

## IDEALIZATION OF A MODULE

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ABSTRACT. Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Nagata introduced the idealization  $R(+M)$  of  $M$ . Here  $R(+M) = R \oplus M$  (direct sum) is a commutative ring with product  $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ . The name comes from the fact that if  $N$  is a submodule of  $M$ , then  $0 \oplus N$  is an ideal of  $R(+M)$ . The idealization can be used to extend results about ideals to modules and to provide interesting examples of commutative rings with zero divisors. We survey known results concerning  $R(+M)$  and give some new ones too. The theme throughout is how properties of  $R(+M)$  are related to those of  $R$  and  $M$ .

**1. Introduction.** Let  $R$  be a commutative ring with 1, and let  $M$  be a unitary  $R$ -module. Then  $R(+M) = R \oplus M$  (direct sum) with coordinate-wise addition and multiplication  $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$  is a commutative ring with 1 (even an  $R$ -algebra) called the *idealization of  $M$*  or the *trivial extension of  $R$  by  $M$* . Note that  $R$  naturally embeds into  $R(+M)$  via  $r \rightarrow (r, 0)$ , if  $N$  is a submodule of  $M$ , then  $0(+N)$  is an ideal of  $R(+M)$ ,  $0(+M)$  is a nilpotent ideal of  $R(+M)$  of index 2, and that  $(R(+M))/(0(+M)) \approx R$ . Idealization is useful for (1) reducing results concerning submodules to the ideal case, (2) generalizing results from rings to modules and (3) constructing examples of commutative rings with zero divisors. The purpose of this article is to survey known results on idealization and to give some new ones and to give a history of the subject and its usefulness.

While we do not know who first constructed an example using idealization, the idea to use idealization to extend results concerning ideals to modules is due to Nagata. The preface to his famous book *Local rings* [49] states: “Among the new methods and new results given

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in the present book, the following four should be noted: (1) A principle, which is called the principle of idealization, and by which modules become ideals, is applied manywhere in the book.” For example, in [49] primary decomposition and the Artin-Rees lemma are proved for ideals and then extended to modules by the principle of idealization.

The purpose of idealization is to put  $M$  inside a commutative ring  $A$  so that the structure of  $M$  as an  $R$ -module is essentially the same as that of  $M$  as an  $A$ -module, that is, as an ideal of  $A$ . We call this a *ringification*. There are two main ways to do this: the idealization  $R(+)M$  and the symmetric algebra  $S_R(M)$ . Both idealization and the symmetric algebra construction give functors from the category of  $R$ -modules to the category of  $R$ -algebras. The symmetric algebra is the freest way to do this, while the idealization has the most relations. Here  $R(+)M$  is naturally isomorphic to  $S_R(M)/\oplus_{n \geq 2} S_R^n(M)$  (and so  $R(+)M$  is a graded  $R$ -algebra). In fact, for any ringification  $A$  of  $M$ , there are epimorphisms  $\theta: S_R(M) \rightarrow A$  and  $\psi: A \rightarrow R(+)M$  with  $\psi\theta$  the natural map, Theorem 2.1. This is covered in Section 2 along with functorial properties of the idealization functor.

Let us mention that there is a third realization of the idealization. Let  $T = \left\{ \begin{bmatrix} r & m \\ 0 & r \end{bmatrix} \mid r \in R, m \in M \right\}$  with the usual matrix addition and multiplication. Then  $T$  is a commutative ring with identity, the map  $r \rightarrow \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$  embeds  $R$  into  $T$  and  $\widehat{M} = \left\{ \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \mid m \in M \right\}$  is an ideal of  $T$ . The map  $R(+)M \rightarrow T$  given by  $(r, m) \rightarrow \begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$  is easily seen to be a ring isomorphism that takes  $0 \oplus M$  to  $\widehat{M}$ . Of course, if you are nervous about where “ $T$  lives,” more properly  $T$  can be viewed as the subring  $\left\{ \begin{bmatrix} (r,0) & (0,m) \\ (0,0) & (r,0) \end{bmatrix} \right\}$  of  $\text{Mat}_2(R(+)M)$ .

In this paper we confine ourselves to the case where  $R$  is a commutative ring with identity. But for any ring  $R$  and  $(R, R)$ -bimodule  $M$ ,  $R(+)M = R \oplus M$  with the product  $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$  is a ring that contains an isomorphic copy of  $M$ , namely  $0 \oplus M$ , as a two-sided ideal. More generally, if  $R$  and  $S$  are rings and  $M$  is an  $(R, S)$ -bimodule, then  $A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{bmatrix} r & m \\ 0 & s \end{bmatrix} \mid r \in R, s \in S, m \in M \right\}$  is a ring under the usual matrix operations. This construction has been used to produce some interesting examples such as a left Artinian ring that is not right Artinian. For more on this construction, see [23, 51]. However, in [51] it is noted that  $A$  may be considered as the idealiza-

tion  $(R \times S)(+)M$  where  $M$  has the natural  $R \times S$ -bimodule structure  $(r, s)m = rm$  and  $m(r, s) = ms$ . There is an extensive literature in the noncommutative case. Noncommutative ring theorists seem to prefer the term “trivial extension” over “idealization” and to some extent in the study of Noetherian commutative rings this is also true (but, of course, Nagata’s book which introduced idealization focuses almost entirely on Noetherian rings). While we will exclusively use the notation  $R(+M)$  for the idealization, other commonly used notations are  $R \ltimes M$  and  $R\alpha M$  and the first author has sometimes used  $R(M)$ . Finally, we mention that the idealization construction can be generalized to what is called a semi-trivial extension. Let  $R$  be a commutative ring,  $M$  an  $R$ -module, and  $\varphi: M \otimes_R M \rightarrow R$  an  $R$ -module homomorphism satisfying  $\varphi(m \otimes m') = \varphi(m' \otimes m)$  and  $\varphi(m \otimes m')m'' = m\varphi(m' \otimes m'')$ . Then  $R\alpha_\varphi M = R \oplus M$  with coordinate-wise addition and multiplication  $(r, m)(r', m') = (rr' + \varphi(m \otimes m'), rm' + r'm)$  is a commutative ring, called a semi-trivial extension of  $R$  by  $M$ . (For  $\varphi = 0$ , we have idealization.) See [57] for details.

In Section 3 we study the ideals of  $R(+M)$  and certain distinguished subsets of  $R(+M)$ . We determine the maximal, prime, homogeneous, primary and radical ideals of  $R(+M)$  as well as the units, idempotents, zero divisors, and nilpotents, and the saturated multiplicatively closed subsets of  $R(+M)$ .

A special role is played by the ideals of  $R(+M)$  of the form  $I \oplus N$ . Now  $R(+M)$  has a natural  $\mathbf{N}$ -grading with  $(R(+M))_0 = R \oplus 0$ ,  $(R(+M))_1 = 0 \oplus M$ , and  $(R(+M))_n = 0$  for  $n \geq 2$ . (This can also be viewed as a  $\mathbf{Z}_2$ -grading since  $(0 \oplus M)^2 = 0$ .) The homogeneous ideals of  $R(+M)$  are the ideals of the form  $I \oplus N$  where  $I$  is an ideal of  $R$ ,  $N$  is a submodule of  $M$ , and  $IM \subseteq N$ . Conditions are given for every ideal of  $R(+M)$  to be homogeneous. In particular, for  $R$  an integral domain, every ideal of  $R(+M)$  is homogeneous if and only if  $M$  is a divisible  $R$ -module.

In Section 4 we study ring-theoretic constructions and properties of  $R(+M)$ , especially how properties for  $R$  and  $M$  relate to properties for  $R(+M)$ . For example, we determine when  $R(+M)$  is Noetherian, Artinian, or a principal ideal ring. We show that  $(R(+M))[X]$  is naturally isomorphic to  $R[X](+M[X])$  with similar results for related ring extensions. Let  $(R, \mathcal{M})$  be a local ring and  $M$  a finitely generated  $R$ -module. So  $R(+M)$  is a local ring with maximal ideal  $\mathcal{M}(+M)$ . We show that

the  $\mathcal{M}(+)$   $M$ -adic completion  $\widehat{R(+)}\widehat{M}$  of  $R(+)$   $M$  is naturally isomorphic to  $\widehat{R}(+)\widehat{M}$  where  $\widehat{R}$ , respectively,  $\widehat{M}$ , is the  $\mathcal{M}$ -adic completion of  $R$ , respectively  $M$ . Also,  $G(R(+)) = \min\{G(R), G(M)\}$  where  $G(\ )$  is the grade (or depth). We end this section with results on chained rings, valuation rings and Prüfer rings.

In Section 5 we study divisibility and factorization in commutative rings with zero divisors and in modules. Some of the topics covered include the notion of associates and irreducible elements, atomic rings, bounded factorization rings and finite factorization rings. We give a number of examples using idealization and discuss using idealization to reduce questions concerning factorization in modules to factorization in commutative rings.

In Section 6 we cover a wide range of topics involving idealization and give some examples (or counterexamples) using idealization. Some of the topics are Buchsbaum, Cohen-Macaulay, and Gorenstein rings, homological dimension, multiplication modules, and Boolean-like rings.

Our notation and terminology are standard and will be introduced as needed. Two general references are Gilmer [28] and Kaplansky [40]. An excellent introduction to idealization and commutative rings with zero divisors is Huckaba [37]. A number of results on idealization are taken from [37], especially some of the material on Prüfer rings. The interested reader may consult [37] for the original sources, usually [35, 36]. One should not take the lack of a reference to mean that a result is new. In fact, many results are folklore. However, we believe that the treatment of  $R(+)$   $M$  as a graded ring is new.

**2. Ringification.** Let  $R$  be a commutative ring and  $M$  an  $R$ -module. The idealization  $R(+)$   $M$  is a commutative  $R$ -algebra containing an isomorphic copy of  $M$ . In fact, idealization induces a functor from the category  ${}_R\mathcal{M}$  of  $R$ -modules to the category  ${}_R\text{Alg}$  of  $R$ -algebras. In this section we discuss various ways of putting  $M$  inside a commutative  $R$ -algebra, that is, a ringification of  $M$ . Besides the idealization  $R(+)$   $M$  we could also use the symmetric algebra  $S_R(M)$ . In some sense any ringification lies between these two with the symmetric algebra, respectively idealization, having the least, respectively most, relations. We also discuss some functorial and related properties of idealization.

Given a commutative ring  $R$  and an  $R$ -module  $M$ , we would like to put  $M$  inside a commutative  $R$ -algebra  $A$  with  $R \cap M = 0$ . So first take  $(A, +) = R \oplus M$ . Since we know how to multiply elements of  $R$  and elements of  $M$  by  $R$ , all that is left is to multiply elements of  $M$ . There are a number of ways to do this. Two such ways are to take  $M^2 = 0$  or to use tensor products. First, suppose we take  $M^2 = 0$ . Then for  $r_i \in R$  and  $m_i \in M$ ,  $(r_1, m_1)(r_2, m_2) = (r_1 + m_1)(r_2 + m_2) = r_1r_2 + r_1m_2 + m_1r_2 + m_1m_2 = r_1r_2 + r_1m_2 + r_2m_1 = (r_1r_2, r_1m_2 + r_2m_1)$  since  $M^2 = 0$  and multiplication is commutative. Here of course  $A$  is just the idealization.

Let  $R$  be a fixed commutative ring. Then idealization induces the functor  $I_R: {}_R\mathcal{M} \rightarrow {}_R\text{Alg}$  with  $I_R(M) = R(+M)$ . It is easily verified that if  $f: M \rightarrow N$  is an  $R$ -module homomorphism, then  $I_R(f): I_R(M) \rightarrow I_R(N)$  given by  $I_R(f)((r, m)) = (r, f(m))$  is an  $R$ -algebra homomorphism and that  $I_R$  is actually a functor. Another ringification functor is given by the symmetric algebra  $S_R(M) = T_R(M)/\langle\{m \otimes n - n \otimes m \mid m, n \in M\}\rangle$  where  $T_R(M)$  is the graded tensor  $R$ -algebra with  $T_R^n(M) = M^{\otimes n}$  and  $\langle\{m \otimes n - n \otimes m \mid m, n \in M\}\rangle$  is the homogeneous ideal of  $T_R(M)$  generated by  $\{m \otimes n - n \otimes m \mid m, n \in M\}$ . Hence  $S_R(M) = \bigoplus_{n=0}^{\infty} S_R^n(M)$  is a graded  $R$ -algebra with  $S_R^0(M) = R$  and  $S_R^1(M) = M$ . Again, if  $f: M \rightarrow N$  is an  $R$ -module homomorphism we get a (graded)  $R$ -algebra homomorphism  $S_R(f): S_R(M) \rightarrow S_R(N)$  and  $S_R: {}_R\mathcal{M} \rightarrow {}_R\text{Alg}$  is a functor. Observe that  $I_R(M)$  and  $S_R(M)/\bigoplus_{n \geq 2} S_R^n(M)$  are isomorphic as  $R$ -algebras (even graded  $R$ -algebras, see below). In fact, if we let  $\pi_M: S_R(M) \rightarrow I_R(M)$  be the natural map, then  $\pi: S_R \rightarrow I_R$  is a natural transformation. Also, note that  $I_R(M) = R(+M)$  has a natural grading with  $I_R^0(M) = R \oplus 0$ ,  $I_R^1(M) = 0 \oplus M$  and  $I_R^n(M) = 0$  for  $n \geq 2$  and with this grading all the homomorphisms discussed are graded. Of course, we could also give  $I_R(M)$  a  $\mathbf{Z}_2$ -grading. For either grading, the homogeneous ideals have the form  $J \oplus N$  where  $J$  is an ideal of  $R$  and  $N$  is a submodule of  $M$  with  $JM \subseteq N$ .

The next result, whose simple proof is shorter than its statement and hence is omitted, shows that the symmetric algebra, respectively idealization, is the ringification having the least, respectively most, relations.

**Theorem 2.1.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Let  $i_1: R \rightarrow S_R(M)$ ,  $j_1: M \rightarrow S_R(M)$  and  $i_2: R \rightarrow I_R(M)$ ,  $j_2: M \rightarrow I_R(M)$  be the natural maps. Let  $A$  be a commutative ring with a ring monomorphism  $i: R \rightarrow A$ , so that  $A$  is an  $R$ -algebra. Suppose there is an  $R$ -module monomorphism  $j: M \rightarrow A$  satisfying  $A = i(R) \oplus \langle j(M) \rangle$  where  $\langle j(M) \rangle$  is the subset of  $A$  generated by sums and products of elements of  $j(M)$  and with product  $i(r)j(m) = j(rm)$  (so that  $\langle j(M) \rangle$  is actually the ideal of  $A$  generated by  $j(M)$ ). Then there are unique  $R$ -algebra epimorphisms  $\theta: S_R(M) \rightarrow A$  and  $\psi: A \rightarrow I_R(M)$  with  $\theta i_1 = i$ ,  $\theta j_1 = j$  and  $\psi i = i_2$ ,  $\psi j = j_2$ . Moreover,  $\psi \theta = \pi_M$ , the natural map.*

If  $F$  is a free  $R$ -module on  $\{x_\alpha\}_{\alpha \in \Lambda}$ , then it is well known that  $S_R(F)$  is naturally isomorphic to the polynomial ring  $R[\{X_\alpha\}_{\alpha \in \Lambda}]$  where  $\{X_\alpha\}_{\alpha \in \Lambda}$  is a set of indeterminates over  $R$  in one-to-one correspondence with  $\{x_\alpha\}_{\alpha \in \Lambda}$ . For the idealization we have the related result:

**Proposition 2.2.** *Let  $R$  be a commutative ring and  $F$  a free  $R$ -module with basis  $\{x_\alpha\}_{\alpha \in \Lambda}$ . Let  $\{X_\alpha\}_{\alpha \in \Lambda}$  be a set of indeterminates over  $R$  in one-to-one correspondence with  $\{x_\alpha\}_{\alpha \in \Lambda}$ . Then  $R(+F)$  is naturally isomorphic to  $R[\{X_\alpha\}_{\alpha \in \Lambda}]/(\{X_\alpha\}_{\alpha \in \Lambda})^2$  via  $(r, \sum r_\alpha x_\alpha) \rightarrow r + \sum r_\alpha X_\alpha + (\{X_\alpha\}_{\alpha \in \Lambda})^2$ . Hence, if  $M$  is an  $R$ -module with generating set of cardinality  $|\Lambda|$ ,  $R(+M)$  is a homomorphic image of  $R[\{X_\alpha\}_{\alpha \in \Lambda}]/(\{X_\alpha\}_{\alpha \in \Lambda})^2$ . Thus, if  $R$  is Noetherian and  $M$  is a finitely generated  $R$ -module,  $R(+M)$  is Noetherian.*

*Proof.* It is easily checked that  $(r, \sum r_\alpha x_\alpha) \rightarrow r + \sum r_\alpha X_\alpha + (\{X_\alpha\}_{\alpha \in \Lambda})^2$  is an isomorphism. For the second statement, if  $\{g_\alpha\}_{\alpha \in \Lambda}$  generates  $M$ , then there is an  $R$ -module epimorphism  $f: F \rightarrow M$  induced by  $f(x_\alpha) = g_\alpha$  and hence an  $R$ -algebra epimorphism  $I_R(f): R(+F) \rightarrow R(+M)$ . The third statement follows from the Hilbert basis theorem.  $\square$

Proposition 2.2 has several generalizations. Let  $A \subseteq B$  be an extension of commutative rings and  $\{X_\alpha\}$  a set of indeterminates over  $B$ . Then  $A + (\{X_\alpha\})B[\{X_\alpha\}] = \{f \in B[\{X_\alpha\}] \mid f(0) \in A\}$  is a subring of  $B[\{X_\alpha\}]$ . It is easily checked that  $A(+B) \approx (A + XB[X])/X^2B[X]$ . This fact has been used by several authors to study  $A + XB[X]$ . More generally, let  $J$  be an ideal of  $B$ . Then  $A + XJB[X]$  is a subring of

$A + XB[X]$  and  $A(+)J \approx (A + XJB[X])/(XJB[X])^2$ . Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Taking  $A = R$ ,  $B = R(+)M$  and  $J = 0(+)M$  gives  $R(+)M \approx (A + XJB[X])/(XJB[X])^2$ ; so every idealization has this form. Also, if we take  $M$  to be a free  $B$ -module with basis  $\{x_\alpha\}$ , then  $A(+)M$  is naturally isomorphic to  $(A + (\{X_\alpha\})B[\{X_\alpha\}]) / ((\{X_\alpha\})B[\{X_\alpha\}])^2$  as in Proposition 2.2. In the above constructions we can use power series instead of polynomials.

If  $M$  and  $N$  are  $R$ -modules, it is well known that  $S_R(M \oplus N)$  is naturally isomorphic to  $S_R(M) \otimes_R S_R(N)$ . So the functor  $S_R: {}_R\mathcal{M} \rightarrow {}_R\text{Alg}$  converts sums to tensor products. This raises the question of whether the functor  $I_R: {}_R\mathcal{M} \rightarrow {}_R\text{Alg}$  also converts sums to tensor products. Observe that  $(I_R(R^n), +) \approx {}_R R^{n+1}$ . Hence  $(I_R(R^n \oplus R^m), +) = (I_R(R^{n+m}), +) \approx {}_R R^{n+m+1}$  while  $(I_R(R^n) \otimes_R I_R(R^m), +) \approx R^{n+1} \otimes_R R^{m+1} \approx {}_R R^{(n+1)(m+1)}$ . Hence  $I_R(R^n \oplus R^m)$  is not isomorphic to  $I_R(R^n) \otimes_R I_R(R^m)$  unless  $n = 0$  or  $m = 0$ . Likewise,  $I_R(R^n \oplus R^m) \not\approx I_R(R^n) \times I_R(R^m)$ ,  $I_R(R^n \otimes_R R^m) \not\approx I_R(R^n) \otimes_R I_R(R^m)$ , and  $I_R(R^n \otimes_R R^m) \not\approx I_R(R^n) \times I_R(R^m)$  (unless  $n = m = 0$  for the second case and  $m = 2, n = 3$  or  $m = 3, n = 2$  for the last case). We end this section with the result that  $R(+) (M \oplus N) = I_R(M \oplus N)$  is an iterated idealization.

**Proposition 2.3.** *Let  $R$  be a commutative ring and  $M$  and  $N$   $R$ -modules. Then  $R(+) (M \oplus N)$  is naturally isomorphic to  $(R(+)M)(+)N$  (or more accurately  $I_R(M \oplus N) \approx I_{I_R(M)}(N)$ ) where  $N$  is considered as an  $R(+)M = I_R(M)$ -module via  $(r, m)n = rn$ .*

*Proof.* Define  $\theta: R(+) (M \oplus N) \rightarrow (R(+)M)(+)N$  by  $\theta((r, (m, n))) = ((r, m), n)$ . Clearly  $\theta$  is an  $R$ -module isomorphism. Also,  $(r_1, (m_1, n_1)) \cdot (r_2, (m_2, n_2)) = (r_1 r_2, r_1(m_2, n_2) + r_2(m_1, n_1)) = (r_1 r_2, (r_1 m_2 + r_2 m_1, r_1 n_2 + r_2 n_1))$  and  $((r_1, m_1), n_1) \cdot ((r_2, m_2), n_2) = ((r_1, m_1), (r_2, m_2), (r_1, m_1)n_2 + (r_2, m_2)n_1) = ((r_1 r_2, r_1 m_2 + r_2 m_1), r_1 n_2 + r_2 n_1)$ . So  $\theta$  is an  $R$ -algebra isomorphism.  $\square$

**3. Ideals and distinguished elements of  $R(+)M$ .** Throughout this section  $R$  is a commutative ring with identity and  $M$  is an  $R$ -module. We determine the maximal, prime, and radical ideals of  $R(+)M$ , the homogeneous primary ideals of  $R(+)M$ , the saturated

multiplicatively closed subsets of  $R(+M)$ , and the units, idempotents, zero divisors, and nilpotents of  $R(+M)$ . We begin with the following result.

