results from [**4**]. We then set off to our main goal, which is to highlight an important difference between the behavior of LO and that of INC.

The major part of this paper is motivated by the fact that prominent motivation for the study of INC-pairs and LO-pairs was provided by the following Folklore Theorem: if $A \subseteq B$ are rings, then B is integral over A if (and only if) (A, B) is both an INC-pair and an LO-pair (cf. [**4**, page 37]). It is natural to ask for sharpenings of the Folklore Theorem, and several have been given: cf. [**5**, Theorem 2.1, Corollary 3.5]. The last-cited result, a consequence of which is reprised in Proposition 2.3 below, is noteworthy as it characterizes the integrality of B over A by the conditions that (A, B) is an LO-pair and $A \subseteq B$ is a MINC-extension. (As in [**5**], a ring extension $A \subseteq B$ is said to satisfy MINC if, whenever comparable prime ideals $Q_1 \subseteq Q_2$ of B are such that $Q_1 \cap A = Q_2 \cap A$ is a maximal ideal of A, then $Q_1 = Q_2$. Recall from [**5**, Example 2.2] that the MINC property is strictly weaker than INC.) Our main focus here concerns the question whether the roles of LO and INC in Proposition 2.3 can be interchanged. Along these lines, there is one relevant result for domains [**1**, Theorem 2.12], but its proof does not extend to the context of arbitrary ring extensions. Nevertheless, the result is valid in the ring-theoretic setting, and as our second order of business, Theorem 2.2 establishes this generalization of [**1**, Theorem 2.12].

However, the full "interchange roles of LO and INC" analogue of Proposition 2.3 is not valid. In fact, our main result, Theorem 2.4, shows that if the base ring A is a one-dimensional Noetherian local domain, then requiring such an analogue would force A to be Henselian (in the sense of [**11**]). Independently of Theorem 2.4 (that is, without reference to the theory of Henselian rings), Remark 2.5 makes the same point, by considering a rational prime that is split in a given quadratic algebraic number field. We recall enough classical algebraic number theory to make Remark 2.5 essentially self-contained.

In addition to the usage mentioned above, we adopt the following conventions. If A is a ring, then by the dimension of A, we mean the Krull dimension of A, denoted by $\dim(A)$; and if $a \in A$, then A_a denotes the ring of fractions A_S, where S is the multiplicatively closed set consisting of the powers of a. Any unexplained material is standard, as in the cited texts.

2. Results Generalizing from the domain-theoretic context of [**1**], we say that a ring extension $D \subseteq E$ is *residually algebraic* if for each prime ideal Q of E, the extension of domains $D/(Q \cap D) \subseteq E/Q$ is algebraic. Hence, by applying the earlier definition of $\mathcal{P}$-pairs, we have that if $A \subseteq B$ are rings, then (A, B) is a *residually algebraic pair* if $D \subseteq E$ is a residually algebraic extension for all rings $A \subseteq D \subseteq E \subseteq B$. Note that in defining a residually algebraic pair of domains, Ayache and Jabbalh [**1**, page 49] restricted to the case $D = A$ in the above notation. The definitions are, in fact, equivalent. Our first result establishes this fact and extends both [**4**, Corollary 4] and [**1**, Theorem 2.3].

Proposition 2.1. *Let $A \subseteq B$ be rings. Then the following conditions are equivalent:*

(1) *For each $u \in B$, there is a polynomial $f \in A[X]$ such that at least one coefficient of f is a unit of A and $f(u) = 0$;*

(2) *For each $u \in B$, the extension $A \subseteq A[u]$ satisfies INC;*

(3) *For each ring E such that $A \subseteq E \subseteq B$, the extension $A \subseteq E$ satisfies INC;*

(4) *(A, B) is an INC-pair;*

(5) *For each $u \in B$, the extension $A \subseteq A[u]$ is residually algebraic;*

(6) *For each ring E such that $A \subseteq E \subseteq B$, the extension $A \subseteq E$ is residually algebraic;*

(7) *(A, B) is a residually algebraic pair.*

Proof. [**4**, Corollary 4] established that conditions (1), (2), (3), and (4) are equivalent. Also, since the property in condition (1) is preserved by passing to images under (unital) ring homomorphisms, it is easy to see that (1) $\Rightarrow$ (7). Of course, it is trivial that (7) $\Rightarrow$ (6) $\Rightarrow$ (5). Thus, it suffices to prove that (5) $\Rightarrow$ (2).

