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A construction of processes with one-dimensional martingale marginals, associated with a Lévy process, via its Lévy sheet

By

Francis Hirsch and Marc Yor

Abstract

We give some adequate extension, in the framework of a general Lévy process, of our previous construction of processes with one-dimensional martingale marginals, done originally in the set-up of Brownian motion. The Lévy process framework allows us to streamline our previous arguments, as well as to reach a larger class of such processes, even in the Brownian case. We give some illustrations of our construction when the Lévy process is either a Gamma process, or a Poisson process. We also work in the fractional Brownian and stable frameworks.

1. Introduction

1.1. Convex order increase and 1-martingales

a) This paper is devoted to investigations about two apparently different classes of processes, which are:

(C_1) the class of processes $(U_t\ ,\ t \geq 0)$ which are *increasing in the convex order*, that is: for any $g : \mathbb{R} \longrightarrow \mathbb{R}$ convex, the function: $t \longrightarrow \mathbb{E}[g(U_t)]$ is increasing;

(C_2) the class of processes $(V_t\ ,\ t \geq 0)$ which are *1-martingales*, that is: there exists, on possibly another probability space, a martingale $(M_t\ ,\ t \geq 0)$ such that, for any given $t \geq 0$, $V_t \overset{\text{(law)}}{=} M_t$.

A few comments about the history of this identity between (C_1) and (C_2) may be helpful: that (C_2) is included in (C_1) is a simple consequence of Jensen's inequality. The inclusion of (C_1) in (C_2) is much more difficult to prove in all generality; three steps may be singled out:

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i) If X_1 and X_2 are two random variables such that
$$\mathbb{E}[g(X_1)] \leq \mathbb{E}[g(X_2)]$$
for every convex function g, then there exists a Markovian kernel Q such that: $\nu_1 Q = \nu_2$, where ν_i $(i = 1, 2)$ is the law of X_i.
Many works have been devoted to this result, by authors such as Hardy-Littlewood, Polya, Blackwell, Stein, Cartier-Fell-Meyer, Strassen, Rotschild-Stiglitz, to name but a few. The main tool used in these works is the Hahn-Banach theorem.
More modern proofs related to Skorokhod embedding theorem, were then given for this problem, along the lines of Rost [13], Falkner-Fitzsimmons [6], and more recently Fitzsimmons [7].

ii) A second step consists in dealing with a sequence $X_1, X_2, \cdots, X_n, \cdots$ of random variables which are increasing in the convex order. This is done, e.g. by Strassen [17].

iii) Finally, Kellerer's work (following those of Strassen [17] and Doob [5]) deals with a continuous time process $(U_t\ ,\ t \geq 0)$ increasing in the convex order, to show that it is a 1-martingale. Kellerer's result hinges on an "integral representation of dilations"; see [11, Theorem 3].

In any case, the proofs offered by Strassen, Doob and especially Kellerer of the identity between the two classes (C_1) and (C_2) are not constructive, and it is an interesting question, given a process $(U_t\ ,\ t \geq 0)$, or the family of its marginals $(\mu_t\ ,\ t \geq 0)$, which are increasing in the convex order, to find, as explicitly and concretely as possible, a martingale $(M_t\ ,\ t \geq 0)$ which admits the same one-dimensional marginals $(\mu_t\ ,\ t \geq 0)$ as $(U_t\ ,\ t \geq 0)$. A connected question is to exhibit large classes of 1-martingales, and so to obtain large classes of processes which are increasing for the convex order.
These aims have already been the topic of the papers by Madan-Yor [12], Baker-Yor [2], Hirsch-Yor [9].
In the present paper we work towards these goals in a general Lévy process framework.

b) In order to understand better the result of Carr et al. [3]:
if $(B_t\ ,\ t \geq 0)$ denotes the standard Brownian motion, and $\lambda \in \mathbb{R}$, then the process
$$\frac{1}{t}\int_0^t \exp\left(\lambda\, B_s - \frac{\lambda^2 s}{2}\right)\, \mathrm{d}s\ ,\ t \geq 0$$
increases in the convex order, Roynette [14] obtained the following general result.