**Theorem 3.1.** *Let  $R$  be a commutative ring,  $I$  an ideal of  $R$ ,  $M$  an  $R$ -module and  $N$  a submodule of  $M$ . Then  $I(+N)$  is an ideal of  $R(+M)$  if and only if  $IM \subseteq N$ . When  $I(+N)$  is an ideal,  $M/N$  is an  $R/I$ -module and  $(R(+M))/(I(+N)) \approx (R/I)(+)(M/N)$ . In particular,  $(R(+M))/(0(+N)) \approx R(+)(M/N)$  and therefore  $(R(+M))/(0(+M)) \approx R$ . So the ideals of  $R(+M)$  containing  $0(+M)$  are of the form  $J(+M)$  for some ideal  $J$  of  $R$ .*

*Proof.* If  $I(+N)$  is an ideal,  $(R(+M))(I(+N)) = I(+)(IM + N)$  gives  $IM \subseteq N$ . Conversely, if  $IM \subseteq N$ ,  $M/N$  is an  $R/I$ -module and the map  $f: R(+M) \rightarrow (R/I)(+)(M/N)$  given by  $f((r, m)) = (r + I, m + N)$  is an epimorphism with  $\ker f = I(+N)$ . So  $I(+N)$  is an ideal of  $R(+M)$  and  $(R(+M))/(I(+N)) \approx (R/I)(+)(M/N)$ . The last statement follows from the Correspondence Theorem.  $\square$

The next result while an immediate corollary of Theorem 3.1 is important enough to be designated a theorem. Several parts of Theorem 3.2 come from [37, Theorem 25.1].

**Theorem 3.2.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module.*

(1) *The maximal ideals of  $R(+M)$  have the form  $\mathcal{M}(+M)$  where  $\mathcal{M}$  is a maximal ideal of  $R$ . So  $R(+M)$  is quasilocal if and only if  $R$  is quasilocal. Also,  $R(+M)$  and  $R$  have the same set of residue fields. The Jacobson radical of  $R(+M)$  is  $J(R(+M)) = J(R)(+M)$ .*

(2) *The prime ideals of  $R(+M)$  have the form  $P(+M)$  where  $P$  is a prime ideal of  $R$ . Hence if  $P$  is a prime ideal of  $R$ ,  $\text{ht}(P(+M)) = \text{ht } P$  and so  $\dim R(+M) = \dim R$ .*

(3) *Radical ideals of  $R(+M)$  have the form  $I(+M)$  where  $I$  is a radical ideal of  $R$ . If  $J$  is an ideal of  $R(+M)$ , then  $\sqrt{J} = \sqrt{I}(+M)$  where  $I = \{r \in R \mid (r, b) \in J \text{ for some } b \in M\}$  is an ideal of  $R$ . In particular, if  $I$  is an ideal of  $R$  and  $N$  is a submodule of  $M$ , then  $\sqrt{I(+N)} = \sqrt{I}(+M)$ ; hence  $\text{nil}(R(+M)) = \text{nil}(R)(+M)$ .*



*Proof.* Let  $A$  be a radical ideal of  $R(+M)$ . Then  $(0(+M))^2 = 0 \subseteq A$  and hence  $0(+M) \subseteq A$ . So by Theorem 3.1  $A = J(+M)$  for some ideal  $J$  of  $R$ . Also,  $(R(+M))/(J(+M)) \approx R/J$  gives that  $J$  is a radical ideal (respectively prime ideal, maximal ideal) if and only if  $J(+M)$  is. Note that  $J(R(+M)) = \cap\{\mathcal{M}(+M) \mid \mathcal{M} \text{ is a maximal ideal of } R\}$ . The remaining statements of (1) and (2) are obvious.

(3) Let  $J$  be an ideal of  $R(+M)$ . Then  $\sqrt{J} = K(+M)$  for some radical ideal  $K$  of  $R$ . Let  $I = \{r \in R \mid (r, b) \in J \text{ for some } b \in M\}$ , so  $I$  is easily seen to be an ideal of  $R$ . Let  $x \in \sqrt{I}$ , so some  $x^n \in I$ ; say  $(x^n, b) \in J$ . Then  $(x^n, b) \in \sqrt{J} = K(+M)$ . Hence  $x^n \in K$ , so  $x \in K$  since  $K$  is a radical ideal. Thus,  $\sqrt{I}(+M) \subseteq K(+M) = \sqrt{J}$ . For the reverse inclusion, let  $x \in K$ . Then  $(x, 0) \in \sqrt{J}$ ; so some  $(x^n, 0) \in J$ . Thus  $x^n \in I$  and hence  $x \in \sqrt{I}$ . So  $K(+M) \subseteq \sqrt{I}(+M)$ . The remaining statements of (3) are immediate.  $\square$

Let  $M$  be an  $R$ -module, and let  $\{m_\alpha\} \subseteq M$ . It is obvious that  $\langle\{m_\alpha\}\rangle = M$  if and only if  $\langle\{(0, m_\alpha)\}\rangle = 0(+M)$ . Thus,  $M$  is finitely generated as an  $R$ -module if and only if  $0(+M)$  is finitely generated as an ideal. If  $I$  is an ideal of  $R$ ,  $I(R(+M)) = I(+M)$ . Thus, if  $I$  is finitely generated, so is  $I(+M)$ . However,  $I(+M)$  can be finitely generated without  $IM$  being finitely generated.

Recall that a ring  $R$  is *graded* if  $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ , a direct sum of Abelian groups, with  $R_i R_j \subseteq R_{i+j}$ ; so each  $R_i$  is an  $R_0$ -module. An  $R$ -module  $M$  is *graded* if  $M = M_0 \oplus M_1 \oplus \cdots$  and  $R_i M_j \subseteq M_{i+j}$ ; so each  $M_i$  is an  $R_0$ -module. Elements of  $M_i$  are said to be *homogeneous of degree  $i$* . A submodule  $N$  of  $M$  is *homogeneous* if one of the following equivalent conditions holds: (1)  $N$  is generated by homogeneous elements, (2) if  $n_0 + n_1 + \cdots + n_i \in N$  where  $n_j$  is homogeneous of degree  $j$ , then each  $n_j \in N$  and (3)  $N = \bigoplus_{n=0}^{\infty} (N \cap M_n)$ . In Section 2 we remarked that  $R(+M)$  is a graded ring with  $(R(+M))_0 = R \oplus 0$ ,  $(R(+M))_1 = 0 \oplus M$ , and  $(R(+M))_n = 0$  for  $n \geq 2$ . So what are the homogeneous ideals of  $R(+M)$ ? Let  $J$  be a homogeneous ideal of  $R(+M)$ . Then  $J = (J \cap R) \oplus (J \cap M)$  where  $J \cap R$  is an ideal of  $R$  and  $J \cap M$  is a submodule of  $M$ ; that is,  $J = I(+N)$  where  $I$  is an ideal of  $R$  and  $N$  is a submodule of  $M$ . By Theorem 3.1,  $IM \subseteq N$ . Conversely, it is easily checked that an ideal of  $R(+M)$  of the form  $I(+N)$  is homogeneous.

However, contrary to [37, Theorem 25.1 (1)] an ideal of  $R(+)M$  need not have the form  $I(+)N$ , that is, need not be homogeneous. (Thus, when reading [37, Section 25] care must be taken since several proofs make this assumption. However, the results of Section 25 are true with the obvious exception of Theorem 25.1 (1).) For example, it is easily checked that  $\langle(2, 2)\rangle$  is not a homogeneous ideal of  $\mathbf{Z}(+)2\mathbf{Z}$ . We next collect some facts about homogeneous ideals of  $R(+)M$ . Recall that a ring  $R$  is *présimplifiable* if for  $x, y \in R$ ,  $xy = x$  implies  $x = 0$  or  $y$  is a unit. It is easy to see that an integral domain or quasilocal ring is *présimplifiable*.

**Theorem 3.3.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module.*

(1) *The homogeneous ideals of  $R(+)M$  have the form  $I(+)N$  where  $I$  is an ideal of  $R$ ,  $N$  is a submodule of  $M$ , and  $IM \subseteq N$ . If  $J$  is a homogeneous ideal, then  $J = I(+)N$  where  $I = \{r \in R \mid (r, b) \in J \text{ for some } b \in M\}$  and  $N = \{m \in M \mid (s, m) \in J \text{ for some } s \in R\}$ .*

(2) *Let  $I(+)N$  and  $I'(+)N'$  be two homogeneous ideals of  $R(+)M$ . Then  $(I(+)N) \cap (I'(+)N') = (I \cap I')(+) (N \cap N')$  and  $(I(+)N)(I'(+)N') = (II')(+) (IN' + I'N)$ .*

(3) *For a principal ideal  $\langle(a, b)\rangle$  of  $R(+)M$ , the following conditions are equivalent:*

- (a)  $\langle(a, b)\rangle$  is homogeneous,
- (b)  $\langle(a, b)\rangle = Ra(+) (Rb + aM)$ ,
- (c)  $(a, 0) \in \langle(a, b)\rangle$ , and
- (d) there exists  $x \in R$  such that  $xa = a$  and  $xb \in aM$ .

*In particular, if  $R$  is *présimplifiable* ( $xy = x \Rightarrow x = 0$  or  $y$  is a unit),  $\langle(a, b)\rangle$  is homogeneous if and only if  $a = 0$  or  $b \in aM$ .*

(4) *Every ideal of  $R(+)M$  is homogeneous if and only if every principal ideal of  $R(+)M$  is homogeneous. Hence, if  $R$  is *présimplifiable*, every ideal of  $R(+)M$  is homogeneous if and only if  $M = aM$  for each nonzero  $a \in R$ . Hence, if  $R$  is an integral domain, every ideal of  $R(+)M$  is homogeneous if and only if  $M$  is divisible. If  $R$  is *présimplifiable* but not an integral domain, every ideal of  $R(+)M$  is homogeneous if and only if  $M = 0$ .*

(5) Suppose that  $M$  is a finitely generated  $R$ -module. Then every ideal of  $R(+)M$  is homogeneous if and only if for each nonzero  $a \in R$ , there exists an  $x_a \in R$  with  $x_a a = a$  and  $x_a M = aM$ .

*Proof.* (1) We have already shown that the homogeneous ideals have the form  $I(+)N$  where  $IM \subseteq N$ . The second statement is obvious.

(2) This is easily checked. (This is [37, Theorem 25.1 (2)].) Of course, this result holds more generally for arbitrary such intersections.

(3) The equivalence of (a) and (b) follows from (1).

If  $\langle(a, b)\rangle$  is homogeneous, certainly  $(a, 0) \in \langle(a, b)\rangle$  and if  $(a, 0) \in \langle(a, b)\rangle$ , we have  $(0, b) \in \langle(a, b)\rangle$  so  $\langle(a, b)\rangle$  is generated by homogeneous elements and hence is homogeneous. So (a) and (c) are equivalent. (c)  $\Leftrightarrow$  (d)  $(a, 0) \in \langle(a, b)\rangle \Leftrightarrow$  there exists  $(x, n) \in R(+)M$  with  $(x, n)(a, b) = (a, 0) \Leftrightarrow xa = a$  and  $xb = -an \in aM$ .

Suppose that  $R$  is présimplifiable. Now when  $a = 0$ ,  $\langle(0, b)\rangle$  is homogeneous and we can take  $x = 0$ . So we can assume that  $a \neq 0$ . Suppose  $xa = a$  and  $xb = -an$ . Now  $xa = a$  gives  $x$  is a unit so  $b = -ax^{-1}n \in aM$ . For the converse, just take  $x = 1$ .

(4) The first statement is clear. Suppose that  $R$  is présimplifiable. By (3) every ideal of  $R(+)M$  is homogeneous if and only if  $M = aM$  for each nonzero  $a \in R$ . So if  $R$  is a domain, this is just  $M$  is divisible. However, if  $R$  has proper zero divisors  $rs = 0$  where  $r, s \neq 0$ , then  $0 = 0M = rsM = r(sM) = rM = M$ .

(5) ( $\Leftarrow$ ). This follows from (3) and does not require  $M$  to be finitely generated. ( $\Rightarrow$ ). Suppose that  $M$  is finitely generated, say  $M = Rm_1 + \cdots + Rm_n$ . Let  $0 \neq a \in R$ . For each  $1 \leq i \leq n$ , by (3) there exists an  $x_i$  with  $x_i a = a$  and  $x_i m_i \in aM$ . Take  $x_a = x_1 \cdots x_n$ ; so  $x_a a = a$  and  $x_a m_i \in aM$ , for each  $i$ . Then  $x_a M \subseteq aM = x_a aM$ , so  $x_a M = aM$ .  $\square$

**Corollary 3.4.** *Let  $R$  be an integral domain and  $M$  an  $R$ -module. Then the following conditions are equivalent:*

- (1) every ideal of  $R(+)M$  is comparable to  $0(+)M$ ,
- (2) every ideal of  $R(+)M$  has the form  $I(+)M$  or  $0(+)N$  for some ideal  $I$  of  $R$  or submodule  $N$  of  $M$ ,

- (3) every ideal of  $R(+)M$  is homogeneous,  
 (4)  $M$  is divisible.

*Proof.* (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) Clear.

(3)  $\Rightarrow$  (4) Theorem 3.3.

(4)  $\Rightarrow$  (1) By Theorem 3.3 every ideal of  $R(+)M$  has the form  $I(+)N$  where  $I$  is an ideal of  $R$ ,  $N$  is a submodule of  $M$  and  $IM \subseteq N$ . Suppose  $I \neq 0$ . Then  $M$  divisible gives  $IM = M$ , or  $M = IM \subseteq N$ . Alternatively, observe that if  $J$  is an ideal of  $R(+)M$  with  $J \not\subseteq 0(+)M$ , then  $J \supseteq 0(+)M$ . For let  $(a, b) \in J$  where  $a \neq 0$ . Let  $m \in M$ , so  $m = am'$  for some  $m' \in M$ . Then  $(0, m) = (a, b)(0, m') \in J$ .  $\square$

We next wish to determine when a homogeneous ideal  $I(+)N$  is primary, but to do so we need to find the zero divisors  $Z(R(+)M)$  of  $R(+)M$ .

**Theorem 3.5.** [37, Theorem 25.3] *Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Then  $Z(R(+)M) = \{(r, m) \mid r \in Z(R) \cup Z(M), m \in M\}$ . Hence  $S(+)M$  where  $S = R - (Z(R) \cup Z(M))$  is the set of regular elements (nonzero divisors) of  $R(+)M$ .*

*Proof.* Let  $r \in Z(R) \cup Z(M)$ . If  $r \in Z(R)$ , there exists a nonzero  $s \in R$  with  $rs = 0$ . So  $(r, 0)(s, 0) = (0, 0)$  and hence  $(r, 0) \in Z(R(+)M)$ . If  $r \in Z(M)$ , there exists a nonzero  $n \in M$  with  $rn = 0$ . So  $(r, 0)(0, n) = (0, 0)$  and hence  $(r, 0) \in Z(R(+)M)$ . Now for any  $m \in M$ ,  $(0, m) \in \text{nil}(R(+)M)$ , so  $(r, m) = (r, 0) + (0, m) \in Z(R(+)M)$ . (This follows since  $Z(R(+)M)$  is a union of prime ideals and  $\text{nil}(R(+)M)$  is contained in each prime ideal.) Conversely, suppose that  $(r, m) \in Z(R(+)M)$ . So  $(0, 0) = (r, m)(s, n) = (rs, rn + sm)$  for some  $(s, n) \neq (0, 0)$ . If  $s \neq 0$ , then  $rs = 0$  and so  $r \in Z(R)$ , while if  $s = 0$ , then  $n \neq 0$  and  $rn = 0$ , so  $r \in Z(M)$ . In either case  $r \in Z(R) \cup Z(M)$ .  $\square$

Alternatively, Theorem 3.5 also follows from Lemma 4.12 since  $Z_R(R(+)M) = Z_R(R \oplus M) = Z(R) \cup Z(M)$ .

While a prime ideal of  $R(+)M$  is homogeneous, a primary ideal need not be homogeneous. For example,  $\langle(\bar{2}, \bar{1})\rangle$  is a primary ideal of  $\mathbf{Z}_4(+) \mathbf{Z}_2$  (as  $\mathbf{Z}_4(+) \mathbf{Z}_2$  is 0-dimensional local), but  $\langle(\bar{2}, \bar{1})\rangle = \{(\bar{0}, \bar{0}), (\bar{2}, \bar{1})\}$  does not have the form  $I(+)N$ . Indeed, for  $R(+)M$  Noetherian, since every proper ideal has a primary decomposition, every primary ideal of  $R(+)M$  being homogeneous is equivalent to every ideal of  $R(+)M$  being homogeneous. Nagata [49, page 24] remarked that if  $N$  is a  $p$ -primary submodule of  $M$ , then for the  $p$ -primary ideal  $q = (N : M)$ ,  $q(+)N$  is  $p(+)M$ -primary. We generalize this result below.

**Theorem 3.6.** [37, Theorem 25.2] *Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Let  $I$  be an ideal of  $R$  and  $N$  a submodule of  $M$ . Then  $I(+)N$  is primary if and only if either*

(a)  $N = M$  and  $I$  is a primary ideal of  $R$  or

(b)  $N \subsetneq M$ ,  $IM \subseteq N$ , and  $I$  and  $N$  are  $P$ -primary where  $P = \sqrt{I}$ . In either case,  $I(+)N$  is  $\sqrt{I}(+)M$ -primary.

*Proof.* Suppose that  $N = M$ . Then by the Correspondence Theorem,  $I(+)M$  is primary if and only if  $I$  is primary. So assume that  $N \subsetneq M$ . For  $I(+)N$  to be an ideal of  $R(+)M$ , we must have  $IM \subseteq N$ . By passing to  $(R(+)M)/(I(+)N)$  we can assume that  $I = 0$  and  $N = 0$ . So we need to show that  $0(+)0$  is a primary ideal of  $R(+)M$  if and only if both  $0$  is a  $P$ -primary ideal of  $R$  and  $0$  is a  $P$ -primary submodule of  $M$  where  $P = \sqrt{0}$ . Now  $0(+)0$  is primary if and only if  $Z(R(+)M) = \text{nil}(R(+)M)$ , or equivalently by Theorem 3.5,  $(Z(R) \cup Z(M)) \oplus M = \sqrt{0}(+)M$ , or just  $Z(R) \cup Z(M) = \sqrt{0}$ . Since  $\sqrt{0}$  is the intersection of all the prime ideals of  $R$  and  $Z(R)$  and  $Z(M)$  are each a union of prime ideals of  $R$ , we have  $Z(R) \cup Z(M) = \sqrt{0}$  if and only if  $Z(R) = Z(M) = \sqrt{0} = P$ ; that is,  $0$  is a  $P$ -primary ideal of  $R$  and  $0$  is a  $P$ -primary submodule of  $M$ . The last statement follows since  $\sqrt{I(+)N} = \sqrt{I}(+)M$ .  $\square$

Suppose that  $R$  is a graded ring. For an ideal  $I$  of  $R$ , let  $I^*$  be the ideal generated by the homogeneous elements of  $I$ . So  $I^* \subseteq I$  and  $I^*$  is the largest homogeneous ideal of  $R$  contained in  $I$ . It is well known that if  $Q$  is a  $P$ -primary ideal of  $R$ , then  $P^*$  is a prime ideal of  $R$  and  $Q^*$

is  $P^*$ -primary. Now let  $R$  be a commutative ring and  $M$  an  $R$ -module. Suppose that  $Q$  is a  $P$ -primary ideal of  $R(+)$   $M$ . Now  $P = p(+)$   $M$  for some prime ideal  $p$  of  $R$  and  $P^* = P$ . Let  $Q^* = I(+)$   $N$ , so  $Q^*$  is  $P$ -primary. By the previous theorem,  $I$  is a  $p$ -primary ideal of  $R$  and either  $N = M$  or  $N$  is a  $p$ -primary submodule of  $M$ . If  $N = M$ , then  $Q = Q^* = I(+)$   $M$ . In either case,  $I = Q \cap R$  and  $N = Q \cap M$ .

The final distinguished elements we determine for  $R(+)$   $M$  are the units, cf. [37, Theorem 25.1 (6)] and idempotents.

**Theorem 3.7.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Then the units of  $R(+)$   $M$  are  $U(R(+)$   $M) = U(R)(+)$   $M$  and the idempotents of  $R(+)$   $M$  are  $Id(R(+)$   $M) = Id(R)(+)$   $0$ .*

*Proof.* Suppose that  $(r, m) \in U(R(+)$   $M)$ . So there exists an  $(s, n)$  with  $(r, m)(s, n) = (1, 0)$ . Hence  $rs = 1$ , so  $r \in U(R)$ . Conversely, suppose that  $r \in R$  is a unit, say  $rs = 1$ . Then  $(r, 0)(s, 0) = (1, 0)$  so  $(r, 0)$  is a unit. For any  $m \in M$ ,  $(0, m)$  is nilpotent and hence  $(r, m) = (r, 0) + (0, m)$  is a unit.