It is enough to show that any residually algebraic ring extension satisfies INC. This, in turn, was established for domains in [**1**, Proposition 1.5], where it was noted that the assertion had already been published by Fontana-Izelgue-Kabbaj [**9**]. The result in [**9**] is actually due to the present author, who provided a proof in response to a question that Fontana had raised (via private communication). That proof, which is valid for arbitrary ring extensions (indeed, for commutative unital algebras), is an easy consequence of the following observation: if $D \subseteq E$ is an algebraic extension of domains and J is a nonzero ideal of E, then $J \cap D \neq 0$. The proof is complete. □

Theorem 2.2 generalizes the Folklore Theorem mentioned in the Introduction that integral extensions can be characterized as the ring extensions inducing both LO-pairs and INC-pairs. The nontrivial part of Theorem 2.2, namely, that (1) ⇒ (3), was established for domains by Ayache-Jaballah [**1**, Theorem 2.12].

Theorem 2.2. *Let $A \subseteq B$ be rings. Let D denote the integral closure of A in B. Then the following conditions are equivalent:*

(1) (A, B) *is an INC-pair and the extension $D \subseteq B$ satisfies LO;*

(2) (A, B) *is an INC-pair and the extension $D \subseteq B$ satisfies GU;*

(3) B *is integral over A.*

Proof. (3) ⇒ (2) since integral extensions satisfy INC and GU [**10**, Theorem 44]; and (2) ⇒ (1) since GU ⇒ LO [**10**, Theorem 42]. It remains to prove that (1) ⇒ (3).

Assume (1). We show that each $u \in B$ is integral over A. Let E denote the integral closure of A in $A[u]$. Then $E = D \cap A[u]$. In particular, $E \subseteq D$ is an integral extension and therefore satisfies LO, by the classical Lying-over Theorem (cf. [**10**, Theorem 44]). By "composing" $E \subseteq D$ with the lying-over extension $D \subseteq B$, we see that $E \subseteq B$ satisfies LO. Since $E \subseteq A[u] \subseteq B$, it follows that $E \subseteq A[u]$ satisfies LO. Moreover, Proposition 2.1 shows that $(E, A[u])$ inherits the property of being an INC-pair from (A, B). Thus, we may replace (A, B) with $(E, A[u])$ (and D with E).

By the above reduction, we now have that $B = A[u]$ is a finite type A-algebra and A is integrally closed in B. It suffices to show that $A = B$. By globalization, it is enough to prove that if P is any prime ideal of A, then the canonical ring homomorphism $A_P \to B_P := B \otimes_A A_P$ is an isomorphism. Since $A \subseteq B$ satisfies LO, we can find a prime ideal Q of B such that $Q \cap A = P$. Note that Q is isolated in its fiber above P since $A \subseteq B$ satisfies INC. Therefore, by Zariski's Main Theorem (as in [**7**, Theorem]), there exists $v \in A \setminus P$ such that the canonical ring homomorphism $A_v \to B_v = B \otimes_A A_v$ is an isomorphism. Tensoring this isomorphism (over A) with A_P leads to the desired isomorphism $A_P \to B_P$, since $A_v \otimes_A A_P \cong A_P$ and $B_v \otimes_A A_P \cong B_P$ canonically. The proof is complete. □

It may seem natural to ask if D could be replaced with A in conditions (1) and (2) in Theorem 2.2. Additional motivation for this question comes from the fact that its "dual", in which the roles of INC and LO (resp., of INC and GU) are interchanged, has an affirmative answer. Indeed, we have the following result, which is an immediate consequence of [**5**, Corollaries 3.5 and 3.2].

Proposition 2.3. *Let $A \subseteq B$ be rings. Then the following conditions are equivalent:*

(1) *(A, B) is an LO-pair and the extension $A \subseteq B$ satisfies INC;*

(2) *(A, B) is a GU-pair and the extension $A \subseteq B$ satisfies INC;*

(3) *B is integral over A.*

Despite Proposition 2.3, the question of whether D could be replaced with A in conditions (1) and (2) in Theorem 2.2 has a negative answer in general, even for domains. In fact, as Theorem 2.4 explains, only very special rings A admit an affirmative answer.

Theorem 2.4. *Let A be a one-dimensional quasilocal domain. Consider the following two conditions:*

(i) *If B is a domain containing A such that (A, B) is an INC-pair and the extension $A \subseteq B$ satisfies LO, then B is integral over A;*

(ii) *A is Henselian.*

Then:

(a) (ii) $\Rightarrow$ (i).

(b) *Suppose also that A is Noetherian. Then* (i) $\Leftrightarrow$ (ii).