Certainly if  $e \in R$  is idempotent,  $(e, 0)$  is idempotent. Conversely, suppose that  $(r, m) \in R(+)$   $M$  is idempotent. Then  $(r, m) = (r, m)^2 = (r^2, 2rm)$ . So  $r = r^2$  is idempotent. Also,  $m = 2rm$ , so  $rm = 2r^2m = 2rm$  and hence  $rm = 0$  so  $m = 2rm = 0$ . (This is a special case of the more general result that any idempotent in a graded ring is homogeneous of degree 0.)  $\square$

The saturated multiplicatively closed subsets of  $R(+)$   $M$  are easy to determine.

**Theorem 3.8.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module.*

(1) *There is a one-to-one correspondence between the saturated multiplicatively closed subsets of  $R$  and those of  $R(+)$   $M$  given by  $S \leftrightarrow S(+)$   $M$ .*

(2) *If  $S$  is a multiplicatively closed subset of  $R$  and  $N$  is a submodule of  $M$ , then  $\overline{S(+)$   $N}$  is a multiplicatively closed subset of  $R(+)$   $M$  with saturation  $\overline{\overline{S(+)$   $N}} = \overline{S(+)$   $M}$ .*

*Proof.* (1) Since saturated multiplicatively closed subsets are just complements of unions of prime ideals, the map  $R - \cup P_\alpha \leftrightarrow (R - \cup P_\alpha)(+)M = R(+)M - \cup(P_\alpha(+)M)$  gives the desired one-to-one correspondence.

(2) Let  $S$  be a multiplicatively closed subset of  $R$ , and let  $N$  be a submodule of  $M$ . It is easily checked that  $\overline{S(+)N}$  is a multiplicatively closed subset of  $R(+)M$ . Now  $\overline{S(+)N} = T(+)M$  for some saturated multiplicatively closed subset  $T$  of  $R$ . Then  $S \subseteq T$ , so  $\overline{S} \subseteq T$ . Hence  $S(+)N \subseteq \overline{S}(+)M \subseteq T(+)M = \overline{S(+)N}$  and  $\overline{S}(+)M$  is saturated multiplicatively closed, so  $S(+)N = \overline{S}(+)M$ .  $\square$

We end this section with a “regular” version of part of Theorem 3.3.

**Theorem 3.9.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Let  $S = R - (Z(R) \cup Z(M))$ . Then the following conditions are equivalent.*

(1) *Every regular ideal of  $R(+)M$  has the form  $I(+)M$  where  $I$  is an ideal of  $R$  with  $I \cap S \neq \emptyset$ .*

(2) *Every regular ideal of  $R(+)M$  is homogeneous.*

(3) *For each  $s \in S$  and  $m \in M$ ,  $\langle (s, m) \rangle$  is homogeneous.*

(4)  *$sM = M$  for all  $s \in S$ , or equivalently,  $M = M_S$ .*

*Hence if  $R(+)M$  is integrally closed, every regular ideal of  $R(+)M$  has the form given in (1).*

*Proof.* (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) Clear.

(3)  $\Rightarrow$  (4). Let  $s \in S$ . By Theorem 3.3, for  $m \in M$ , there exists  $x \in R$  (depending on  $m$ ) with  $xs = s$  and  $xm \in sM$ . Since  $s$  is a regular element of  $R$ ,  $x = 1$ . Hence,  $m \in sM$ . So,  $M = sM$ .

(4)  $\Rightarrow$  (1). Let  $J$  be a regular ideal of  $R(+)M$ . So  $(s, m) \in J$  for some  $s \in S$  and  $m \in M$ . By Theorem 3.3 (3) (with  $a = s$  and  $x = 1$ ),  $\langle (s, m) \rangle = Rs(+) (Rm + sM) = Rs(+)M$ . So  $0(+)M \subseteq J$  and hence  $J = I(+)M$  for some ideal  $I$  of  $R$  necessarily with  $I \cap S \neq \emptyset$ .

Suppose that  $R(+)M$  is integrally closed or more generally just root closed. Let  $m \in M$  and  $s \in S$ . Then  $(0, m/s) = (0, m)/(s, 0)$

is contained in the total quotient ring of  $R(+M)$ , see Theorem 4.1. Now  $(0, m/s)^2 = (0, 0)$ , so  $R(+M)$  root closed gives that  $(0, m/s) \in R(+M)$ . Hence  $M = M_S$ .  $\square$

**4. Some ring constructions and properties of  $R(+M)$ .** In this section we study some common ring constructions such as localization, adjunction of indeterminates, and completion as they apply to the idealization. In several of these cases the construction commutes with the idealization, for example,  $(R(+M)[X])$  is naturally isomorphic to  $R[X](+M[X])$ . We also determine when  $R(+M)$  has certain properties such as being Noetherian, Artinian or a principal ideal ring. The general theme is how properties of  $R$  and  $M$  relate to those of  $R(+M)$ . For example,  $R(+M)$  is Noetherian, respectively Artinian, if and only if  $R$  is Noetherian, respectively Artinian, and  $M$  is finitely generated. We first look at localization. The first three results are from [37].

**Theorem 4.1.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module.*

(1) *Let  $S$  be a multiplicatively closed subset of  $R$  and  $N$  a submodule of  $M$ . Then  $(R(+M))_{S(+N)}$  is naturally isomorphic to  $R_S(+M_S)$ . In the case where  $N = 0$ , the isomorphism is simply  $(r, m)/(s, 0) \rightarrow (r/s, m/s)$ .*

(2) *Let  $P$  be a prime ideal of  $R$ . Then  $(R(+M))_{P(+M)} \approx R_P(+M_P)$ .*

(3) *The total quotient ring  $T(R(+M))$  of  $R(+M)$  is naturally isomorphic to  $R_S(+M_S)$  where  $S = R - (Z(R) \cup Z(M))$ .*

*Proof.* (1) The map  $f: (R(+M))_{S(+N)} \rightarrow R_S(+M_S)$  given by  $f((r, m)/(s, n)) = (r/s, (sm - rn)/s^2)$  is the desired isomorphism. (To see why the map is defined this way, observe that  $(r, m)/(s, n) = (s, -n)(r, m)/(s, -n)(s, n) = (sr, sm - rn)/(s^2, 0)$ .)

(2) This follows immediately from (1) with  $S = R - P$  and  $N = M$ .

(3) Here if  $S = R - (Z(R) \cup Z(M))$ ,  $S(+M)$  is the set of regular elements of  $R(+M)$ , Theorem 3.5. So the total quotient ring of  $R(+M)$  is  $(R(+M))_{S(+M)}$ . The result follows from (1).  $\square$



Concerning Theorem 4.1 (1), note that for any submodule  $N$  of  $M$ ,  $S(+N)$  and  $S(+0)$  have the same saturation, so  $(R(+M))_{S(+N)}$  and  $(R(+M))_{S(+0)}$  are isomorphic, but the isomorphism depends on  $N$ .

We next determine the integral closure of  $R(+M)$  in  $T(R(+M))$ .

**Theorem 4.2.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Let  $S = R - (Z(R) \cup Z(M))$ . If  $R'$  is the integral closure of  $R$  in  $T(R)$ , then  $(R' \cap R_S)(+M_S)$  is the integral closure of  $R(+M)$  in  $T(R(+M))$ .*

*Proof.* We have  $R(+M) \subseteq (R' \cap R_S)(+M_S) \subseteq R_S(+M_S) = T(R(+M))$ . If  $r \in R' \cap R_S$ , then  $r$  is integral over  $R$ . It easily follows that  $(r, 0)$  is integral over  $R(+M)$ . If  $b \in M_S$ ,  $(0, b)^2 = (0, 0)$  and so  $(0, b)$  is integral over  $R(+M)$ . Hence,  $(r, b) = (r, 0) + (0, b)$  is integral over  $R(+M)$ , that is,  $(R' \cap R_S)(+M_S) \subseteq (R(+M))'$ . Conversely, suppose that  $(r, b) \in (R(+M))'$ . Since  $(0, b)^2 = (0, 0)$ ,  $(r, 0) \in (R(+M))'$ . It easily follows that  $r$  is integral over  $R$ , so  $r \in R' \cap R_S$ .  $\square$

**Corollary 4.3.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Let  $S = R - (Z(R) \cup Z(M))$ .*

(1) *If  $R$  is integrally closed, then  $R(+M_S)$  is the integral closure of  $R(+M)$  in  $T(R(+M))$ .*

(2) *If  $Z(M) \subseteq Z(R)$ , then  $R(+M_S)$  is integrally closed if and only if  $R$  is integrally closed.*

*Proof.* (1) Here  $R = R'$ , so  $R' \cap R_S = R$ .

(2) ( $\Leftarrow$ ). This follows from (1).

( $\Rightarrow$ ) Suppose that  $R(+M_S)$  is integrally closed. Note that  $Z(M) \subseteq Z(R)$  gives that  $T(R(+M_S)) = T(R)(+M_S)$ . Let  $r \in T(R)$  be integral over  $R$ . Then  $(r, 0) \in T(R(+M_S))$  is integral over  $R(+M_S)$ . Since  $R(+M_S)$  is integrally closed,  $r \in R$ . So  $R$  is integrally closed.

However, in general  $R(+M_S)$  integrally closed does not imply that  $R$  is integrally closed. Indeed [37, page 166], if  $R$  is a nonintegrally-closed ring and  $M = \bigoplus\{R/P \mid P \in \text{Spec}(R)\}$ , then  $R(+M)$  is its own total quotient ring and hence is integrally closed.

Suppose that  $R_1$  and  $R_2$  are commutative rings and  $M_i$  is an  $R_i$ -module,  $i = 1, 2$ . Then  $M_1 \times M_2$  is an  $R_1 \times R_2$ -module with action  $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$ . Conversely, let  $R = R_1 \times R_2$  and suppose that  $M$  is an  $R$ -module. Put  $M_1 = (R_1 \times 0)M$  and  $M_2 = (0 \times R_2)M$ . So  $M_i$  is an  $R_i$ -module and  $M$  is the internal direct sum of  $M_1$  and  $M_2$ , so  $M \approx M_1 \times M_2$ .

**Theorem 4.4.** *Let  $R_1$  and  $R_2$  be commutative rings, and let  $M_i$  be an  $R_i$ -module,  $i = 1, 2$ . Then  $(R_1 \times R_2)(+) (M_1 \times M_2) \approx (R_1(+)) M_1 \times (R_2(+)) M_2$ .*

*Proof.* It is easily checked that the map  $((r_1, r_2), (m_1, m_2)) \rightarrow ((r_1, m_1), (r_2, m_2))$  is an isomorphism.  $\square$

We have already remarked that  $R(+M)$  is a graded ring. We next show that if  $R$  is a graded ring and  $M$  is a graded  $R$ -module, then  $R(+M)$  has a natural grading. This is given in [49, Exercise 1, page 24].

**Theorem 4.5.** *Let  $R = R_0 \oplus R_1 \oplus \dots$  be a graded commutative ring and  $M = M_0 \oplus M_1 \oplus \dots$  a graded  $R$ -module. Then  $R(+M)$  is a graded ring with  $(R(+M))_n = R_n \oplus M_n$ .*

*Proof.* Additively  $R(+M) = (R_0 \oplus R_1 \oplus \dots) \oplus (M_0 \oplus M_1 \oplus \dots) = (R_0 \oplus M_0) \oplus (R_1 \oplus M_1) \oplus \dots = (R(+M))_0 \oplus (R(+M))_1 \oplus \dots$ . Observe that  $(R(+M))_i (R(+M))_j = (R_i \oplus M_i)(R_j \oplus M_j) = R_i R_j \oplus (R_i M_j + R_j M_i) \subseteq R_{i+j} \oplus M_{i+j} = (R(+M))_{i+j}$ .  $\square$

As a special case we have the polynomial ring over  $R(+M)$  and the related, but not graded, power series case.

**Corollary 4.6.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module.*

(1)  $(R(+M))[\{X_\alpha\}] \approx R[\{X_\alpha\}] (+) M[\{X_\alpha\}]$  for any set of indeterminates  $\{X_\alpha\}$  over  $R$ .

(2)  $(R(+M))[[\{X_\alpha\}]] \approx R[[\{X_\alpha\}]] (+) M[[\{X_\alpha\}]]$  for any set of power series indeterminates  $\{X_\alpha\}$  over  $R$ .

*Proof.* (1) and (2). The map  $f: (R(+)M)[\{X_\alpha\}] \rightarrow R[\{X_\alpha\}](+)M[\{X_\alpha\}]$  given by  $\sum (r_i, m_i)f_i \rightarrow (\sum r_i f_i, \sum m_i f_i)$  where  $f_i$  is a form of degree  $i$  in  $\{X_\alpha\}$  is the desired isomorphism. Note that  $f((R(+)M)[\{X_\alpha\}]) = R[\{X_\alpha\}](+)M[\{X_\alpha\}]$ .  $\square$

Recall that for  $f = a_0 + a_1X + \cdots + a_nX^n \in R[X]$ , the *content*  $A_f$  of  $f$  is the ideal  $(a_0, \dots, a_n)$  of  $R$ . The set  $N = \{f \in R[X] \mid A_f = R\}$  is a saturated multiplicatively closed subset of  $R[X]$ ; in fact  $N = R[X] - \cup \mathcal{M}[X]$  where the union runs over all maximal ideals  $\mathcal{M}$  of  $R$ . Then  $R(X) := R[X]_N$  and if  $M$  is an  $R$ -module,  $M(X) := M[X]_N$ . So  $M(X)$  is an  $R(X)$ -module. Observe that for  $f = (r_0, m_0) + \cdots + (r_n, m_n)X^n \in (R(+)M)[X]$ ,  $A_f = R(+)M$  if and only if  $(r_0, r_1, \dots, r_n) = R$ . So  $\{f \in (R(+)M)[X] \mid A_f = R(+)M\} = N(+)M[X]$ . While the next result is true for any set of indeterminates, we content ourselves with the one-variable case.

**Corollary 4.7.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Then  $(R(+)M)(X)$  is naturally isomorphic to  $R(X)(+)M(X)$ .*

*Proof.* Now  $(R(+)M)[X]$  is naturally isomorphic to  $R[X](+)M[X]$ . So

$$\begin{aligned} (R(+)M)(X) &= ((R(+)M)[X])_{N(+)M[X]} \\ &\approx (R[X](+)M[X])_{N(+)M[X]} \\ &\approx R[X]_N(+)M[X]_N = R(X)(+)M(X). \quad \square \end{aligned}$$

We next determine when  $R(+)M$  is Noetherian or Artinian.

**Theorem 4.8.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Then  $R(+)M$  is Noetherian, respectively Artinian, if and only if  $R$  is Noetherian, respectively Artinian, and  $M$  is finitely generated.*

*Proof.* Suppose that  $R(+)M$  is Noetherian. Then  $R$  being a homomorphic image of  $R(+)M$  is Noetherian. Now  $0(+)M$  is a finitely generated ideal of  $R(+)M$  since  $R(+)M$  is Noetherian. Observe that  $(0, m_1), \dots, (0, m_n)$  generate  $0(+)M$  as an ideal if and only if

$m_1, \dots, m_n$  generated  $M$  as an  $R$ -module. Hence  $M$  is a finitely generated  $R$ -module. If  $R(+M)$  is Artinian, then  $R$  being a homomorphic image of  $R(+M)$  is Artinian and since an Artinian ring is Noetherian,  $M$  is finitely generated.

Conversely, suppose that  $R$  is Noetherian and  $M$  is finitely generated. By Proposition 2.2,  $R(+M)$  is Noetherian. Alternatively, let  $P(+M)$  be a prime ideal of  $R(+M)$ . Since  $R$  is Noetherian,  $P$  is finitely generated. Then since  $M$  is a finitely generated  $R$ -module,  $P(+M)$  is a finitely generated ideal of  $R(+M)$ . Since every prime ideal of  $R(+M)$  is finitely generated, Cohen's theorem gives that  $R(+M)$  is Noetherian. Suppose that  $R$  is Artinian. Then  $R$  is Noetherian with  $\dim R = 0$ . So  $M$  finitely generated gives that  $R(+M)$  is Noetherian and we also have  $\dim R(+M) = \dim R = 0$ , Theorem 3.2; hence,  $R(+M)$  is Artinian.  $\square$

Recall that a ring  $R$  is a *generalized ZPI-ring*, respectively  $\pi$ -ring, if every proper ideal, respectively proper principal ideal, of  $R$  is a product of prime ideals. An integral domain which is a  $\pi$ -ring is called a  $\pi$ -domain. Of course an integral domain is a generalized ZPI-ring if and only if it is a Dedekind domain. It is well known (for example, see [28, Sections 39 and 46]) that  $R$  is a  $\pi$ -ring (respectively generalized ZPI-ring, principal ideal ring (PIR)) if and only if  $R$  is a finite direct product of the following types of rings: (1)  $\pi$ -domains (respectively Dedekind domains, PIDs) which are not fields, (2) special principal ideal rings (SPIRs), that is, a local principal ideal ring, not a field, whose maximal ideal is nilpotent, and (3) fields. We next characterize when  $R(+M)$  is a  $\pi$ -ring, a generalized ZPI-ring, or a PIR.

**Lemma 4.9.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Suppose that  $R(+M)$  is a  $\pi$ -ring (respectively generalized ZPI-ring, PIR). Then  $R$  is a  $\pi$ -ring (respectively generalized ZPI-ring, PIR). Hence  $R = R_1 \times \dots \times R_n$  where  $R_i$  is either (1) a  $\pi$ -domain (respectively Dedekind domain, PID) but not a field, (2) an SPIR, or (3) a field. Let  $M_i = (0 \times \dots \times 0 \times R_i \times 0 \times \dots \times 0)M$ , so  $M_i$  is an  $R_i$ -module and  $M = M_1 \times \dots \times M_n$ . If  $R_i$  is a domain or SPIR, but not a field, then  $M_i = 0$  while if  $R_i$  is a field,  $M_i = 0$  or  $M_i \approx R_i$  (that is,  $M_i$  is cyclic).*

*Conversely, if  $R = R_1 \times \dots \times R_n$  and  $M = M_1 \times \dots \times M_n$  are as above and  $R$  is a  $\pi$ -ring (respectively generalized ZPI-ring, PIR), then  $R(+M)$  is a  $\pi$ -ring (respectively generalized ZPI-ring, PIR).*

*Proof.* Suppose that  $R(+M)$  is a  $\pi$ -ring (respectively generalized ZPI-ring, PIR), then its homomorphic image is  $R$ . Write  $R = R_1 \times \cdots \times R_n$  and  $M_1 \times \cdots \times M_n$  as above. Then by Theorem 4.4,  $R(+M) \approx (R_1(+M_1)) \times \cdots \times (R_n(+M_n))$ , so  $R_i(+M_i)$  is a homomorphic image of  $R(+M)$ . Thus, we may assume that  $R(+M)$  is a  $\pi$ -ring (respectively generalized ZPI-ring, PIR) where either (1)  $R$  is a domain, and hence a  $\pi$ -domain, (respectively Dedekind domain or PID) but not a field, (2)  $R$  is an SPIR, or (3)  $R$  is a field. First, suppose that  $R$  is a domain, not a field. Then  $R(+M)$  is an indecomposable  $\pi$ -ring (respectively generalized ZPI-ring, PIR) with  $\dim R(+M) = \dim R \geq 1$ . Thus,  $R(+M)$  must be an integral domain. Hence,  $M = 0$ . Next, suppose that  $R$  is an SPIR. Let  $(\pi)$  be the maximal ideal of  $R$  and suppose  $\pi^n \neq 0$  but  $\pi^{n+1} = 0$ ,  $n \geq 1$ . Now  $R(+M)$  is an indecomposable  $\pi$ -ring (or generalized ZPI-ring or PIR) with  $\dim R(+M) = \dim R = 0$ . Hence,  $R(+M)$  is also an SPIR. Thus  $M$  is finitely generated, even cyclic. Now  $(\pi)(+M)$  is the maximal ideal of  $R(+M)$ . So  $0(+M)$  is a power of  $(\pi)(+M)$ , say  $0(+M) = ((\pi)(+M))^s = (\pi)^s(+M) \pi^{s-1}M$ . Then  $\pi^s = 0$  and  $M = \pi^{s-1}M$ . Since  $\pi^s = 0$ ,  $s \geq 2$ , and hence  $s-1 \geq 1$ . So Nakayama's lemma gives  $M = 0$ . Finally, suppose that  $R$  is a field. Then as in the SPIR case,  $R(+M)$  is an SPIR and hence  $M$  is cyclic. So  $M = 0$  or  $M \approx R$ .

Conversely, let  $R = R_1 \times \cdots \times R_n$  and  $M = M_1 \times \cdots \times M_n$ . Then  $R(+M) \approx (R_1(+M_1)) \times \cdots \times (R_n(+M_n))$ . Since a direct product of  $\pi$ -rings (respectively generalized ZPI-rings, PIR's) is again of the same type, we can assume that  $R_i$  is either a domain with  $\dim R_i > 0$ , an SPIR or a field. In the first case  $M_i = 0$  and so  $R_i(+M_i) = R_i$  is a  $\pi$ -domain (respectively Dedekind domain, PID). Next suppose that  $R_i$  is an SPIR. Then again  $M_i = 0$ , so  $R_i(+M_i) = R_i$  is an SPIR. Finally, suppose that  $R_i$  is a field. If  $M_i = 0$ ,  $R_i(+M_i) = R_i$  is a field, while if  $M_i \approx R_i$ ,  $R_i(+M_i) \approx R_i[X]/(X^2)$  (by Proposition 2.2) is an SPIR.  $\square$

**Theorem 4.10.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Then  $R(+M)$  is a  $\pi$ -ring (respectively generalized ZPI-ring, PIR) if and only if  $R$  is a  $\pi$ -ring (respectively generalized ZPI-ring, PIR) and  $M$  is cyclic with annihilator  $P_1 \cdots P_s$  where  $P_1, \dots, P_s$  are some idempotent maximal ideals of  $R$  (if  $s = 0$ ,  $\text{ann}(M) = R$ , that is,  $M = 0$ ).*

*Proof.* The proof amounts to translating Lemma 4.9 to a “coordinate-free” version. For  $R(+M)$  to be a  $\pi$ -ring (respectively generalized ZPI-ring, PIR),  $R$  must also be a  $\pi$ -ring (respectively generalized ZPI-ring, PIR). So write  $R = R_1 \times \cdots \times R_n$  and  $M = M_1 \times \cdots \times M_n$  where  $M_i$  is an  $R_i$ -module and  $R_i$  is either an integral domain with  $\dim R_i > 0$ , an SPIR, or a field. Note that an idempotent maximal ideal of  $R_1 \times \cdots \times R_n$  has the form  $R_1 \times \cdots \times R_{i-1} \times 0 \times R_{i+1} \times \cdots \times R_n$  where  $R_i$  is a field. Now  $M_i = 0$  unless  $R_i$  is a field and in this case  $M_i = 0$  or  $M_i = R_i$ : But this translates to  $M$  is cyclic and  $\text{ann}(M) = I_1 \times \cdots \times I_n$  where  $I_i = R_i$  unless  $R_i$  is a field and  $M_i = R_i$  in which case  $I_i = 0$ . But  $I_1 \times \cdots \times I_n$  has this form if and only if it is a product of idempotent maximal ideals.  $\square$

An alternative approach to Lemma 4.9 and Theorem 4.10 is to treat  $R(+M)$  as a graded ring and use results from [11]. We next give two results on local rings. While the first result is given for the local case, it is clearly true in more generality.

**Theorem 4.11.** *Let  $(R, \mathcal{M})$  be a local ring and  $M$  a finitely generated  $R$ -module. So  $R(+M)$  is a local ring with maximal ideal  $\mathcal{M}(+M)$ . Let  $\widehat{\phantom{x}}$  denote the  $\mathcal{M}$ -adic, respectively  $\mathcal{M}(+M)$ -adic, completion of  $R$  and  $M$ , respectively  $R(+M)$ . Then  $\widehat{R(+M)} \approx \widehat{R}(+)\widehat{M}$ .*

*Proof.* Note that  $(\mathcal{M}(+M))^n = \mathcal{M}^n(+M^{n-1})$ . So  $\widehat{R(+M)} = \varprojlim (R(+M)/(\mathcal{M}(+M))^n) = \varprojlim (R(+M)/(\mathcal{M}^n(+M^{n-1}))) \approx \varprojlim (R/\mathcal{M}^n)(+M/M^{n-1}) \approx (\varprojlim R/\mathcal{M}^n)(+)\varprojlim M/M^{n-1} = \widehat{R}(+)\widehat{M}$ .  $\square$

Let  $R$  be a commutative ring and  $M$  an  $R$ -module. The two ring homomorphisms  $R \rightarrow R(+M)$  ( $r \rightarrow (r, 0)$ ) and  $R(+M) \rightarrow R$  ( $(r, m) \rightarrow r$ ) induce functors  ${}_{R(+M)}\mathcal{M} \rightarrow {}_R\mathcal{M}$  and  ${}_R\mathcal{M} \rightarrow {}_{R(+M)}\mathcal{M}$  where the respective “scalar products” are  $ra := (r, 0)a$  and  $(r, m)a := ra$ . Observe that the map  ${}_R\mathcal{M} \rightarrow {}_{R(+M)}\mathcal{M} \rightarrow {}_R\mathcal{M}$  is the identity map. For if  $A$  is an  $R$ -module, the  $R(+M)$ -action is  $(r, m)a := ra$  and hence the induced  $R$ -action on  $A$  is  $ra := (r, 0)a = ra$ , the original

action. If  $A$  is an  $R$ -module or  $R(+)$   $M$ -module, we need to know how  $Z_R(A)$  and  $Z_{R(+)}M(A)$  relate.

**Lemma 4.12.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Let  $A$  be either an  $R$ -module or an  $R(+)$   $M$ -module, then with the ring-actions defined in the previous paragraph,*

$$Z_{R(+)}M(A) = Z_R(A)(+)M.$$

*Proof.* First, let  $A$  be an  $R(+)$   $M$ -module. So  $A$  is an  $R$ -module with  $ra := (r, 0)a$ . Let  $(r, m) \in Z_{R(+)}M(A)$ . Now since  $Z_{R(+)}M(A)$  is a union of prime ideals of  $R(+)$   $M$  and since  $0(+)$   $M$  is in each of these prime ideals,  $(r, 0) \in Z_{R(+)}M(A)$ . So there is a nonzero  $a \in A$  with  $(r, 0)a = 0$ . But then as an  $R$ -module,  $ra := (r, 0)a = 0$ , so  $r \in Z_R(A)$ . Hence,  $(r, m) \in Z_R(A)(+)M$ . For the reverse inclusion, let  $r \in Z_R(A)$ . So there is a nonzero  $a \in A$  with  $0 = ra := (r, 0)a$ . So  $(r, 0) \in Z_{R(+)}M(A)$ . Hence, as before,  $(r, m) \in Z_{R(+)}M(A)$  for any  $m \in M$ ; so  $Z_R(A)(+)M \subseteq Z_{R(+)}M(A)$ . So if  $A$  is an  $R(+)$   $M$ -module,  $Z_{R(+)}M(A) = Z_R(A)(+)M$ .

Next, let  $A$  be an  $R$ -module. Then  $A$  is an  $R(+)$   $M$ -module. But then considering  $A$  an  $R$ -module returns the original ring-action. So from the first paragraph  $Z_{R(+)}M(A) = Z_R(A)(+)M$ .  $\square$

Suppose that  $R$  is a Noetherian ring,  $I$  is an ideal of  $R$  and  $M$  is a finitely generated  $R$ -module with  $IM \neq M$ . Then  $x_1, \dots, x_n \in I$  is an  $R$ -sequence of  $I$  on  $M$  if  $x_i \notin Z(M/(x_1, \dots, x_{i-1})M)$  for  $i = 1, \dots, n$ . And  $x_1, \dots, x_n$  is a maximal  $R$ -sequence of  $I$  on  $M$  if  $I \subseteq Z(M/(x_1, \dots, x_n)M)$ . Now maximal  $R$ -sequences of  $I$  on  $M$  exist and any two have the same length (for example, see [40, Section 3.1]). This length is called the *grade of  $I$  on  $M$*  and denoted  $G(I, M)$ . If  $(R, \mathcal{M})$  is local, then  $G(R) = G(\mathcal{M}, R)$  and  $G(M) = G(\mathcal{M}, M)$ . A Noetherian ring  $R$  is *Cohen-Macaulay* if  $\text{ht } \mathcal{M} = G(\mathcal{M}, R)$  for each maximal ideal  $\mathcal{M}$  of  $R$ .

**Theorem 4.13.** *Let  $(R, \mathcal{M})$  be a local ring and  $M$  a finitely generated nonzero  $R$ -module. Then  $G_{R(+)}M(R(+))M = G_R(R \oplus M) = \min\{G(R), G(M)\}$ .*

*Proof.* Let  $a_1, \dots, a_n \in \mathcal{M}$ . We show that  $(a_1, 0), \dots, (a_n, 0)$  is a maximal  $R$ -sequence of  $\mathcal{M}(+)M$  on  $R(+)M$  if and only if  $a_1, \dots, a_n$  is a maximal  $R$ -sequence of  $\mathcal{M}$  on  $R \oplus M$ . By Lemma 4.12,  $Z_{R(+)M}((R(+)M)/\langle\langle a_1, 0 \rangle\rangle, \dots, \langle\langle a_i, 0 \rangle\rangle) = Z_R((R(+)M)/\langle\langle a_1, 0 \rangle\rangle, \dots, \langle\langle a_i, 0 \rangle\rangle)(+)M$ . But  $Z_R((R(+)M)/\langle\langle a_1, 0 \rangle\rangle, \dots, \langle\langle a_i, 0 \rangle\rangle) = Z_R((R(+)M)/\langle\langle a_1, \dots, a_i \rangle\rangle(+)M) = Z_R(R/(a_1, \dots, a_i)(+)M/(a_1, \dots, a_i)M) = Z_R(R/(a_1, \dots, a_i) \oplus M/(a_1, \dots, a_i)M) = Z_R(R/(a_1, \dots, a_i) \cup Z_R(M/(a_1, \dots, a_i)M))$  and also  $= Z_R((R \oplus M)/\langle\langle a_1, \dots, a_i \rangle\rangle \cdot (R \oplus M))$ . Hence  $(a_{i+1}, 0) \notin Z_{R(+)M}((R(+)M)/\langle\langle a_1, 0 \rangle\rangle, \dots, \langle\langle a_i, 0 \rangle\rangle) \Leftrightarrow a_{i+1} \notin Z_R(R/(a_1, \dots, a_i) \cup Z_R(M/(a_1, \dots, a_i)M)) = Z_R((R \oplus M)/\langle\langle a_1, \dots, a_i \rangle\rangle(R \oplus M))$ . Also,  $\mathcal{M}(+)M \subseteq Z_{R(+)M}((R(+)M)/\langle\langle a_1, 0 \rangle\rangle, \dots, \langle\langle a_n, 0 \rangle\rangle) \Leftrightarrow \mathcal{M} \subseteq Z_R((R \oplus M)/\langle\langle a_1, \dots, a_n \rangle\rangle(R \oplus M)) \Leftrightarrow \mathcal{M} \subseteq Z_R(R/(a_1, \dots, a_n) \cup Z_R(M/(a_1, \dots, a_n)M)) \Leftrightarrow \mathcal{M} \subseteq Z_R(R/(a_1, \dots, a_n))$  or  $\mathcal{M} \subseteq Z_R(M/(a_1, \dots, a_n))$ . Thus,  $(a_1, 0), \dots, (a_n, 0)$  is a maximal  $R$ -sequence of  $\mathcal{M}(+)M$  on  $R(+)M$  if and only if  $a_1, \dots, a_n$  is a maximal  $R$ -sequence of  $\mathcal{M}$  on  $R \oplus M$  if and only if  $a_1, \dots, a_n$  is a maximal  $R$ -sequence of  $\mathcal{M}$  on both  $R$  and  $M$  and is a maximal  $R$ -sequence on either  $R$  or  $M$ . The result follows.  $\square$

**Corollary 4.14.** *Let  $R$  be a local ring and  $M$  a finitely generated nonzero  $R$ -module. Then  $R(+)M$  is Cohen-Macaulay if and only if  $R$  is Cohen-Macaulay and  $G(M) \geq G(R)$ .*

*Proof.* We need  $\dim R(+)M = G(R(+)M)$ . Since  $\dim R = \dim R(+)M \geq G(R(+)M) = \min\{G(R), G(M)\}$  and  $\dim R \geq G(R)$ , we have the desired equality if and only if  $\dim R = G(R)$  and  $G(M) \geq G(R)$ .  $\square$

**Example 4.15.** Let  $(R, \mathcal{M})$  be an  $n$ -dimensional regular local ring with  $x_1, \dots, x_n$  a minimal basis for  $\mathcal{M}$ . Let  $M_i = R/(x_1, \dots, x_i)$ ,  $0 \leq i \leq n$  (for  $i = 0$ ,  $M_i = R$ ). Then  $\dim R(+)M_i = n$  and  $G(R(+)M_i) = n - i$ .

Arguably some of the most important work concerning non-Noetherian commutative rings with zero divisors has been the extension of valuation theory and the theory of Prüfer domains to commutative rings with zero divisors. For a detailed treatment of these topics, see [37, 42]. We first recall the following pertinent definitions and facts.



Let  $R$  be a subring of a ring  $T$ , and let  $P$  be a prime ideal of  $R$ . Then  $(R, P)$  is called a *valuation pair on  $T$*  (or just  $R$  is a *valuation ring on  $T$* ) if there is a surjective valuation  $v: T \rightarrow G \cup \{\infty\}$  ( $v(xy) = v(x) + v(y)$ ,  $v(x + y) \geq \min\{v(x), v(y)\}$ ,  $v(1) = 0$ , and  $v(0) = \infty$ ),  $G$  a totally ordered Abelian group, with  $R = \{x \in T \mid v(x) \geq 0\}$  and  $P = \{x \in T \mid v(x) > 0\}$ . This is equivalent to if  $x \in T - R$ , then there exists  $x' \in P$  with  $xx' \in R - P$ . A valuation ring  $R$  is called a (*Manis*) *valuation ring* if  $T = T(R)$ . Unlike the domain case,  $P$  need not be a maximal ideal of  $R$  and we may have  $0 \subsetneq v^{-1}(\infty)$ . If the map  $v$  is not assumed to be onto,  $R$  is called a *paravaluation ring*. Also,  $R$  is called a *Prüfer ring* if every finitely generated regular ideal of  $R$  is invertible. This is equivalent to every overring of  $R$  being integrally closed or to  $(R_{[\mathcal{M}]}, [\mathcal{M}]R_{[\mathcal{M}]})$  being a valuation pair for each maximal ideal  $\mathcal{M}$  of  $R$  where  $R_{[\mathcal{M}]} = \{z \in T(R) \mid sz \in R \text{ for some } s \in R - \mathcal{M}\}$  is the large quotient ring of  $R$  with respect to  $\mathcal{M}$  and  $[\mathcal{M}]R_{[\mathcal{M}]} = \{z \in T(R) \mid sz \in \mathcal{M} \text{ for some } s \in R - \mathcal{M}\}$ .

Now, in the domain case,  $V$  is a valuation domain if and only if the set of (principal) ideals of  $V$  is totally ordered by inclusion. A ring with this property is called a *chained ring*. Now a chained ring is a Manis valuation ring but not conversely. One of the many characterizations of a Prüfer domain is that its lattice of ideals is distributive. A ring with this property is called an *arithmetical ring*. So  $R$  is arithmetical if and only if  $R_{\mathcal{M}}$  is a chained ring for each maximal ideal  $\mathcal{M}$  of  $R$ . Now an arithmetical ring is Prüfer but not conversely. We next give the following theorem which characterizes when  $R(+M)$  is a valuation ring, Prüfer ring, chained ring, or arithmetical ring. The first two parts of the theorem are from [37], but the statement of (2) and its proof [43, 45] are somewhat different. For a result related to (3), see [55].

**Theorem 4.16.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Let  $S = R - (Z(R) \cup Z(M))$ .*

(1) [37, Theorem 25.13]  *$R(+M)$  is a Manis valuation ring if and only if  $R$  is a valuation ring on  $R_S$  and  $M = M_S$  (that is,  $sM = M$  for each  $s \in S$ ).*

(2) [37, Theorem 25.11]  *$R(+M)$  is a Prüfer ring if and only if for each finitely generated ideal  $I$  of  $R$  with  $I \cap S \neq \emptyset$ ,  $I$  is invertible, and  $M = M_S$ .*

(3)  $R(+M)$  is a chained ring if and only if either (a)  $R$  is a chained ring and  $M = 0$  or (b)  $R$  is a valuation domain and  $M$  is a nonzero divisible  $R$ -module whose (cyclic) submodules are totally ordered by inclusion.

(4)  $R(+M)$  is arithmetical if and only if  $R$  is an arithmetical ring,  $M$  is an arithmetical  $R$ -module (the lattice of submodules of  $M$  is distributive), and for each maximal ideal  $\mathcal{M}$  of  $R$  with  $M_{\mathcal{M}} \neq 0_{\mathcal{M}}$ ,  $R_{\mathcal{M}}$  is a (valuation) domain and  $M_{\mathcal{M}}$  is a divisible  $R_{\mathcal{M}}$ -module.

*Proof.* (1) Assume that  $(R(+M), P(+M))$  is a valuation ring on  $T(R(+M)) = R_S(+M_S)$ . Since  $R(+M)$  is integrally closed,  $M = M_S$ . If  $(x, m) \in R_S(+M_S) - R(+M)$ , then there exists  $(r, c) \in P(+M_S)$  such that  $(x, m)(r, c) \in R(+M_S) - P(+M_S)$ . Hence if  $x \in R_S - R$ , there is some  $r \in P$  with  $sr \in R - P$ . So  $(R, P)$  is a valuation pair of  $R_S$ . The argument is reversible.

(2) ( $\Rightarrow$ ). Suppose that  $R(+M)$  is Prüfer. Since  $R(+M)$  is integrally closed,  $M = M_S$ . Let  $I$  be a finitely generated ideal of  $R$  with  $I \cap S \neq \emptyset$ . Then  $I(+M)$  is a finitely generated regular ideal of  $R(+M)$ . (Let  $I = (i_1, \dots, i_n)$  where  $i_1 \in S$ . Then  $(i_1, 0) \in I(+M)$  is regular and  $i_1 \in S$  gives  $i_1M = M$ , so  $\langle (i_1, 0) \rangle = Ri_1(+M)$  and hence  $\langle (i_1, 0), \dots, (i_n, 0) \rangle = I(+M)$ .) So  $I(+M)$  is invertible. So there is an ideal  $J'$  of  $R(+M)$  with  $J'(I(+M)) = \langle (i_1, 0) \rangle$ . Since  $M = M_S$ , by Theorem 3.9,  $J' = J(+M)$  for some ideal  $J$  of  $R$ . So  $JI = Ri_1$ . Hence,  $I$  is invertible.

( $\Leftarrow$ ). Since  $M = M_S$ , by Theorem 3.9 every finitely generated regular ideal of  $R(+M)$  is homogeneous and hence has the form  $I(+M)$  where  $I$  is a finitely generated ideal of  $R$  with  $I \cap S \neq \emptyset$ . So by hypothesis  $I$  is invertible. Let  $s \in I \cap S$ , so  $Rs = IJ$  for some ideal  $J$  of  $R$ . Then  $(I(+M))(J(+M)) = IJ(+M) = Rs(+M)$  is a regular principal ideal, so  $I(+M)$  is invertible. Hence,  $R(+M)$  is a Prüfer ring.

(3) ( $\Rightarrow$ ). Suppose that  $R(+M)$  is a chained ring and  $M \neq 0$ . Then  $R \approx (R(+M))/(0(+M))$  is a chained ring and since the ideals of  $R(+M)$  are totally ordered, the submodules of  $M$  are totally ordered. Suppose that  $0 \neq a \in R$ . Then for any  $m \in M$ ,  $\langle (a, m) \rangle \supseteq 0(+M)$ . So every ideal of  $R(+M)$  is homogeneous. Since  $R$  is quasilocal and  $M \neq 0$ , Theorem 3.3 (4) gives that  $R$  is an integral domain and  $M$  is divisible.

( $\Leftarrow$ ). Clearly  $R(+)M$  is a chained ring if (a) holds. So suppose that  $R$  is a valuation domain and  $M$  is a nonzero divisible  $R$ -module whose (cyclic) submodules are totally ordered. By Corollary 3.4 the ideals of  $R(+)M$  are totally ordered.

(4) ( $\Rightarrow$ ). Suppose that  $R(+)M$  is arithmetical. As in (3)  $R$  and  $M$  are arithmetical. Let  $\mathcal{M}$  be a maximal ideal of  $R$  with  $M_{\mathcal{M}} \neq 0_{\mathcal{M}}$ . Then  $R_{\mathcal{M}}(+)M_{\mathcal{M}} \approx (R(+)M)_{\mathcal{M}(+)_{\mathcal{M}}}$  is a chained ring. By (3)  $R_{\mathcal{M}}$  is a valuation domain and  $M_{\mathcal{M}}$  is a divisible  $R_{\mathcal{M}}$ -module.

( $\Leftarrow$ ). It suffices to show that  $R(+)M$  is locally a chained ring. Let  $\mathcal{M}(+)M$  be a maximal ideal of  $R(+)M$  where  $\mathcal{M}$  is a maximal ideal of  $R$ . Now  $(R(+)M)_{\mathcal{M}(+)_{\mathcal{M}}} \approx R_{\mathcal{M}}(+)M_{\mathcal{M}}$ . Now  $R$  arithmetical gives that  $R_{\mathcal{M}}$  is a chained ring and  $M$  arithmetical gives that the  $R_{\mathcal{M}}$ -submodules of  $M_{\mathcal{M}}$  are totally ordered. If  $M_{\mathcal{M}} = 0_{\mathcal{M}}$ ,  $R_{\mathcal{M}}(+)M_{\mathcal{M}}$  is a chained ring, while if  $M_{\mathcal{M}} \neq 0_{\mathcal{M}}$ , then by hypothesis  $R_{\mathcal{M}}$  is a valuation domain and  $M_{\mathcal{M}}$  is  $R_{\mathcal{M}}$ -divisible. So by (3)  $R_{\mathcal{M}}(+)M_{\mathcal{M}}$  is a chained ring.  $\square$

Let  $T$  be a commutative ring,  $M$  a  $T$ -module,  $G$  a totally ordered Abelian group, and  $v: T \rightarrow G \cup \{\infty\}$  a paravaluation. Let  $R_v = \{x \in T \mid v(x) \geq 0\}$  and  $P_v = \{x \in T \mid v(x) > 0\}$ . Then  $v_M: T(+)M \rightarrow G \cup \{\infty\}$  given by  $v_M((t, m)) = v(t)$  is a paravaluation on  $T(+)M$

with  $\text{im } v = \text{im } v_M$  (so  $v_M$  is a valuation  $\Leftrightarrow v$  is a valuation),  $R_{v_M} = \{x \in T(+)M \mid v_M(x) \geq 0\} = R_v(+)M$ ,  $P_{v_M} = \{x \in T(+)M \mid v_M(x) > 0\} = P_v(+)M$ , and  $v_M^{-1}(\infty) = v^{-1}(\infty)(+)M$ . Conversely, suppose that  $w: T(+)M \rightarrow G \cup \{\infty\}$  is a paravaluation. Since  $(0, m)$  is nilpotent,  $w((0, m)) = \infty$ . Hence,  $w((t, m)) = w((t, 0))$ . Thus,  $w_T: T \rightarrow G \cup \{\infty\}$  given by  $w_T(t) = w((t, 0))$  is a paravaluation on  $T$  with  $(w_T)_M = w$ . These observations can be used to give an alternative proof of Theorem 4.16 (1).