Proof. (a) Assume (ii). Let B be a domain containing A such that (A, B) is an INC-pair and the extension $A \subseteq B$ satisfies LO. By [**4**, Theorem] (or Proposition 2.1), B is algebraic over A. Let D denote the integral closure of A in B. Since A is Henselian by assumption, it follows from [**11**, (30.5)] that D is quasilocal. Furthermore, by [**6**, Lemma 2.2 (b)], D inherits the Henselian property from A. Now, let M (resp., N) denote the unique maximal ideal of A (resp., D), so that the set of prime ideals of A (resp., D) is $\{0, M\}$ (resp., $\{0, N\}$). (By integrality (cf. [**10**, Theorems 42, 44, 47 and 48]), $N \cap A = M$ and $\dim(D) = \dim(A) = 1$.) Therefore, by considering the tower $A \subseteq D \subseteq B$, for which $A \subseteq B$ satisfies LO, we see that $D \subseteq B$ also satisfies LO. Moreover, by Proposition 2.1, (D, B) inherits the property of being an INC-pair from (A, B). Thus, without loss of generality, we may replace A with D. In other words, we may also assume that A is integrally closed in B, and our task is to show that $A = B$. This, in turn, follows from [**1**, Theorem 2.5, $(i) \Rightarrow (iv)$] (and, thus, essentially from [**4**, Theorem], the (u, u^{-1}) Lemma [**10**, Theorem 67], and an argument of Visweswaran [**12**]), which applies since A is quasilocal, A is integrally closed in B and (thanks to Proposition 2.1) (A, B) is a residually algebraic pair.

(b) By (a), we need only show that if A is Noetherian, then (i) $\Rightarrow$ (ii). We prove the contrapositive. Consider, then, a Noetherian local one-dimensional non-Henselian domain (A, M). We shall produce an extension domain B of A such that (A, B) is an INC-pair, the extension $A \subseteq B$ satisfies LO and B is not integral over A.

Since A is not Henselian, [**11**, (30.5)] supplies an extension domain D of A such that D is integral over A and D is not quasilocal. Pick distinct maximal ideals Q_1, Q_2 of D and an element $u \in Q_1 \setminus Q_2$. By integrality, $Q_1 \cap A = M = Q_2 \cap A$. There is no harm in replacing D with $A[u]$. As D is then module-finite over A, we have $[L : K] < \infty$, where L (resp., K) denotes the quotient field of D (resp., A). Therefore, by the Krull-Akizuki Theorem (as formulated in [**2**, Proposition 5, page 500]), each ring between A and D_{Q_1} is (Noetherian and) of dimension at most 1. In view of the final comment in the proof of Proposition 2.1, algebraicity now ensures that (A, D_{Q_1}) satisfies condition (3) in the statement of Proposition 2.1 and, thus, is an INC-pair. Moreover, it is evident that the extension $A \subseteq D_{Q_1}$ satisfies LO. However, the extension $D \subseteq D_{Q_1}$ does not satisfy LO, since no prime ideal of D_{Q_1}

meets D in Q_2. (The point is that $A \subseteq D$, being an integral extension, satisfies INC, whence $Q_2 \not\subseteq Q_1$.) By the classical Lying-over Theorem, D_{Q_1} is not integral over D (a fact that could also be seen via [**10**, Exercise 10, page 24]) and, *a fortiori*, is therefore not integral over A. Thus, $B := D_{Q_1}$ has the asserted properties, to complete the proof. $\square$

In view of Theorem 2.4, one might well wonder, given a one-dimensional Noetherian local non-Henselian domain A, how to produce an extension domain B so that condition (i) in Theorem 2.4 fails to hold. Accordingly, it seems worthwhile to present a particular example illustrating the main point of Theorem 2.4. Remark 2.5 does so independently of Theorem 2.4, with an example that depends on classical algebraic number theory.

Remark 2.5. It is possible to give a concrete example that exhibits a negative answer to the question that was raised following the proof of Theorem 2.2. In fact, we next produce an extension $A \subseteq B$ of domains such that (A, B) is an INC-pair, the extension $A \subseteq B$ satisfies LO and B is not integral over A. (The reader will observe that in this example, A is a Noetherian local one-dimensional domain; in the closing comment, we also verify directly that this A is not Henselian.)

The desired example is shaped strategically so that, to use the notation figuring in the statement of Theorem 2.2, the extension $A \subseteq D$ has fibers that are not singleton sets. Its details depend on a consequence of the fundamental equation of ramification theory (a nice ring-theoretic exposition of which can be found in [**14**, Corollary, page 287]). In a given quadratic algebraic number field L, this consequence classifies the (rational) prime numbers as being either split (sometimes known as "decomposed"), inertial or ramified. These mutually exclusive cases are characterized in [**14**, Theorem 32 (c), page 313] (cf. also [**13**, Theorem 6-2-1, Corollary 6-2-3]). As noted in [**14**, Remark (1), page 289], any L as above has infinitely many split prime numbers, but our construction needs only one instance of a split prime. For simplicity, we might focus on the prime 2, which, according to the above-cited criteria, is split in the quadratic algebraic number field $L = \mathbf{Q}(\sqrt{m})$, where m is a squarefree integer, if and only if $m \equiv 1 \pmod 8$. (Thus, L can be taken either real or complex, with m as, for instance, 17 or -7.)