We next give several examples concerning valuation rings and Prüfer rings involving idealization. Recall that a commutative ring  $R$  is a *Marot ring* if every regular ideal of  $R$  is generated by regular elements.

**Example 4.17.** [12, Example 3.5], [37, Examples 9, 10, pages 183–184] A Prüfer valuation ring that is not a Marot ring. Let  $v$  be the rank-one discrete valuation on  $\mathbf{Q}(X)$  given by  $v(f/g) = \deg g - \deg f$ ;

so  $Q[X^{-1}]_{(X^{-1})}$  is the valuation ring associated with  $v$ . Let  $p \in Q[X]$  be an irreducible polynomial of degree  $n > 1$ . Restricting  $v$  to  $Q[X][1/p]$  gives a rank-one discrete valuation with valuation ring  $D = \mathbf{Q}[X][1/p] \cap Q[X^{-1}]_{(X^{-1})}$ . Let  $A = \oplus\{\mathbf{Q}[X]/\mathcal{M} \mid \mathcal{M} \text{ is a maximal ideal of } \mathbf{Q}[X] \text{ with } \mathcal{M} \neq (p)\}$ . Let  $v_A$  be the valuation defined by  $v_A((t, m)) = v(t)$ . Then  $v_A$  is a rank-one discrete valuation on  $\mathbf{Q}[X][1/p](+)A$  with valuation ring  $D(+)A$ . The image of the regular elements under  $v_A$  is  $n\mathbf{Z}$ , so  $P_{v_A}$  is a regular prime ideal of  $D(+)A$  (since it contains  $(1/p, 0)$ ) that is not generated by regular elements. (The ideal generated by the regular elements of  $P_{v_A}$  is  $(1/p)(+)A$ , but  $(X^{n-1}/p, 0) \in P_{v_A}$ .) Hence,  $D(+)A$  is not a Marot ring. Now  $D$  is Dedekind, so  $P_{v_A} = P(+)A$  where  $P = D \cap (X^{-1})_{(X^{-1})}$  is invertible. Hence  $D(+)A$  is a rank-one discrete Prüfer valuation ring. Note that  $P_{v_A}$  is a nonprincipal invertible ideal. This example (and the next) answer in the negative the question of Griffin as to whether an invertible ideal in a Prüfer valuation ring must be principal.

**Example 4.18.** [12, Example 3.6], [37, Example 11, page 187] Let  $D$  be a Dedekind domain with maximal ideal  $\mathcal{M}$  that is not principal, but some power of  $\mathcal{M}$  is principal. Say  $m > 1$  is the least positive integer with  $\mathcal{M}^m = (t)$  principal. Let  $A = \oplus\{D/Q \mid Q \text{ is a maximal ideal of } D, Q \neq \mathcal{M}\}$  and  $R = D(+)A$ . Then  $\{t^n R\}_{n=0}^{\infty}$  is the set of ideals of  $R$  generated by regular elements. Let  $P = \mathcal{M}(+)A$ . Now  $P$  is the unique regular prime ideal of  $R$  and  $P^m = tR$ , so  $P$  is invertible but not principal. Now  $(R, P)$  is a Prüfer valuation ring that is not Marot. Also, while  $P$  is divisorial ( $P = (P^{-1})^{-1}$ ),  $P$  is not the intersection of the regular principal fractional ideals containing  $P$ . (See [10] for the import of a divisorial ideal not to be the intersection of the regular principal ideals containing it.)

**Example 4.19.** [27], [12, Example 4.2] A ring  $R_0$  in which every regularly generated ideal is invertible, but  $R_0$  is not a Prüfer ring and a ring  $R_1$  in which the intersection of two regular principal ideals is not generated by regular elements. Let  $D = K[X, Y]$  where  $K$  is a field and let  $A = \oplus\{D/\mathcal{M} \mid \mathcal{M} \text{ is a maximal ideal with } Y \notin \mathcal{M}\}$ . Let  $R_0 = D(+)A$ . The regular elements of  $R_0$  are  $\{(\alpha Y^m, a) \mid \alpha \in K - \{0\}, m \geq 0, a \in A\}$ . Here  $R_0$  is a rank-one discrete Manis valuation ring that is not a Prüfer ring [27] but every ideal of  $R_0$  generated by regular

elements is invertible. Let  $D_1 = K[X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3]$ , the subring of  $D$  of polynomials with no linear term. Now in  $D_1$  we have  $Y^2D_1 \cap Y^3D_1 = \{Y^5, Y^6, XY^4, XY^5, X^2Y^3, X^2Y^4, X^3Y^3\}D_1$ . Let  $R_1 = D_1(+)A$ , so  $R_1$  is a subring of  $R_0$ . Note that the regular elements of  $R_1$  are  $\{(\alpha Y^m, a) \mid \alpha \in K - \{0\}, m = 0 \text{ or } m \geq 2, a \in A\}$ . Now  $Y^2R_1 = Y^2D_1(+)A$  and  $Y^3R_1 = Y^3D_1(+)A$  are two regular principal ideals of  $R_1$ . However,  $Y^2R_1 \cap Y^3R_1 = (Y^2D_1 \cap Y^3D_1)(+)A$  cannot be generated by regular elements [12].

**Example 4.20.** [26], [37, Example 20, pages 193–194] A valuation ring  $V$  whose total quotient ring  $T(V)$  is chained, but  $V$  is not chained. Let  $T = \mathbf{Z}_{(2)}(+) \mathbf{Z}_{2^\infty}$ , so by Theorem 4.16  $T$  is a chained ring. Since  $Z(T) = (2)(+) \mathbf{Z}_{2^\infty}$ ,  $T$  is a total quotient ring. Let  $\omega$  be the 3-adic valuation on  $\mathbf{Q}$ . Define a valuation  $v$  on  $T$  by  $v((z, n)) = \omega(z)$ . Then  $V = (\mathbf{Z}_{(2)} \cap \mathbf{Z}_3)(+) \mathbf{Z}_{2^\infty}$  is the valuation ring for  $v$  on  $T(V) = T$ . Since  $V$  has two maximal ideals,  $V$  is not chained.

Let  $R$  be a commutative ring. Then  $R$  is *strongly Prüfer* if each finitely generated ideal  $I$  with  $\text{ann}(I) = 0$  is locally principal. Also,  $R$  is *additively regular* if for each  $z \in T(R)$ , there exists  $u \in R$  with  $z+u \in T(R)$  regular,  $R$  satisfies *Property A* if for each finitely generated ideal  $I$  of  $R$  consisting of zero divisors,  $\text{ann}(I) \neq 0$ , and  $R$  satisfies the *annihilator condition* (a.c.) if for  $a, b \in R$  there exists  $c \in R$  with  $\text{ann}(a, b) = \text{ann}(c)$ . For more on these last three properties and their role in the study of commutative rings with zero divisors, see [37].

**Example 4.21.** [37, Examples 5, 19], [36, 44] A nonreduced Prüfer ring (even a total quotient ring) that is not strongly Prüfer. Let  $K$  be an algebraically closed field, and let  $D = K[X, Y]$ . Let  $B = \bigoplus\{D/P \mid P \text{ is a nonzero principal prime of } D\}$ . So  $R = D(+)B$  is a total quotient ring and hence is Prüfer. However,  $R$  is not strongly Prüfer since  $R(Z)$  is not Prüfer. See [37, Example 19] for details. Here  $R$  satisfies (a.c.) and has minimum spectrum compact but does not satisfy Property A.

We remark that [12, Example 2.1] uses idealization to give an example of a total quotient ring with few zero divisors not satisfying Property A and that [37, Example 4], from [44], uses idealization to give an exam-

ple of a nonreduced ring with Property A whose minimum spectrum is compact, but in which (a.c.) does not hold. Also [37, Example 12], from [48], uses idealization to give an example of an additively regular ring which is not Marot. Idealization is used to give some examples [12, Examples 3.4, 3.16, 4.4] of sublattices of the lattice of ideals of a commutative ring that exhibit bad behavior (such as the sublattice of faithful ideals not being compactly generated).

**Example 4.22.** [59, Example 3] A quasilocal Manis valuation ring  $R_v$  with  $P_v$  not maximal and  $R_v$  is not  $v$ -closed ( $R_v$  is  $v$ -closed if  $R_v/v^{-1}(\infty)$  is a valuation domain). Let  $K$  be a field and  $S = K[X, Y]_{(X, Y)}$ . Let  $S^Y = S[1/Y]$  and  $M = \bigoplus_{s \in N} S^Y / sS^Y$  where  $N$  is the set of nonunits of  $S^Y$ . Let  $T = S^Y (+) M$ , so  $T$  is a total quotient ring. Let  $v: T \rightarrow \mathbf{Z} \cup \{\infty\}$  be given by  $v((r, m)) = \omega(r)$  where  $\omega$  is the  $Y$ -adic valuation on  $K(X, Y)$  restricted to  $S^Y$ ; so  $v$  is a valuation on  $T$ . Then  $R_v = S (+) M$  is quasilocal, but  $P_v = YS (+) M$  is not maximal. Also,  $R_v$  is not  $v$ -closed since  $v^{-1}(\infty) = 0 (+) M$ .

We end this section with some results from [60]. Consider the following classes of rings:  $\mathcal{VR}$  (Manis valuation rings),  $\mathcal{P}$  (Manis valuation Prüfer rings),  $\mathcal{Cl}$  ( $v$ -closed Manis valuation rings),  $\mathcal{Z}$  (Manis valuation rings with  $Z(R_v) = v^{-1}(\infty)$ ),  $\mathcal{Ch}$  (chained rings),  $\mathcal{RC}$  (rings whose regular principal ideals are totally ordered),  $\mathcal{M}$  (Marot Manis valuation rings), and  $\mathcal{S}$  (surjective Manis valuation rings, i.e., for each  $r \in R_v$  there exists a regular element  $s \in R_v$  with  $v(r) = v(s)$ ). We always have  $\mathcal{M} \cup \mathcal{Cl} \subseteq \mathcal{P}$ ,  $\mathcal{P} \cap \mathcal{S} = \mathcal{M}$ ,  $\mathcal{Ch} \subseteq \mathcal{Z} \subseteq \mathcal{Cl} \cap \mathcal{M}$ , and  $\mathcal{P} \cup \mathcal{S} \subseteq \mathcal{VR}$ . The next theorem shows that  $\mathcal{Ch} \subsetneq \mathcal{Z} \subsetneq \mathcal{Cl} \cup \mathcal{M}$ ,  $\mathcal{Cl} \not\subseteq \mathcal{M}$ ,  $\mathcal{M} \not\subseteq \mathcal{Cl}$ ,  $\mathcal{Cl} \cup \mathcal{M} \subsetneq \mathcal{P}$ ,  $\mathcal{S} \not\subseteq \mathcal{P}$ ,  $\mathcal{P} \not\subseteq \mathcal{S}$ , and  $\mathcal{P} \cup \mathcal{S} \subsetneq \mathcal{VR}$ ; moreover, the inequalities are realized by Manis valuation rings with arbitrary value groups.

**Theorem 4.23.** (1) [60, Theorem 5] *For each ordered abelian group  $G$ , there exists a Manis valuation ring  $R_v$  with value group  $G$  which satisfies exactly one of the following conditions:*

- (a)  $R_v$  is chained and it is not a domain;
- (b)  $Z(R_v) = v^{-1}(\infty)$  and  $R_v$  is not chained;
- (c)  $R_v$  is a  $v$ -closed Marot ring and  $Z(R_v) \neq v^{-1}(\infty)$ ;

- (d)  $R_v$  is surjective and  $R_v$  is not a Prüfer ring;
- (e)  $R_v$  is neither surjective nor Prüfer;
- (f)  $R_v$  is a Marot ring and  $R_v$  is not  $v$ -closed;
- (g)  $R_v$  is  $v$ -closed and  $R_v$  is not Marot; and
- (h)  $R_v$  is a Prüfer ring, neither Marot nor  $v$ -closed.

(2) [60, Theorem 6]. *The class  $\mathcal{RC}$  of rings whose regular principal ideals are comparable properly contains the class  $\mathcal{VR}$  of Manis valuation rings.*

*Proof.* (1) Parts (d)–(h) are done using idealization.

(d) We sketch the proof of (d). Let  $K$  be a field,  $F = K(Y)$ ,  $G$  a totally ordered Abelian group, and  $F(X^G)$  the quotient field of the semigroup ring  $F[X; G]$ . Let  $\delta: F[X; G] \rightarrow G$  be given by  $\delta(aX^g) = g$  and  $\delta(f)$  is the minimal degree of the monomials of  $f$ . So  $\delta$  extends to a valuation on  $F(X^G)$  with value group  $G$ . Let  $D = K[X; G][Y] \subseteq F(X^G)$ ; let  $D'$  be the localization of  $D$  at the maximal ideal  $(Y, \{X^g \mid g > 0\})$ . Put  $D_1 = \{f/X^g \mid f \in D', g \geq 0\}$ ,  $B = \bigoplus\{D_1/dD_1 \mid d \notin U(D_1)\}$  and  $T = D_1(+ )B$ . Then  $T$  is a total quotient ring and  $v: T \rightarrow G \cup \{\infty\}$  given by  $v((d, k)) = \delta(d)$  is a valuation. Then  $R_v = D'(+ )B$ . For each  $g \in G$  with  $g \geq 0$ ,  $(X^g, 0)$  is a regular element of  $R_v$ , so  $R_v$  is surjective with value group  $G$ . But  $R_v$  is not a Prüfer ring since  $P_v$  is not maximal  $((Y, 0) \notin P_v$  since  $v((Y, 0)) = 0$ , but  $(Y, 0)R_v + P_v \neq R_v$ ). The proofs of (e)–(h) are similar.

(2) Let  $(V, P)$  be a valuation domain with rank  $V \geq 2$ , and let  $P_1$  be a nonzero nonmaximal prime ideal of  $V$ . Let  $D = V + YP_1[Y]$ . Let  $K$  be the quotient field of  $V$ , and let  $B = \bigoplus\{K[Y]/\mathcal{M} \mid \mathcal{M} \text{ is a maximal ideal of } K[Y]\}$ . Then  $R = D(+ )B$  is the desired example. Note that  $(d, b)$  is a regular element of  $R \Leftrightarrow d \in V - \{0\}$ . It is easily checked that  $V$  a valuation domain gives that  $R \in \mathcal{RC}$ . The proof that  $R$  is not a Manis valuation ring is more involved, see [60] for details.  $\square$

We remark that [12, Example 4.3] also gives an example using idealization of a ring where regular principal ideals are comparable but that is not integrally closed and hence not a Manis valuation ring.

**5. Factorization in commutative rings and modules.** In this section we first review some of the basic definitions and results concerning factorization in integral domains and then show how they can be extended to commutative rings with zero divisors and to modules. We give a number of examples using idealization and discuss using idealization to reduce questions concerning factorization in modules to factorization in commutative rings.

Let  $D$  be an integral domain with quotient field  $K$ . Two elements  $a, b \in D$  are *associates*, written  $a \sim b$ , if  $a \mid b$  and  $b \mid a$ , or equivalently,  $Da = Db$  or  $b = ua$  for some unit  $u \in D$ . A nonzero nonunit  $a \in D$  is *irreducible* or an *atom* if  $a = bc$  ( $b, c \in D$ ) implies  $b$  or  $c$  is a unit of  $D$ , and  $D$  is *atomic* if every nonzero nonunit of  $D$  is a finite product of atoms. If  $D$  satisfies the ascending chain condition on principal ideals (ACCP), then  $D$  is atomic, but not conversely. A domain  $D$  is a *half-factorial domain* (HFD), (respectively *finite factorization domain* (FFD), *bounded factorization domain* (BFD)) if  $D$  is atomic and any two factorizations of a nonzero nonunit into atoms have the same length (respectively each nonzero nonunit has only a finite number of nonassociate divisors, for each nonzero nonunit  $x \in D$  there is a positive integer  $N(x)$  so that if  $x = x_1 \cdots x_n$ , a product of atoms (or of just nonunits), then  $n \leq N(x)$ ). We have  $\text{UFD} \Rightarrow \text{FFD} \Rightarrow \text{BFD} \Rightarrow \text{ACCP} \Rightarrow \text{atomic}$  and  $\text{UFD} \Rightarrow \text{HFD} \Rightarrow \text{BFD}$ , but there are no other implications. For an introduction to factorization in integral domains, the reader is referred to [7].

When studying factorization in commutative rings with zero divisors or modules one must decide what an irreducible element should be and there are several choices; each, while equivalent in the domain case, is different once zero divisors are allowed. The approach taken in [1, 14, 15] is via different “associate” relations. We outline that approach.

Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Two elements  $m, n \in M$  are *associates* ( $m \sim n$ ) (respectively *strong associates* ( $m \approx n$ ), *very strong associates* ( $m \cong n$ )) if  $Rm = Rn$  (respectively  $m = un$  for some  $u \in U(R)$ ,  $m \sim n$  and either  $m = n = 0$  or  $m = rn$  implies  $r \in U(R)$ ). Taking  $M = R$  gives the notions of “associates” in  $R$ . We say that  $M$  is *strongly associate* if for  $m, n \in M$ ,  $m \sim n \Rightarrow m \approx n$  and  $R$  is *strongly associate* if  $R$  is strongly associate as an  $R$ -module.  $M$  is  *$R$ -présimplifiable* if for  $r \in R$  and  $m \in M$ ,  $rm = m \Rightarrow r \in U(R)$  or  $m = 0$ . This generalizes the previous



definition of  $R$  being présimplifiable. It is not hard to show that  $M$  is  $R$ -présimplifiable  $\Leftrightarrow m \sim n$  (or  $m \approx n$ )  $\Rightarrow m \cong n$ . So  $M$   $R$ -présimplifiable  $\Rightarrow M$  is strongly associate. Strongly associate rings and modules are investigated in [8].

**Theorem 5.1.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module.*

(1)  $R(+)$   $M$  is présimplifiable  $\Leftrightarrow R$  and  $M$  are présimplifiable.

(2)  $R(+)$   $M$  strongly associate  $\Rightarrow R$  and  $M$  are strongly associate.

(3) Suppose that  $R$  is présimplifiable. Then  $R(+)$   $M$  is strongly associate  $\Leftrightarrow M$  is strongly associate. ( $R$  being présimplifiable is already strongly associate.)

(4) Let  $G$  be an Abelian group. Then  $G$  is présimplifiable (respectively strongly associate)  $\Leftrightarrow G$  is torsion-free ( $G = F \oplus T$  where  $F$  is torsion-free and  $T$  is torsion with  $4T = 0$  or  $6T = 0$ ). Hence,  $\mathbf{Z}(+)G$  is présimplifiable, respectively strongly associate,  $\Leftrightarrow G$  is présimplifiable, respectively strongly associate.

(5) Let  $p$  be a prime number. Every ideal of  $\mathbf{Z}(+)\mathbf{Z}_p$  can be generated by two elements.

*Proof.* (1) [15, Proposition 3.1]. (2), (3) [8, Theorem 14]. (4) The first equivalence is [8, Theorem 15]. The second equivalence follows from (1) and (2), respectively. (5) [8, Lemma 17].  $\square$

Now a PIR is strongly associate since it is a finite direct product of integral domains, even PIDs, and quasilocal rings, SPIRs, each of which is strongly associate. However, we have the following examples.

**Example 5.2.** [8, Example 18] Let  $p \geq 5$  be prime. Then every ideal of  $\mathbf{Z}(+)\mathbf{Z}_p$  is two-generated, but  $\mathbf{Z}(+)\mathbf{Z}_p$  is not strongly associate. This follows from (4) and (5) of Theorem 5.1.

**Example 5.3.** [14, Example 6.1], [8, Example 19] A ring  $R$  that is strongly associate but  $R[X]$  is not strongly associate. Let  $R = \mathbf{Z}_{(2)}(+)\mathbf{Z}_4$ . So  $R$  is a one-dimensional local ring and hence is présimplifiable and thus strongly associate. Let  $a = (0, 1)$  and  $f = (1, 0) + (2, 0)X$ . Then  $a \sim af$ , but  $a \not\approx af$ .

A nonunit  $a \in R$  is *irreducible* (respectively *strongly irreducible*, *very strongly irreducible*) if  $a = bc$  implies  $a \sim b$  or  $a \sim c$  (respectively  $a \approx b$  or  $a \approx c$ ,  $a \cong b$  or  $a \cong c$ ) and  $a$  is *m-irreducible* if  $Ra$  is a maximal element of the set of proper principal ideals of  $R$ . For a nonzero nonunit  $a \in R$ ,  $a$  very strongly irreducible  $\Rightarrow a$  is *m-irreducible*  $\Rightarrow a$  is strongly irreducible  $\Rightarrow a$  is irreducible, but none of these implications can be reversed. Also  $a$  is *prime*, respectively *weakly prime*, if  $a \mid bc \Rightarrow a \mid b$  or  $a \mid c$  ( $a \mid bc \neq 0 \Rightarrow a \mid b$  or  $a \mid c$ ). So  $a$  prime  $\Rightarrow a$  is weakly prime  $\Rightarrow a$  is irreducible.

In the case of an  $R$ -module  $M$ , we say that  $m \in M$  is *R-primitive* (respectively *strongly R-primitive*, *very strongly R-primitive*) if for  $a \in R$  and  $n \in M$ ,  $m = an \Rightarrow m \sim n$  (respectively  $m \approx n$ ,  $m \cong n$ ). And  $m$  is *R-superprimitive* if  $bm = an$  for  $a, b \in R$  implies  $a \mid b$ . Note that (1)  $m$  is *R-primitive*  $\Leftrightarrow Rm$  is a maximal cyclic  $R$ -submodule of  $M$ , (2)  $m$  *R-superprimitive*  $\Rightarrow m$  is very strongly *R-primitive*  $\Rightarrow m$  is strongly *R-primitive*  $\Rightarrow m$  is *R-primitive*, (3) if  $\text{ann}(m) = 0$ ,  $m$  *R-primitive*  $\Rightarrow m$  is very strongly *R-primitive*, and (4)  $m$  *R-superprimitive*  $\Rightarrow \text{ann}(m) = 0$ .

A commutative ring  $R$  is *atomic* if every (nonzero) nonunit of  $R$  is a product of irreducibles; there are similar definitions of *strongly atomic*, *very strongly atomic*, and *m-atomic*. As in the domain case, ACCP implies atomic. A ring  $R$  is a *half-factorial ring* (HFR) (respectively *bounded factorial ring* (BFR)) if  $R$  is atomic and any two factorizations of a nonzero nonunit into atoms have the same length (respectively for each nonzero nonunit  $x \in R$ , there is a natural number  $N(x)$  so that for any factorization  $x = x_1 \cdots x_n$ , where each  $x_i$  is a nonunit, we have  $n \leq N(x)$ ). And  $R$  is called a *finite factorization ring* (FFR) (respectively *weak finite factorization ring* (WFFR), *atomic idf-ring*) if each nonzero nonunit of  $R$  has only a finite number of factorizations up to order and associates (respectively every nonzero nonunit of  $R$  has only a finite number of nonassociate divisors,  $R$  is atomic and each nonzero element of  $R$  has at most a finite number of nonassociate irreducible divisors). Here  $\text{FFR} \Rightarrow \text{WFFR} \Rightarrow \text{atomic idf-ring}$  and all three are the same in the domain case. But  $\mathbf{Z}_{(2)} \times \mathbf{Z}_{(2)}$  is an atomic idf-ring that is not a WFFR and  $\mathbf{Z}_2 \times \mathbf{Z}_2$  is a WFFR that is not an FFR.

Now if a ring has a nontrivial idempotent  $e$ , then  $e = e^2$  shows that  $R$  is not an HFR, a BFR or an FFR; let alone a “unique factorization

ring.” To get around this trivial case of nonunique factorization, Fletcher [24, 25] introduced the notion of a  $U$ -decomposition. Let  $a \in R$  be a nonunit, possibly 0. By a *factorization of  $a$*  we mean  $a = a_1 \cdots a_n$  where each  $a_i$  is a nonunit. By an  $\alpha$ -*factorization*,  $\alpha \in \{\text{irreducible, strongly irreducible, } m\text{-irreducible, very strongly irreducible, weakly prime, prime}\}$ , of  $a$  we mean a factorization  $a = a_1 \cdots a_n$  where each  $a_i$  is  $\alpha$ . Recall from [24] that for  $a \in R$ ,  $U(a) = \{r \in R \mid \exists s \in R \text{ with } rsa = a\} = \{r \in R \mid r(a) = (a)\}$ . A  $U$ -*factorization of  $a$*  is a factorization  $a = a_1 \cdots a_n b_1 \cdots b_m$  where  $a_i \in U(b_1 \cdots b_m)$  for  $1 \leq i \leq n$  and  $b_i \notin U(b_1 \cdots \hat{b}_i \cdots b_m)$ , for  $1 \leq i \leq m$ . We denote this  $U$ -factorization by  $a = a_1 \cdots a_n [b_1 \cdots b_m]$  and call  $a_1, \dots, a_n$ , respectively  $b_1, \dots, b_m$ , the irrelevant, respectively relevant, factors. A  $U$ -factorization is called an  $\alpha$ - $U$ -*factorization* if each  $a_i, b_j$  is  $\alpha$ . An irreducible  $U$ -factorization is called a  $U$ -*decomposition*.

Using the  $U$ -factorization concept, we can give another generalization of atomic, HFD, et al. to commutative rings with zero divisors. The idea is to take these definitions for domains and apply them to the relevant factors of  $U$ -factorizations of nonzero nonunits. For example,  $R$  is  $U$ -*atomic* if for each nonzero nonunit  $a \in R$ ,  $a = a_1 \cdots a_n [b_1 \cdots b_m]$  where each  $b_i$  is irreducible, and  $R$  is a  $U$ -*HFR* if  $R$  is  $U$ -atomic and for any two  $U$ -factorizations with irreducible relevant factors  $a_1 \cdots a_n [b_1 \cdots b_m] = a = a'_1 \cdots a'_n [b'_1 \cdots b'_{m'}]$  of a nonzero nonunit  $a$ ,  $m = m'$ .

As previously mentioned, see [7] for a survey of factorization in integral domains. Reference [14] began the study of factorization in commutative rings and defined the various associate relations and types of irreducible elements. Factorization in modules is given in [15] while  $U$ -factorizations and the rings such as  $U$ -BFRs defined via  $U$ -factorizations are introduced in [1]. The study of  $U$ -factorizations is carried forward in [17, 18].

As expected, there is a close relationship between the factorization properties of an element  $m$  in an  $R$ -module  $M$  and the ring element  $(0, m)$  of  $R(+M)$ . We summarize several of these in the following theorem.

**Theorem 5.4.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module.*

(1) *Let  $m, n \in N$ . Then  $m \sim n$  (respectively  $m \approx n$ ,  $m \cong n$ ) in  $M \Leftrightarrow (0, m) \sim (0, n)$  (respectively  $(0, m) \approx (0, n)$ ,  $(0, m) \cong (0, n)$ ) in  $R(+)M$ .*

(2) *Let  $R$  be an integral domain and  $0 \neq m \in M$ . Then  $m$  is primitive (respectively strongly primitive, very strongly primitive) in  $M \Leftrightarrow (0, m)$  is irreducible (respectively strongly irreducible, very strongly irreducible) in  $R(+)M$ .*

(3) *Suppose that  $R$  has a nontrivial idempotent and  $M \neq 0$ . Then no element  $(0, m)$  of  $0(+)M$  is irreducible in  $R(+)M$ .*

(4) *Suppose that  $R$  is indecomposable and  $0 \neq m \in M$  is superprimitive. Then  $(0, m)$  is very strongly irreducible in  $R(+)M$ .*

*Proof.* (1) is [15, Proposition 3.1] whose proof there is left to the reader. (2) is [14, Proposition 5.1] (in different terminology) or [15, Theorem 3.3]. The proofs are straightforward. (3) and (4) are given in [15, Theorem 3.4]. For (3), note that if  $e \neq 0, 1$  is idempotent, then for  $m \in M$ ,  $(0, m) = (e, m)(1 - e, m)$ , but  $(0, m) \approx (e, m)$  and  $(0, m) \approx (1 - e, m)$ . So  $(0, m)$  is not irreducible.  $\square$

Thus, (2) above shows that in the case where  $R$  is an integral domain, our definitions of primitive, strongly primitive, and very strongly primitive seem to be appropriate. But (3) shows that in general there does not appear to be a reasonable definition of primitive so that  $m$  is primitive in  $M$  if and only if  $(0, m)$  is irreducible in  $R(+)M$ . As for global properties of  $R(+)M$ , we have the following. For simplicity we take  $R$  to be an integral domain. Some generalizations are mentioned after the theorem.

**Theorem 5.5.** *Let  $R$  be an integral domain and  $M$  an  $R$ -module.*

(1) *If  $R$  satisfies ACCP, then every ascending chain of principal ideals of  $R(+)M$  containing a principal ideal of the form  $\langle (a, m) \rangle$  where  $0 \neq a \in R$  stops.*

(2)  *$R(+)M$  satisfies ACCP  $\Leftrightarrow R$  satisfies ACCP and  $M$  satisfies ACC on cyclic submodules.*

(3)  *$R(+)M$  is a BFR  $\Leftrightarrow R$  is a BFD and  $M$  is a BF-module, i.e.,*

for each nonzero  $n \in M$ , there exists a natural number  $N(n)$  so that  $n = a_1 \cdots a_{s-1} n_s$  (each  $a_i$  a nonunit)  $\Rightarrow s \leq N(n)$ .

(4)  $R(+)$   $M$  is atomic if  $R$  satisfies ACCP and  $M$  satisfies MCC, i.e., every cyclic submodule of  $M$  is contained in a maximal (not necessarily proper) cyclic submodule.

(5)  $R(+)$   $M$  is a  $U$ -FFR or equivalently, a  $U$ -WFFR  $\Leftrightarrow$  (i)  $R$  is an FFD, (ii)  $M$  is a  $U$ -FF module, i.e., for every  $0 \neq m \in M$ , there are only finitely many reduced submodule factorizations such that  $Rm = Rd_1 \cdots d_n m_j$ , up to order and associates on the  $d_i$  and  $m_j$ s and (iii) for every nonzero nonunit  $d \in R$ , there are only finitely many distinct principal ideals  $\langle (d, m) \rangle$  in  $R(+)$   $M$ .

(6)  $R(+)$   $M$  is a  $U$ -BFR if and only if  $R$  is a BFD and  $M$  is a  $U$ -BF module, i.e., for every  $0 \neq m \in M$ , there exists a natural number  $N(m)$  so that if  $Rm = Rd_1 \cdots d_t \bar{m}$  where  $d_j \notin U(R)$ ,  $t > N(m)$ , and  $\bar{m} \in M$ ; then after cancellation and reordering of some of the  $d_j$ s we have  $Rm = Rd_1 \cdots d_s \bar{m}$  where  $s \leq N(m)$ .

(7) Let  $R$  be an integral domain with ACCP. Then  $R(+)$   $M$  is atomic  $\Leftrightarrow R(+)$   $M$  is  $U$ -atomic.

*Proof.* (1)–(4) are part of [14, Theorem 5.2]. (1) Suppose  $a \neq 0$  and  $\langle (a, n) \rangle \subsetneq \langle (b, m) \rangle$ . Then  $(a, n) = (b, m)(c, l)$  for some  $(c, l)$ . Now  $a = bc$  and  $c$  cannot be a unit for this gives that  $(c, l) \in U(R(+)$   $M)$ . So  $Ra \subsetneq Rb$ . Thus, if  $R$  has ACCP, every ascending chain of principal ideals of  $R(+)$   $M$  containing a principal ideal of the form  $\langle (a, n) \rangle$  where  $a \neq 0$  stops.

(2) ( $\Rightarrow$ ). If  $R(+)$   $M$  satisfies ACCP, then  $R(+)$   $M$  satisfies ACCP on ideals of the form  $\langle (a_1, 0) \rangle \subseteq \langle (a_2, 0) \rangle \subseteq \cdots$  and  $\langle (0, n_1) \rangle \subseteq \langle (0, n_2) \rangle \subseteq \cdots$ . Thus,  $R$  satisfies ACCP and  $M$  satisfies ACC on cyclic submodules.

( $\Leftarrow$ ). Let  $\langle (a_1, n_1) \rangle \subseteq \langle (a_2, n_2) \rangle \subseteq \cdots$  be an ascending chain. If every  $a_i = 0$ , the chain gives rise to the chain  $Rn_1 \subseteq Rn_2 \subseteq \cdots$  which stops by ACC on cyclic submodules and hence the original chain in  $R(+)$   $M$  stops. If some  $a_i \neq 0$ , then (1) gives that the chain stops.

(3) ( $\Rightarrow$ ). Clear.

( $\Leftarrow$ ). Let  $(0, 0) \neq (a, n) \in R(+)$   $M$  be a nonunit and suppose we have a factorization into nonunits  $(a, n) = (a_1, n_1) \cdots (a_s, n_s)$ . If

$a = 0$ ,  $(0, n) = (a_1, n_1) \cdots (a_s, n_s)$  forces say  $a_s = 0$  and hence  $n = a_1 \cdots a_{s-1} n_s$  so  $s \leq N(n)$ . If  $a \neq 0$ ,  $a = a_1 \cdots a_s$  so  $s \leq N(a)$  since  $R$  is a BFD.

(4) Let  $(0, 0) \neq (a, n) \in R(+)M$  be a nonunit. Suppose  $a \neq 0$ . By (1),  $(a, n)$  is a product of irreducibles. Suppose  $a = 0$ . Then  $Rn \subseteq Rm$  where  $Rm$  is a maximal cyclic submodule (and so  $m$  is primitive) and  $n = cm$  for some nonzero  $c \in R$ . By Theorem 5.4 (2),  $(0, m)$  is irreducible. Since  $(0, n) = (c, 0)(0, m)$  and  $(c, 0)$  is either a unit or a product of irreducibles,  $(0, n)$  is a product of irreducibles.

(5) See [17, Theorem 4.2].

(6) See [17, Theorem 4.4].

(7) See [17, Theorem 4.6].  $\square$

Some of the parts of Theorem 5.5 admit generalizations to the case where  $R$  is not an integral domain. For example, (1) and (7) can be generalized to the case where  $R$  is présimplifiable [18, Lemma 3.14, Theorem 3.15], (5)  $(\Rightarrow)$  holds for  $R$  an FFR [18, Theorem 3.6], (3)  $(\Rightarrow)$  holds if  $R$  is présimplifiable [18, Theorem 3.7] and (6)  $(\Rightarrow)$  holds where  $R$  is a  $U$ -BFR and  $M$  is a  $U$ -BFM (with the obvious definition) [18, Lemma 3.8].

We previously defined a nonzero nonunit  $p \in R$  to be weakly prime if  $p \mid ab \neq 0 \Rightarrow p \mid a$  or  $p \mid b$ . More generally, call a proper ideal  $I$  weakly prime if  $0 \neq ab \in I \Rightarrow a \in I$  or  $b \in I$ . So a prime ideal is weakly prime, but if  $(R, \mathcal{M})$  is quasilocal with  $\mathcal{M}^2 = 0$ , every ideal of  $R$  is weakly prime. For a detailed study of weakly prime ideals and their application to factorization see [13] from where the next result is taken.

**Theorem 5.6.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Let  $I$  be a proper ideal of  $R$ . Then  $I' = I(+)M$  is weakly prime if and only if  $I$  is weakly prime and for  $a, b \in R$  with  $ab = 0$  but  $a \notin I$  and  $b \notin I$ ,  $a, b \in \text{ann}(M)$ .*

*Proof.* See [13, Theorem 17].  $\square$

We next use idealization to give some examples. While it is nontrivial to give examples of atomic integral domains that don't satisfy ACCP, the first example gives a one-dimensional quasilocal atomic ring  $R$  that doesn't satisfy ACCP. This ring is also an LCM ring, i.e., each two elements have an LCM. Now an atomic LCM-domain is a UFD, but  $R$  does not have unique factorization (see [14] for various notions of unique factorization rings).

**Example 5.7.** [14, Example 5.3] A one-dimensional quasilocal ring  $R$  that is atomic but does not satisfy ACCP. Also,  $R$  is an LCM ring, but not a unique factorization ring. Take  $R = \mathbf{Z}_{(2)} (+)(\mathbf{Z}_2 \oplus \mathbf{Z}_{2^\infty})$ . Here  $\mathbf{Z}_2 \oplus \mathbf{Z}_{2^\infty}$  does not satisfy ACC on cyclic submodules, so  $R$  does not satisfy ACCP. But  $\mathbf{Z}_2 \oplus \mathbf{Z}_{2^\infty}$  satisfies MCC so by Theorem 5.5 (4),  $R$  is atomic. For the second part see [14].

**Example 5.8.** [14, Example 5.5] A ring  $R$  which is not atomic but 0 and every regular element of  $R$  is a product of irreducible elements. Take  $R = \mathbf{Z} (+)(\mathbf{Z}_2 \oplus \mathbf{Q})$ . Now  $R$  is not atomic since no subgroup of  $\mathbf{Z}_2 \oplus \mathbf{Q}$  other than  $\langle (1, 0) \rangle$  is contained in a maximal cyclic subgroup. Now  $(0, (1, 0))$  is irreducible and hence  $(0, (0, 0)) = (0, (1, 0))^2$  is a product of irreducibles. By Theorem 5.5 (1) every element of the form  $(a, (b, c))$  where  $a \neq 0, 1$ , is a product of irreducibles.

**Example 5.9.** [14, Example 5.7] An irreducible element that is neither prime nor  $m$ -irreducible. Let  $R = \mathbf{Z} (+)(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ . Then  $(0, (0, 1))$  is irreducible since  $\langle (0, 1) \rangle$  is a maximal cyclic subgroup of  $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ , but  $R(0, (0, 1))$  is certainly not prime. Also,  $(0, (0, 1))$  is not  $m$ -irreducible since  $R(0, (0, 1)) \subsetneq R(3, (0, 0))$ .

Let  $R$  be a commutative ring. An element  $a \in R$  is  $U$ -bounded if there exists an  $N(a)$  so that for each  $U$ -factorization  $a = a_1 \cdots a_n [b_1 \cdots b_m]$ ,  $m \leq N(a)$ . So  $R$  is a  $U$ -BFR if each nonzero (nonunit) of  $R$  is  $U$ -bounded. If  $R$  is Noetherian, then  $R$  is a  $U$ -BFR and 0 is  $U$ -bounded [1, Theorem 4.17] and a decomposable  $U$ -BFR has 0  $U$ -bounded [1, Corollary 4.10].

**Example 5.10.** [1, Example 4.12] A quasilocal BFR (and hence an indecomposable  $U$ -BFR) in which  $0$  is not  $U$ -bounded. Let  $R = k[[X, Y]] (+) M$  where  $k$  is a field and  $M = \bigoplus \{k[[X, Y]]/P \mid P \text{ a ht-one prime of } k[[X, Y]]\}$ . Now  $\bigcap_{n=1}^{\infty} ((X, Y) (+) M)^n = 0$ , so  $R$  is a quasilocal BFR. But  $0$  is not  $U$ -bounded since  $(0, 0) = [(p_1, 0) \cdots (p_n, 0)(0, e_{p_1} + \cdots + e_{p_n})]$  for each  $n \geq 1$  where  $\{p_i\}$  is a countable set of nonassociate nonzero principal primes of  $k[[X, Y]]$  and  $e_{p_i} = 1_{k[[X, Y]]} + (p_i)$  in  $M$ .

For more results on factorization in modules including unique factorization and factorization in torsion-free modules, the reader is referred to [15] which also discusses the role of  $S_R(M)$  in factorization. We next consider another extension of factorization in integral domains, the factorization of regular elements.

Let  $R$  be a commutative ring and  $\text{reg}(R)$  the monoid of regular elements (nonzero-divisors) of  $R$ . One simplification in dealing with only the regular elements is that the three associate relations  $\sim$ ,  $\approx$  and  $\cong$  all agree for regular elements. Hence for a regular nonunit  $a \in R$ , the notions of irreducible, strongly irreducible, very strongly irreducible, and  $m$ -irreducible all coincide, so we simply use the term irreducible. We say that  $R$  is  $r$ -atomic if every regular nonunit of  $R$  is a product of irreducibles and that  $R$  satisfies  $r$ -ACCP if every ascending chain of regular principal ideals stabilizes. In a similar manner we can define  $r$ -UFR ( $r$ -unique factorization ring),  $r$ -HFR,  $r$ -FFR, and  $r$ -BFR.

The following theorem shows how idealization can be used to give examples of rings satisfying the various factorization properties for the regular elements.

**Theorem 5.11.** *Let  $R$  be an integral domain and  $M$  an  $R$ -module.*

- (1) *If  $R$  satisfies ACCP,  $R (+) M$  satisfies  $r$ -ACCP.*
- (2) *If  $R$  is a BFD,  $R (+) M$  is an  $r$ -BFR.*

*Suppose further that  $M = M_S$  where  $S = R - (Z(R) \cup Z(M)) = R - Z(M)$ .*

- (3) *If  $R$  is atomic,  $R (+) M$  is  $r$ -atomic.*
- (4) *If  $R$  is an HFD,  $R (+) M$  is an  $r$ -HFR.*
- (5) *If  $R$  is an FFD,  $R (+) M$  is an  $r$ -FFR.*



*Proof.* (1) This follows immediately from Theorem 5.5 (1).

(2) Suppose that  $R$  is a BFD. So for each nonzero nonunit  $a \in R$ , there exists a nonnegative integer  $N(a)$  with the property that if  $a = a_1 \cdots a_n$  where  $a_i$  is a nonunit, then  $n \leq N(a)$ . Suppose that  $(a, m)$  is a regular nonunit of  $R(+)M$  and  $(a, m) = (a_1, m_1) \cdots (a_n, m_n)$  where each  $(a_i, m_i)$  is a nonunit of  $R(+)M$ . Then  $a$  is a nonzero nonunit of  $R$  and  $a = a_1 \cdots a_n$  where each  $a_i$  is a nonunit of  $R$ ; so  $n \leq N(a)$ . Thus,  $R(+)M$  is an  $r$ -BFR.