Suppose then that p is a prime number that is split in $L = \mathbf{Q}(\sqrt{m})$, with m as above. Let E denote the ring of algebraic integers of L, that is, the integral closure of $\mathbf{Z}$ in L. Since p is split in L, the ideal $P := p\mathbf{Z}$ is such that $PE = Q_1Q_2$ for some distinct prime ideals Q_1, Q_2 of E. In particular, $Q_1 \cap \mathbf{Z} = P = Q_2 \cap \mathbf{Z}$. We show that $(A, B) := (\mathbf{Z}_P, E_{Q_1})$ has the asserted properties.

The verification proceeds much as in the proof of Theorem 2.4 (b), essentially since the "Henselian" assumption was used in that earlier proof mainly to obtain distinct Q_i, a situation that we have achieved concretely here by using ramification theory. Indeed, it is evident that the extension $\mathbf{Z}_P \subseteq E_{Q_1}$ satisfies LO; and Proposition 2.1 and the Krull-Akizuki Theorem can be used as before to show that $(\mathbf{Z}_P, E_{Q_1})$ is an INC-pair. Also, note via [**2**, Proposition 16, page 314] that the integral closure of $\mathbf{Z}_P$ in L is $E_{\mathbf{Z}\setminus P}$, which differs from $E_{Q_1} = B$ by having a prime ideal that meets E in Q_2. In particular, B is not integral over A, and the proof is complete.

In closing, we note that Remark 2.5 is compatible with Theorem 2.4, since $A := \mathbf{Z}_{2\mathbf{Z}}$ is a one-dimensional Noetherian local non-Henselian domain. All but possibly the non-Henselian assertion is clear. To verify this, we need only apply the definition of a Henselian ring, as given in [**11**, pages 103–104]. Indeed, note that the residue field of A is canonically isomorphic to $\mathbf{Z}/2\mathbf{Z}$, that $(X+1, X, 2) = \mathbf{Z}_{2\mathbf{Z}}[X]$, and that the factorization $X^2+X+2 = (X+1)X$ in $(\mathbf{Z}/2\mathbf{Z})[X]$ does not lift to a factorization via monic polynomials in $\mathbf{Z}_{2\mathbf{Z}}[X]$. The verification comes down to the fact that neither $X^2 + X + 2$ nor 1 is congruent to $X + 1$ (or to X) modulo $2\mathbf{Z}_{2\mathbf{Z}}[X]$. Thus, as is often the case in verifying ring-theoretic assertions in number-theoretic contexts, the assertion follows from the fact that $\frac{1}{2} \notin \mathbf{Z}$.

REFERENCES

1. A. Ayache and A. Jaballah, *Residually algebraic pairs of rings*, *Math. Z.* **225** (1997), 49–65.

2. N. Bourbaki, *Commutative Algebra,*, *Addison-Wesley, Reading, Mass.*, (1972).

3. J. Coykendall and D. E. Dobbs, *Survival-pairs of commutative rings have the lying-over property*, *Comm. Algebra* **31** (2003), 259–270.

4. D. E. Dobbs, *On INC-extensions and polynomials with unit content*, *Can. Math. Bull.* **23** (1980), 37–42.

5. D. E. Dobbs, *Lying-over pairs of commutative rings*, *Can. J. Math.* **33** (1981), 454–475.

6. D. E. Dobbs, *Locally Henselian going-down domains*, *Comm. Algebra* **24** (1996), 1621–1635.

7. E. G. Evans, Jr., *A generalization of Zariski's main theorem*, *Proc. Amer. Math. Soc.* **26** (1970), 45–48.

8. M. Fontana, J. Huckaba and I. J. Papick, *Prüfer Domains*, *Dekker, New York* (1997).

9. M. Fontana, L. Izelgue and S. Kabbaj, *Quelques propriétés des chaînes d'idéaux dans les anneaux $A + XB[X]$*, *Comm. Algebra* **22** (1994), 9–27.

10. I. Kaplansky, *Commutative Rings, Revised Edn.*, *Univ. of Chicago Press, Chicago* (1974).

11. M. Nagata, *Local Rings*, *Wiley-Interscience, New York*, (1962).

12. S. Visweswaran, *Intermediate rings between $D+I$ and $K[y_1, \ldots, y_t]$*, *Comm. Algebra.* **18** (1990), 309–345.

13. E. Weiss, *Algebraic Number Theory*, *McGraw-Hill, New York*, (1963).

14. O. Zariski and P. Samuel, *Commutative Algebra*, *Volume I, Van Nostrand, Princeton*, (1958).

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