Now suppose that  $M = M_S$  where  $S = R - Z(M)$ . Let  $b \in R - Z(M)$  and  $m \in M$ . Then  $m = bm'$  for some  $m' \in M$ , so  $(b, m) = (b, 0)(1, m')$ . Hence  $(b, 0) \sim (b, m)$  for each  $m \in M$ . So  $(b, m)$  is an atom  $\Leftrightarrow (b, 0)$  is an atom  $\Leftrightarrow b$  is an atom.

(3) Let  $(r, m)$  be a regular nonunit of  $R(+)M$ , so  $r \in R - Z(M)$  and  $r$  is a nonunit of  $R$ . Then  $r = r_1 \cdots r_n$  where each  $r_i$  is an atom of  $R$ . Now each  $(r_i, 0)$  is a regular atom of  $R(+)M$  and since  $M = M_S$ ,  $(r, m) = u(r_1, 0) \cdots (r_n, 0)$  where  $u$  is a unit of  $R(+)M$ . So  $R(+)M$  is  $r$ -atomic.

(4) By (3),  $R(+)M$  is  $r$ -atomic. Suppose that  $(r_1, m_1) \cdots (r_n, m_n) = (r'_1, m'_1) \cdots (r'_{n'}, m'_{n'})$  where each  $(r_i, m_i), (r'_i, m'_i)$  is a regular atom of  $R(+)M$ . Then  $r_1 \cdots r_n = r'_1 \cdots r'_{n'}$  where each  $r_i, r'_i$  is an atom of  $R$ . So  $n = n'$  since  $R$  is an HFD. Hence,  $R(+)M$  is an  $r$ -HFR.

(5) Suppose that  $R$  is an FFD. Let  $(d, m)$  be a regular nonunit of  $R(+)M$ . So  $d$  is a nonzero nonunit of  $R$ . Let  $d_1, \dots, d_n$  be a set of all nonassociate divisors of  $d$ . Suppose that  $(d', m')$  is a divisor of  $(d, m)$ . Then  $d'$  is a divisor of  $d$ . So  $d' \sim d_i$  for some  $i$ . Then  $M = M_S$  gives  $(d', m) \sim (d_i, 0)$ . So  $(d_1, 0), \dots, (d_n, 0)$  is a set of all nonassociate divisors of  $(d, m)$ . Hence  $R(+)M$  is an  $r$ -FFR.  $\square$

Much of the theory of factorization in integral domains has been generalized to cancelative monoids and hence can be applied to  $\text{reg}(R)$ . We end this section with examples concerning Krull rings and Krull monoids which use idealization. A commutative ring  $R$  is a *Krull ring* if  $R = \bigcap_{\alpha} (V_{\alpha}, P_{\alpha})$  where each  $(V_{\alpha}, P_{\alpha})$  is a rank-one discrete Manis valuation ring on  $T(R)$ , each  $P_{\alpha}$  is a regular prime ideal, and each regular element of  $R$  is a unit in almost all  $V_{\alpha}$ . Equivalently,  $R$  is a Krull ring if  $R$  is completely integrally closed and  $R$  has ACC on integral divisorial ideals. A cancelative monoid  $S$  is a *Krull monoid*

if there exists a family  $(v_i)_{i \in I}$  of discrete valuations on the quotient monoid  $\langle S \rangle$  of  $S$  (that is, each  $v_i: \langle S \rangle \rightarrow \mathbf{Z}$  is a group homomorphism) such that  $S = \bigcap V_i$  where  $V_i = \{x \in \langle S \rangle \mid v_i(x) \geq 0\}$  and for every  $x \in S$ , the set  $\{i \in I \mid v_i(x) > 0\}$  is finite. Here  $S$  is a Krull monoid if and only if  $S$  is completely integrally closed and  $S$  has ACC on integral divisorial ideals.

Now an integral domain  $R$  is a Krull domain if and only if  $(R - \{0\}, \cdot)$  is a Krull monoid. Unfortunately, the theorem we would like, namely  $R$  is a Krull ring if and only if  $\text{reg}(R)$  is a Krull monoid, is not true. It is easily seen that if  $R$  is a Krull ring, then  $\text{reg}(R)$  is a Krull monoid, and if  $R$  is a Marot ring and  $\text{reg}(R)$  is a Krull monoid, then  $R$  is a Krull ring [15, Theorem 5.1].

**Example 5.12.** [15, Example 5.2] A ring  $R$  with  $\text{reg}(R)$  a Krull monoid, even factorial, but  $R$  is not a Krull ring. Let  $D = K[X, Y]$ ,  $K$  a field, and  $A = \bigoplus \{D/\mathcal{M} \mid \mathcal{M} \text{ is a maximal ideal of } D, Y \notin \mathcal{M}\}$ . Put  $D_2 = K[Y, X^2, XY, X^3]$  and  $R_2 = D_2 (+) A$ . Since  $(X^2, 0) \in R_2$  but  $(X, 0) \notin R_2$ ,  $R_2$  is not (completely) integrally closed and hence  $R_2$  is not a Krull ring. However,  $\text{reg}(R_2) = \{(\alpha Y^m, a) \mid \alpha \in K - \{0\}, m \geq 0, a \in A\}$ . Since  $(\alpha Y^m, a) \sim (Y^m, 0)$ ,  $\text{reg}(R_2) \approx U(R_2) \times \mathbf{N}_0$  is a Krull monoid, even factorial.

In a manner analogous to Krull domains, we can define the divisor class group  $Cl(R)$  of a Krull ring  $R$  and the divisor class group  $Cl(S)$  of a Krull monoid  $S$ . There is a natural monomorphism  $\bar{\psi}: Cl(\text{reg}(R)) \rightarrow Cl(R)$  given by  $\bar{\psi}([A]) = [(RA)_v]$  where  $[A]$  represents the class of a divisorial ideal  $A$  of  $\text{reg}(R)$  and  $(RA)_v = ((RA)^{-1})^{-1}$ . If  $R$  is a Marot ring, the map  $\bar{\psi}$  is surjective [15, Theorem 5.3].

**Example 5.13.** [15, Example 5.4] A Krull ring  $R$  with  $Cl(\text{reg}(R)) \subsetneq Cl(R)$ . We take  $R$  to be the ring of Example 4.18. Let  $D$  be a Dedekind domain with maximal ideal  $\mathcal{M}$  that is not principal, but  $\mathcal{M}^m = (t)$  is principal,  $m$  minimal,  $A = \bigoplus \{D/Q \mid Q \text{ is a maximal ideal of } D, Q \neq \mathcal{M}\}$ , and  $R = D (+) A$ . Then  $(R, P)$ ,  $P = \mathcal{M} (+) A$ , is a rank-one discrete Manis valuation ring and hence a Krull ring. Here  $\{t^n R\}_{n=0}^\infty$  is the set of regular principal ideals of  $R$ , so  $\text{reg}(R) \approx U(R) \times (\mathbf{N}_0, +)$  and  $Cl(\text{reg}(R)) = 0$  since  $\text{reg}(R)$  is factorial. Now  $P$  is invertible but not principal, so  $Cl(R) \neq 0$ , in fact  $Cl(R) = \langle [P] \rangle \approx \mathbf{Z}_m$ .

**6. Miscellaneous topics.** In this section we cover a number of different topics involving idealization. The first topic involves generalizations of cyclic modules including finitely generated locally cyclic modules, multiplication modules and cancellation modules. In most cases idealization allows us to reduce to the case of ideals. We begin with the relevant definitions.

Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Then  $M$  is a *multiplication module* if for each submodule  $N$  of  $M$ ,  $N = IM$  for some ideal  $I$  of  $R$ . In this case we can take  $I = (M : N)$ . A multiplication module is locally cyclic and if  $M$  is finitely generated the converse is true. Multiplication modules and ideals have been extensively studied; see [5] for some references and a number of characterizations.

An  $R$ -module  $M$  is a (*weak*) *cancellation module* if for ideals  $I$  and  $J$  of  $R$ ,  $IM = JM$  implies  $I = J$  ( $I + (0 : M) = J + (0 : M)$ ) and  $M$  is a *restricted cancellation module* if  $IM = JM \neq 0$  implies  $I = J$ . Clearly a cancellation module  $M$  is a restricted cancellation module, a restricted cancellation module  $M$  is a weak cancellation module, and the notions coincide when  $M$  is faithful. Moreover,  $M$  is a restricted cancellation module if and only if  $M$  is a weak cancellation module and  $(0 : M)$  is comparable to each ideal of  $R$ . An ideal  $I$  is a cancellation ideal if and only if for each maximal ideal  $\mathcal{M}$  of  $R$ ,  $I_{\mathcal{M}}$  is a regular principal ideal of  $R_{\mathcal{M}}$ . A finitely generated locally cyclic module is a weak cancellation module. For results on cancellation modules and their generations along with additional references, the reader is referred to [6].

The following theorem shows that when studying these various generalizations of locally cyclic modules we can usually reduce to the ideal case via idealization.

**Theorem 6.1.** [6, Theorem 3.1] *Let  $R$  be a commutative ring,  $M$  an  $R$ -module and  $N$  a submodule of  $M$ .*

- (1)  *$N$  is a cyclic  $R$ -module  $\Leftrightarrow 0(+)N$  is a principal ideal of  $R(+)M$ .*
- (2)  *$N$  is a (finitely generated) locally cyclic  $R$ -module  $\Leftrightarrow 0(+)N$  is a (finitely generated) locally principal ideal of  $R(+)M$ .*
- (3)  *$N$  is a multiplication module  $\Leftrightarrow 0(+)N$  is a multiplication ideal of  $R(+)M$ .*

(4)  $N$  is a weak cancellation module  $\Leftrightarrow 0(+)$   $N$  is a weak cancellation ideal of  $R(+)$   $M$ .

(5)  $N$  is a cancellation module  $\Leftrightarrow 0(+)$   $N$  is a weak cancellation ideal of  $R(+)$   $M$  and  $(0 : (0(+)$   $N)) = 0(+)$   $M$ .

(6)  $0(+)$   $N$  is a restricted cancellation ideal of  $R(+)$   $M$  if and only if  $N$  is a restricted cancellation module and for  $r \in R$ ,  $rN \neq 0$  implies  $rM = M$ .

*Proof.* (1), (2). Just observe that  $0(+)$   $N$  is finitely generated, respectively cyclic, if and only if  $N$  is, and that for a maximal ideal  $\mathcal{M}$  of  $R$ ,  $(0(+)$   $N)_{\mathcal{M}(+)M} \approx 0_{\mathcal{M}(+)N_{\mathcal{M}}}$ .

(3) ( $\Rightarrow$ ). Let  $N$  be a multiplication submodule of  $M$ , and let  $N'$  be a submodule of  $N$ . Then  $N' = JN$  for some ideal  $J$  of  $R$  and hence  $0(+)$   $N' = (J(+)$   $M)(0(+)$   $N)$ . Since every ideal contained in  $0(+)$   $N$  has the form  $0(+)$   $N'$  for some submodule  $N'$  of  $N$ ,  $0(+)$   $N$  is a multiplication ideal.

( $\Leftarrow$ ). Suppose that  $0(+)$   $N$  is a multiplication ideal of  $R(+)$   $M$ , and let  $N'$  be a submodule of  $N$ . So  $0(+)$   $N' = J'(0(+)$   $N)$  for some ideal  $J'$  of  $R(+)$   $M$ . Since  $(0(+)$   $M)(0(+)$   $N') = 0(+)$   $0 \subseteq 0(+)$   $N'$ , we can assume that  $J' \supseteq 0(+)$   $M$ . So  $J' = J(+)$   $M$  for some ideal  $J$  of  $R$ . Then  $0(+)$   $N' = (J(+)$   $M)(0(+)$   $N) = 0(+)$   $JN$ . Hence,  $N' = JN$ . Thus  $N$  is a multiplication module.

(4) The proof of (4) involves techniques similar to the proof of (3), see [6] for details. Note that (5) follows from (4) since  $N$  is faithful if and only if  $(0 : (0(+)$   $N)) = 0(+)$   $M$ . The proof of (6) is more involved, see [6].  $\square$

We end this topic with an application and an example. For a finitely generated  $R$ -module  $M$ , let  $\mu(M)$  be the minimal number of elements necessary to generate  $M$ . The following result [4, Corollary 2] uses the fact that a module  $M$  is finitely generated locally cyclic if and only if  $0(+)$   $M$  is. Let  $I_1, \dots, I_{k-1}$  be finitely generated ideals of  $R$  and  $M$  a finitely generated  $R$ -module with at least  $k-1$  of  $I_1, \dots, I_{k-1}, M$  being locally cyclic. Then  $\mu(I_1 \cdots I_{k-1}M) \leq \mu(I_1) + \cdots + \mu(I_{k-1}) + \mu(M) - k + 1$ .

We next give an example of a weak cancellation ideal in a local ring that has a homomorphic image that is not a weak cancellation ideal.

**Example 6.2.** [6, Example 3.2] Let  $(R, \mathcal{M})$  be a local domain that is not a DVR or field, and let  $M = R \oplus \mathcal{M}$ . Hence  $M$  is a cancellation  $R$ -module. Then  $R(+)\mathcal{M}$  is a local ring with unique minimal prime ideal  $P = 0(+)\mathcal{M}$  and  $P^2 = 0$ . Since  $M$  is a cancellation  $R$ -module,  $P$  is a weak cancellation ideal of  $R(+)\mathcal{M}$ . Since  $\mathcal{M}$  is not a cancellation ideal of  $R$  (it is not principal),  $0 \oplus \mathcal{M}$  is not a weak cancellation submodule of  $M = R \oplus \mathcal{M}$ . So  $P/(0(+)(R \oplus 0)) = (0(+)(R \oplus \mathcal{M}))/((0(+)(R \oplus 0)) \approx 0(+)(0 \oplus \mathcal{M})$  is not a weak cancellation ideal of  $(R(+)\mathcal{M})/(0(+)(R \oplus 0)) \approx R(+)(0 \oplus \mathcal{M})$ .

We next discuss Boolean rings, von Neumann regular rings and their generalizations. This material is taken from [3] which the reader may consult for more details, references and a history of the topic. It is well known that Boolean rings, Boolean algebras, and complemented distributive lattices are essentially the same things. Thus, the duality for Boolean algebras can also be stated for Boolean rings. Less well known is that such a general duality theory, due to Foster, can be given for arbitrary commutative rings with the Boolean ring duality as a special case. From this duality one is naturally led to Boolean-like rings. *Boolean-like rings* are characterized as the commutative rings  $R$  with identity satisfying  $2x = 0$  and  $xy(1+x)(1+y) = 0$  for all  $x, y \in R$ . More generally a commutative ring  $R$  is an  $n$ -Boolean ring if  $\text{char } R = 2$  and  $x_1 \cdots x_n(1+x_1) \cdots (1+x_n) = 0$  for all  $x_1, \dots, x_n \in R$ . Thus, Boolean rings are just the 1-Boolean rings and Boolean-like rings are the 2-Boolean rings. Also, a commutative ring  $R$  is  $n$ -von Neumann regular if, given  $x_1, \dots, x_n \in R$ , there exist  $a_1, \dots, a_n \in R$  with  $(x_1 a_1 x_1 - x_1) \cdots (x_n a_n x_n - x_n) = 0$ . So  $R$  is 1-von Neumann regular if and only if  $R$  is von Neumann regular. It can be shown that  $R$  is  $n$ -Boolean, respectively  $n$ -von Neumann regular, if and only if  $R/\text{nil}(R)$  is Boolean, respectively von Neumann regular, and  $\text{nil}(R)^n = 0$ . There is also a  $T$ -nilpotent version of both concepts. We have the following result which gives a structure theory for Boolean-like rings using idealization.

**Theorem 6.3.** (1) [3, Theorem 9] *Let  $R$  be a commutative ring with identity and  $N$  an  $R$ -module. If  $R$  is  $n$ -Boolean, respectively  $n$ -von Neumann regular, then  $R(+)\mathcal{N}$  is  $(n+1)$ -Boolean, respectively  $(n+1)$ -von Neumann regular. Moreover,  $R$  is  $n$ -Boolean, respectively  $n$ -von Neumann regular, if and only if  $\text{nil}(R)^{n-1}N = 0$ .*

(2) *Structure theory for Boolean-like rings* [3, Theorem 10]. If  $B$  is a Boolean ring and  $N$  is a  $B$ -module, then  $B(+N)$  is a Boolean-like ring. Conversely, suppose that  $R$  is a Boolean-like ring. Then  $\overline{R} = R/\text{nil}(R)$  is a Boolean ring and  $R \approx \overline{R}(+)\text{nil}(R)$  where  $\text{nil}(R)$  is considered as an  $\overline{R}$ -module (since  $\text{nil}(R)^2 = 0$ ). Equivalently, if  $B = \{b \in R \mid b = b^2\}$ , then  $B$  is a Boolean subring of  $R$  (with  $B \approx \overline{R}$ ) and  $R = B(+)\text{nil}(R)$  where  $\text{nil}(R)$  is considered as a  $B$ -module.

*Proof.* (1) Suppose that  $R$  is  $n$ -Boolean. Put  $R^* = R(+N)$ . Since  $\text{char } R = 2$ ,  $2x = 0$  for all  $x \in N$ , so  $\text{char } R^* = 2$ . Now  $R^*/\text{nil}(R^*) \approx R/\text{nil}(R)$  is a Boolean ring. Since  $\text{nil}(R^*)^m = \text{nil}(R)^m(+)\text{nil}(R)^{m-1}N$  for each natural number  $m$ ,  $R$   $n$ -Boolean  $\Rightarrow \text{nil}(R)^n = 0 \Rightarrow \text{nil}(R^*)^{n+1} = 0$ . So  $R$   $n$ -Boolean implies that  $R^*$  is  $(n+1)$ -Boolean and  $R^*$  is  $n$ -Boolean  $\Leftrightarrow \text{nil}(R^*)^n = 0 \Leftrightarrow \text{nil}(R)^{n-1}N = 0$ . The proof of the  $n$ -von Neumann regular result is similar.

(2) The first part of (2) follows from (1) since a Boolean-like ring is just a 2-Boolean ring. Conversely, suppose that  $R$  is a Boolean-like ring. Since  $\text{char } R = 2$ , it is easily checked that  $B$  is a Boolean subring of  $R$ . Also,  $R = B + \text{nil}(R)$  [3, Theorem 6] (in fact,  $R = B \oplus \text{nil}(R)$ ) [3, Theorem 8]. So the map  $B \rightarrow R \rightarrow R/\text{nil}(R)$  is an isomorphism. Since every element  $r \in R$  has a unique representation in the form  $r = b + n$  where  $b \in B$  and  $n \in \text{nil}(R)$ , it is easily checked that the map  $R \rightarrow B(+)\text{nil}(R)$  given by  $r = b + n \rightarrow (b, n)$  is an isomorphism.  $\square$

Unfortunately, Theorem 6.3 cannot be extended to  $n$ -Boolean rings for  $n > 2$  nor to  $n$ -von Neumann regular rings. For if  $R^* = \mathbf{Z}/4\mathbf{Z}$ , then  $R^*$  is 2-von Neumann regular, but  $R^*$  does not have the form  $R^* = R(+N)$  where  $R$  is von Neumann regular and  $N$  is an  $R$ -module. For since  $R^*$  is not von Neumann regular, we must have  $R = \mathbf{Z}_2$  and hence if  $R^* = R(+N)$ , then  $R^* = \mathbf{Z}/4\mathbf{Z}$  would have characteristic 2. The same example shows that a ring satisfying the identity  $x_1x_2(1+x_1)(1+x_2) = 0$  need not be the idealization of a ring satisfying the identity  $x_1(1+x_1) = 0$ . Next, let  $R^* = (\mathbf{Z}_2[X]/(X^3)) \times \mathbf{Z}_2$ , so  $R^*$  is 3-Boolean, but not 2-Boolean. It can be shown that  $R^*$  is not the idealization of a 2-Boolean ring, see [3, pages 74–75] for details.

We next consider commutative clean rings and some generalizations. Nicholson [50] defined a (not necessarily commutative) ring  $R$  to be *clean* if each element of  $R$  can be written as the sum of a unit and an idempotent. McGovern defined a commutative ring to be *almost clean* if every element is the sum of a regular element and an idempotent and in [2] a commutative ring  $R$  is said to be  $(\{0, 1\}$ -) *weakly clean* if for each  $x \in R$ , either  $x = u + e$  or  $x = u - e$  for some unit  $u$  and some idempotent  $e$  ( $e \in \{0, 1\}$ ). The reader is referred to [2] for relevant references. A quasilocal ring and any zero-dimensional ring is clean and a direct product or ultraproduct of clean rings is clean. Any integral domain is almost clean but is clean if and only if it is quasilocal. McGovern determined when certain rings of continuous functions are (almost) clean. An indecomposable weakly clean ring is either quasilocal or has exactly two maximal ideals and has 2 as a unit. Thus  $\mathbf{Z}_{(3)} \cap \mathbf{Z}_{(5)}$  is weakly clean but not clean.

**Theorem 6.4.** *Let  $R$  be a commutative ring and  $M$  an  $R$ -module.*

(1) [2, Theorem 1.10]. *Then  $R(+M)$  is clean (respectively weakly clean,  $\{0, 1\}$ -weakly clean) if and only if  $R$  is clean (respectively weakly clean,  $\{0, 1\}$ -weakly clean).*

(2) [2, Theorem 2.11]. *Then  $R(+M)$  is almost clean if and only if each  $x \in R$  can be written in the form  $x = r + e$  where  $x \in R - (Z(R) \cup Z(M))$  and  $e \in \text{Id}(R)$ .*

*Proof.* Recall that  $U(R(+M)) = \{(r, m) \mid r \in U(R), m \in M\}$ ,  $\text{Id}(R(+M)) = \{(e, 0) \mid e \in \text{Id}(R)\}$ , and  $\text{reg}(R(+M)) = \{(r, m) \mid r \in R - (Z(R) \cup Z(M)), m \in M\}$ . If  $R$  is clean, then for  $r \in R$ ,  $r = u + e$  where  $u$  is a unit and  $e$  is an idempotent. Then  $(r, m) = (u, m) + (e, 0)$  where  $(u, m)$  is a unit and  $(e, 0)$  is idempotent. So  $R(+M)$  is clean. A similar argument shows that if  $R$  is weakly clean (respectively  $\{0, 1\}$ -weakly clean, has each element the sum of an element from  $R - (Z(R) \cup Z(M))$  and an idempotent), then  $R(+M)$  is weakly clean (respectively  $\{0, 1\}$ -weakly clean, almost clean). The converse of each statement is also similar.  $\square$

Recently a number of papers concerning Armendariz rings have appeared. Rege and Chhawchharia [52] defined a ring  $R$  (not necessarily commutative) to be Armendariz if for  $f, g \in R[X]$  with  $fg = 0$ ,  $a_i b_j = 0$

for each coefficient  $a_i$  of  $f$  and  $b_j$  of  $g$ . In a similar manner one can define an Armendariz module (take  $f \in R[X]$  and  $g \in M[X]$ ). A reduced ring or a commutative arithmetical ring is Armendariz,  $R[X]$  is Armendariz if and only if  $R$  is, and  $R[X]/(X^n)$ ,  $n \geq 2$ , is Armendariz if and only if  $R$  is reduced. For these results and the following theorem, see [9].

**Theorem 6.5.** [9, Theorem 12] *Let  $R$  be a commutative ring and  $M$  an  $R$ -module.*

(1) *If  $R(+M)$  is Armendariz, then  $R$  is an Armendariz ring and  $M$  is an Armendariz  $R$ -module.*

(2) *Suppose that  $R$  is an integral domain. Then  $R(+M)$  is Armendariz if and only if  $M$  is an Armendariz  $R$ -module. In particular, if  $M$  is torsion-free,  $R(+M)$  is Armendariz.*

*Proof.* We identify  $(R(+M)[X])$  with  $R[X](+M[X])$ .

(1) Suppose that  $R(+M)$  is Armendariz. Let  $f \in R[X]$  and  $g \in M[X]$  with  $fg = 0$ . Then in  $(R(+M)[X])$ ,  $(f, 0)(0, g) = (0, 0)$ . Let  $a_i$  be a coefficient of  $f$  and  $b_j$  a coefficient of  $g$ ; so  $(a_i, 0)$  is a coefficient of  $(f, 0)$  and  $(0, b_j)$  is a coefficient of  $(0, g)$ . So  $(0, 0) = (a_i, 0)(0, b_j) = (0, a_i b_j)$ . Thus,  $a_i b_j = 0$  and hence  $M$  is an Armendariz  $R$ -module. A similar proof shows that  $R$  is an Armendariz ring.

(2) Suppose that  $R$  is an integral domain. Now  $(\Rightarrow)$  follows from (1).

$(\Leftarrow)$ . Let  $f, g \in (R(+M)[X])$  with  $fg = 0$ . Write  $f = \sum (r_i, m_i)X^i = (f_1, f_2)$  and  $g = \sum (s_i, n_i)X^i = (g_1, g_2)$  in  $(R(+M)[X]) = R[X](+M[X])$ . Now  $0 = f_1 g_1$  in  $R[X]$ , so  $R$  a domain gives, say,  $f_1 = 0$ . So  $0 = f_1 g_2 + g_1 f_2 = g_1 f_2$ . So  $M$  Armendariz gives each  $s_i m_j = 0$ . So  $(r_j, m_j)(s_i, n_i) = (0, m_j)(s_i, n_i) = (0, s_i m_j) = (0, 0)$ .  $\square$

We note that the converse of (1) above, namely  $R$  Armendariz and  $M$  an Armendariz  $R$ -module  $\Rightarrow R(+M)$  is Armendariz, is not true. For take  $R = \mathbf{Z}/4\mathbf{Z}$ . Then  $R$  being a PIR is Armendariz and so  $R$

is an Armendariz  $R$ -module. However,  $R(+R) \approx R[X]/(X^2)$  is not Armendariz since  $R$  is not reduced.

We next briefly mention four papers concerning the idealization of rings that are related to Bezout domains.



Mahdou [47] considered the transfer of Steinitz, Hermite, semi-Steinitz and weakly semi-Steinitz properties to idealization.

Faith raised the question of whether a commutative ring  $R$  with the property that the endomorphism ring of each ideal is commutative forces  $R$  to be self-injective. This is the case if  $R$  is Noetherian or reduced. Clark [20] showed that the idealization  $R = A(+)(K/A)$  where  $A$  is a discrete rank-one noncomplete valuation domain with quotient field  $K$  gives a negative answer.

Call a commutative ring  $R$  stable if  $\text{Hom}_R(M, E) = 0$  implies  $\text{Hom}_R(E(M), E) = 0$  for all  $R$ -modules  $M$  and injective  $R$ -modules  $E$  where  $E(M)$  is the injective envelope of  $M$ . Damiano and Shapiro [22] showed that if  $R$  is a stable ring with Gabriel dimension and  $M$  is an  $R$ -module such that  $\text{Hom}_R(M, E)$  is isomorphic to a submodule of  $\oplus E$  for each injective  $R$ -module  $E$ , then the idealization  $R(+M)$  is stable. This result is used to give examples of stable rings that are neither Noetherian nor perfect.

A commutative ring  $R$  is an FGC ring if every finitely generated  $R$ -module is a direct sum of cyclic modules. A commutative ring  $R$  is an FGC ring if and only if  $R$  is a finite direct product of maximal chained rings, almost maximal Bezout domains, and torch rings. Here  $R$  is a torch ring if  $R$  is not quasilocal,  $R$  has a unique minimal prime ideal  $P$ ,  $P$  is a nonzero uniserial  $R$ -module, and  $R/P$  is an  $h$ -local locally almost maximal Bezout domain. Shores and Wiegand [56] showed that if  $S$  is an FGC domain with quotient field  $K$  that is not quasilocal and which is not a maximal domain, then for  $\mathcal{M}$  a maximal ideal of  $S$ ,  $R = S(+)(K/S_{\mathcal{M}})$  is a torch ring.

We have yet to consider modules over the idealization. An  $R(+M)$ -module can be identified with a pair  $(U, f)$  where  $U$  is an  $R$ -module and  $f: M \otimes_R U \rightarrow U$  is an  $R$ -module map satisfying  $f \circ (1_M \otimes f) = 0$ . Here  $U$  is given an  $R(+M)$ -module structure via  $f$ , namely,  $(r, m)u = ru + f(m \otimes u)$ . For a thorough study of modules over  $R(+M)$  (where  $R$  is not assumed to be commutative), see [51]. In particular, that cited paper is concerned with finding the global homological dimension of  $R(+M)$ . A number of very technical results are given. For example, if  $\text{Tor}_i^R(M, M) = 0$  for all  $i \geq 0$ , then  $\max(\text{ID}(R), \max(\text{whd}(M_R), \text{hd}({}_R M) + 1)) \leq \text{ID}(R(+M)) \leq \text{ID}(R) + \min(\text{whd}(M_R), \text{hd}({}_R M) + 1)$  where  $\text{ID}(\ )$  is the left global

homological dimension,  $\text{hd}(\ )$  is the homological (projective) dimension and  $\text{whd}(\ )$  is the weak homological (flat) dimension. Of course, in the case where  $R$  is a commutative Noetherian ring and  $M$  is a finitely generated  $R$ -module,  $\text{ID}(R(+)\ M) = \infty$  unless  $M = 0$  and  $R$  is a finite-dimensional regular ring since  $\text{ID}(R(+)\ M) < \infty$  implies that  $R(+)\ M$  is locally a regular local ring and hence locally an integral domain and  $\text{ID}(R(+)\ M) = \sup\{\text{ID}((R(+)\ M)_{\mathcal{M}(+)\ \mathcal{M}}) \mid \mathcal{M} \text{ is a maximal ideal of } R\}$ .

Suppose that  $R$  is a commutative ring. A finite  $n$ -presentation for an  $R$ -module  $M$  is an exact sequence  $F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$  where each  $F_i$  is a finitely generated free  $R$ -module. The  $\lambda$ -dimension of  $R$ , denoted  $\lambda\text{-dim } R$ , is the least positive integer  $n$ , or  $\infty$  if none exists, with the property that whenever an  $R$ -module  $M$  has a finite  $n$ -presentation, then  $M$  has a finite  $m$ -presentation for all  $m \geq n$ . So  $\lambda\text{-dim } R = 0$ , respectively  $\lambda\text{-dim } R \leq 1$ , if and only if  $R$  is Noetherian, respectively coherent. The notion of  $\lambda$ -dimension was introduced by Vasconcelos, and he raised the problem of giving a commutative ring  $R_n$  for each positive integer  $n$  with  $\lambda\text{-dim } R_n = n$ . This problem was solved by Roos [54] using idealization. He showed that (1) if  $R$  is Noetherian and  $M$  is a free  $R$ -module of infinite rank, then  $\lambda\text{-dim}(R(+)\ M) = 2$  and (2) if  $(R, \mathcal{M})$  is a local  $n$ -dimensional Gorenstein ring, then  $\lambda\text{-dim}(R(+)\ E(R/\mathcal{M})) = n$ . Conditions are given for  $R(+)\ M$  to be coherent. Also, see [29].

Let  $n$  and  $d$  be nonnegative integers. Costa [21] defined a commutative ring  $R$  to be an  $(n, d)$ -ring if every  $n$ -presented  $R$ -module has projective dimension at most  $d$ . He raised a number of open problems including whether there are examples of  $(n, d)$ -rings which are neither  $(n, d-1)$ -rings nor  $(n-1, d)$ -rings for all nonnegative integers  $n$  and  $d$ . Using trivial extensions of fields, Mahdou [46] constructed a  $(2, 0)$ -ring which is not a  $(1, 0)$ -ring. Soon after Kabbaj and Mahdou [38] used idealization to construct a class of  $(3, d)$ -rings which are neither  $(3, d-1)$ -rings nor  $(2, d)$ -rings for arbitrary  $d$ . These ideas are pushed forward in [39] which investigates coherent-like conditions in an idealization.

The majority of this survey has dealt with non-Noetherian rings. As seen by Example 4.15, idealization can be used to produce examples of local rings with  $\dim R = r$  and  $G(R) = s$  where  $r \geq s \geq 0$ . Thus, it

is not surprising that idealization can be used to provide examples of Cohen-Macaulay rings and related rings. We briefly mention some of the work. Reiten [53] showed that if  $R$  is a local ring with a Gorenstein module  $M$  of rank 1, then  $R(+M)$  is Gorenstein and thus  $R$  is a quotient of a local Gorenstein ring. Goto [30] showed that if  $R$  is a Cohen-Macaulay ring of dimension  $d$  and  $M$  a Cohen-Macaulay module of dimension  $d-1$ , then  $R(+M)$  is an approximately Cohen-Macaulay ring. Goto [31] and Yamagishi [58] gave conditions for the idealization to be a Buchsbaum or quasi-Buchsbaum ring. Aoyama [16] showed that  $R(+M)$  is quasi-Gorenstein if and only if  $\widehat{R}$  is  $(S_2)$  and  $M$  is a canonical module. Gulliksen [34] showed that the idealization of a local complete intersection and a finitely generated module has a Poincaré series that is a rational function, but Bøgvad [19] using Gulliksen's result that  $(R, \mathcal{M})$  local Artinian implies  $R(+E(R/\mathcal{M}))$  is Gorenstein gave an example of a local Gorenstein ring with transcendental Poincaré series. Goto et al. [32, 33] used idealization to study “good ideals” (see papers for definition) in local Gorenstein rings.

We end this survey with a brief discussion of an interesting paper by Lambert and Lucas [41]. Among other things, they showed that if  $R \subseteq S$  are commutative rings and  $\phi, \phi': S \rightarrow R$  are  $R$ -module maps with  $\phi(1) = 1 = \phi'(1)$ , then  $R(+\ker \phi)$  and  $R(+\ker \phi')$  are isomorphic as rings. Techniques from idealization are used to prove the following theorem: Let  $W$  be an Abelian von Neumann algebra of operators acting on a separable Hilbert space  $H$ . Then there is an algebra  $N$  of nilpotent operators of index 2 such that  $W + N$  is a maximal Abelian algebra of operators on  $H$ .

## REFERENCES

1. A.G. Ağargün, D.D. Anderson and S. Valdes-Leon, *Factorization in commutative rings with zero divisors*, III, Rocky Mountain J. Math. **31** (2001), 1–21.
2. M.S. Ahn and D.D. Anderson, *Weakly clean rings and almost clean rings*, Rocky Mountain J. Math. **30** (2006), 783–798.
3. D.D. Anderson, *Generalizations of Boolean rings, Boolean-like rings and von Neumann regular rings*, Comment. Math. Univ. St. Paul. **35** (1986), 69–76.
4. ———, *On a result of M. Martin concerning the number of generators of products of invertible ideals*, Comm. Algebra **15** (1987), 1765–1768.
5. ———, *Some remarks on multiplication ideals*, II, Comm. Algebra **28** (2000), 2577–2583.

6. ———, *Cancellation modules and related modules*, in *Ideal theoretic methods in commutative algebra*, D.D. Anderson and I.J. Papick, eds., Lecture Notes Pure Appl. Math. **220**, Marcel Dekker, New York, 2001.
7. D.D. Anderson, D.F. Anderson and M. Zafrullah, *Factorization in integral domains*, J. Pure Appl. Algebra **69** (1990), 1–19.
8. D.D. Anderson, M. Axtell, S.J. Forman and J. Stickles, *When are associates unit multiples?*, Rocky Mountain J. Math. **34** (2004), 811–828.
9. D.D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra **26** (1998), 2265–2272.
10. D.D. Anderson and R. Markanda, *Unique factorization rings with zero divisors*, Corrigendum, Houston J. Math. **11** (1985), 423–426.
11. D.D. Anderson and J. Matijevic, *Graded  $\pi$ -rings*, Canadian J. Math. **31** (1979), 449–457.
12. D.D. Anderson and J. Pascual, *Regular ideals in commutative rings, sublattices of regular ideals, and Prüfer rings*, J. Algebra **111** (1987), 404–426.
13. D.D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Math. **29** (2003), 831–840.
14. D.D. Anderson and S. Valdes-Leon, *Factorization in commutative rings with zero divisors*, Rocky Mountain J. Math. **26** (1996), 439–480.
15. ———, *Factorization in commutative rings with zero divisors, II*, in *Factorization in integral domains*, D.D. Anderson, ed., Lecture Notes Pure Appl. Math. **189**, Marcel Dekker, New York, 1997.
16. Y. Aoyama, *Some basic results on canonical modules*, J. Math. Kyoto Univ. **23** (1983), 85–94.
17. M. Axtell, *U-factorizations in commutative rings with zero divisors*, Comm. Algebra **30** (2002), 1241–1255.
18. M. Axtell, S. Forman, N. Roersma and J. Stickles, *Properties of U-factorizations*, Inter. J. Commut. Rings **2** (2003), 83–99.
19. R. Bøgvad, *Gorenstein rings with transcendental Poincaré-series*, Math. Scand. **53** (1983), 5–15.
20. J. Clark, *On a question of Faith in commutative endomorphism rings*, Proc. Amer. Math. Soc. **98** (1986), 196–198.
21. D.L. Costa, *Parameterizing families of non-Noetherian rings*, Comm. Algebra **22** (1994), 3997–4011.
22. R. Damiano and J. Shapiro, *Commutative torsion stable rings*, J. Pure Appl. Algebra **32** (1984), 21–32.
23. C. Faith, *Algebra: Rings, modules and categories I*, Die Grundlehren der mathematischen Wissenschaften **190**, Springer-Verlag, New York, 1973, corrected reprint 1981.
24. C.R. Fletcher, *Unique factorization rings*, Math. Proc. Cambridge Philos. Soc. **65** (1969), 579–583.
25. ———, *The structure of unique factorization rings*, Math. Proc. Cambridge Philos. Soc. **67** (1970), 535–540.
26. P.A. Froeschl, III, *Chained rings*, Pacific J. Math. **65** (1976), 47–53.

27. R. Gilmer, *On Prüfer rings*, Bull. Amer. Math. Soc. **78** (1972), 223–224.
28. ———, *Multiplicative ideal theory*, Queen's Papers Pure Appl. Math. **90**, Queen's University, Kingston, Ontario, 1992.
29. S. Glaz, *Commutative coherent rings*, Lecture Notes Math. **1371**, Springer-Verlag, Berlin, 1989.
30. S. Goto, *Approximately Cohen-Macaulay rings*, J. Algebra **76** (1982), 214–225.
31. ———, *Maximal Buchsbaum modules over regular local rings and a structure theorem for generalized Cohen-Macaulay modules*, Commutative Algebra Combinatorics, Adv. Stud. Pure Math. **11**, North-Holland, Amsterdam, 1987.
32. S. Goto and S. Haraikawa, *Good ideals in Artinian Gorenstein local rings obtained by idealization*, Tokyo J. Math. **25** (2002), 493–499.
33. S. Goto, S.-I. Iai and M.-K. Kim, *Good ideals in Gorenstein local rings obtained by idealization*, Proc. Amer. Math. Soc. **130** (2002), 337–344.
34. T.H. Gulliksen, *A change of ring theorem with applications to Poincaré series and intersection multiplicity*, Math. Scand. **34** (1974), 167–183.
35. G. Hinkle, *The generalized Kronecker function ring and the ring  $R(X)$* , dissertation, University of Missouri-Columbia, 1975.
36. G. Hinkle and J. Huckaba, *The generalized Kronecker function ring and the ring  $R(X)$* , J. reine angew. Math. **292** (1977), 25–36.
37. J.A. Huckaba, *Commutative rings with zero divisors*, Monographs and Textbooks in Pure and Applied Mathematics **117**, Marcel Dekker, Inc., New York, 1988.
38. S. Kabbaj and N. Mahdou, *Trivial extensions of local rings and a conjecture of Costa*, in *Commutative ring theory and applications*, M. Fontana, S. Kabbaj and S. Wiegand, eds., Lecture Notes Pure Appl. Math. **231**, Marcel Dekker, New York, 2003.
39. ———, *Trivial extensions defined by coherent-like conditions*, Comm. Algebra **32** (2004), 3937–3953.
40. I. Kaplansky, *Commutative rings*, revised ed., Polygonal Publishing House, Washington, N.J., 1994.
41. A. Lambert and T.G. Lucas, *Nagata's principle of idealization in relation to module homomorphisms and conditional expectations*, Kyungpook Math. J. **40** (2000), 327–337.
42. M.D. Larsen and P.J. McCarthy, *Multiplicative theory of ideals*, Academic Press, New York, 1971.
43. T. Lucas, *The annihilator conditions property (A) and (A C)*, dissertation, University of Missouri-Columbia, 1983.
44. ———, *Two annihilator conditions: Property (A) and (A C)*, Comm. Algebra **14** (1986), 557–580.
45. ———, *Some results on Prüfer rings*, Pacific J. Math. **124** (1986), 333–343.
46. N. Mahdou, *On Costa's conjecture*, Comm. Algebra **29** (2001), 2775–2785.
47. ———, *Steinitz properties in trivial extensions of commutative rings*, Arab. J. Sci. Eng. **26** (2001), Commutative Algebra, 119–125.

48. R. Matsuda, *On Marot rings*, Proc. Japan Acad. Math. Sci. **60** (1984), 134–137.
49. M. Nagata, *Local rings*, Interscience Publishers, New York, 1962.
50. W.K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269–278.
51. I. Palmér and J.-E. Roos, *Explicit formulae for the global homological dimensions of trivial extensions of rings*, J. Algebra **27** (1973), 380–413.
52. M.B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Math. Sci. **73** (1997), 14–17.
53. I. Reiten, *The converse to a theorem of Sharp on Gorenstein modules*, Proc. Amer. Math. Soc. **32** (1972), 417–420.
54. J.-E. Roos, *Finiteness conditions in commutative algebra and solution of a problem of Vasconcelos*, London Math. Soc. Lecture Note **72**, Cambridge University Press, Cambridge, 1982.
55. H. Sekiyama, *Trivial extension of a commutative ring with balanced condition*, Proc. 12th Symposium on Ring Theory Okayama Univ., Okayama, 1980.
56. T.S. Shores and R. Wiegand, *Rings whose finitely generated modules are direct sums of cyclics*, J. Algebra **32** (1974), 152–172.
57. E. Valtonen, *Some homological properties of commutative semitrivial ring extensions*, Manus. Math. **63** (1989), 45–68.
58. K. Yamagishi, *Idealizations of maximal Buchsbaum modules over a Buchsbaum ring*, Math. Proc. Cambridge Philos. Soc. **104** (1988), 451–478.
59. P. Zanardo, *On  $v$ -closed Manis valuation rings*, Comm. Algebra **18** (1990), 775–788.
60. ———, *Constructions of Manis valuation rings*, Comm. Algebra **21** (1993), 4183–4194.

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