Theorem 1.1. *Let $(M_t\ ,\ t \geq 0)$ denote a H^1_{loc} martingale, i.e. for each $t > 0$, $\mathbb{E}[\sup_{s\leq t} |M_s|] < \infty$. Then, the processes $\left(\frac{1}{t}\int_0^t M_s\, \mathrm{d}s\ ,\ t \geq 0\right)$ and $\left(\int_0^t (M_s - M_0)\, \mathrm{d}s\ ,\ t \geq 0\right)$ increase in the convex order.*

1.2. The Brownian "guiding example"

In our previous paper [9], we worked in a Brownian motion framework, and used in an essential manner the Wiener (or Brownian) sheet $(W_{u,t}\ ;\ u \geq 0, t \geq 0)$ in order to construct martingales with respect to

$$\mathcal{W}_t = \sigma\{W_{u,s}\ ;\ u \geq 0, s \leq t\}\ , \quad t \geq 0$$

The two key properties we used are:

a) $(W_{\bullet,t}\ ,\ t \geq 0)$ is a Lévy process, taking values in $C([0,\infty);\mathbb{R})$ (in fact, it may be called a $C([0,\infty);\mathbb{R})$-Brownian motion);

b) for any fixed $t \geq 0$, $\quad B_{t\,\bullet} \overset{\text{(law)}}{=} W_{\bullet,t}$.

The "guiding example" in Baker-Yor [2] and Hirsch-Yor [9] has been the identity in law, which follows from b): for fixed $t > 0$,

$$\text{(G)} \quad \frac{1}{t}\int_0^t \exp\left(\lambda\, B_s - \frac{\lambda^2 s}{2}\right)\,\mathrm{d}s \overset{\text{(law)}}{=} \int_0^1 \exp\left(\lambda\, W_{u,t} - \frac{\lambda^2 u t}{2}\right)\,\mathrm{d}u$$

and the fact that the RHS in (G) is a $(\mathcal{W}_t)$-martingale.

In this paper, we develop some adequate extensions of (G), with Brownian motion being replaced by a general Lévy process (then, λ often needs to be assumed purely imaginary).
Thus, for this purpose, to a Lévy process $(L_t\ ,\ t \geq 0)$, we associate a Lévy sheet $(X_{u,t}\ ;\ u \geq 0, t \geq 0)$ and the exact analogues of a) and b) are satisfied. This is the content of Section 2.

1.3. Extending (G) in the Lévy framework

We now explain how (G) above may be developed in the Lévy framework; we do so in the hope that it will facilitate the reader's understanding of our construction of martingale processes $(\Phi_t^m(X))$ and 1-martingale processes $(\Phi_t^\sharp(L))$, as indicated briefly in Subsection 1.4 below, and developed thoroughly in Sections 3 and 4 of the paper.

Let $\mathbb{D}_0$ be the Skorokhod space consisting of all càdlàg functions ε from $\mathbb{R}_+$ into $\mathbb{R}$ such that $\varepsilon(0) = 0$. Searching for some adequate extension of (G), and encouraged by Theorem 1.1, we would like to find a reasonable class of functionals $U(\varepsilon, s)$ $(\varepsilon \in \mathbb{D}_0\ ,\ s \geq 0)$ such that the process:

$$V_t = \frac{1}{t}\int_0^t U(L_{s\,\bullet}, s)\,\mathrm{d}s\ , \quad t \geq 0$$

is a 1-martingale. We show in particular (see Example 4 in Subsection 4.3) that this is the case for $U(\varepsilon, s) = f(\varepsilon(1), s)$, where $f(x, s)$ is a *space-time harmonic function for* L. In general, concerning V_t, we note that, from b) written for the pair (L, X) instead of (B, W), we get, for fixed t,

$$V_t = \int_0^1 U(L_{u\,t\,\bullet}, u\,t)\,\mathrm{d}u \overset{\text{(law)}}{=} \int_0^1 U(X_{u\,\bullet,t}, u\,t)\,\mathrm{d}u$$

Thus, in order to show that $(V_t\ ,\ t \geq 0)$ is a 1-martingale, it suffices to find U such that, for any given $u \in (0,1)$, the process

$$(U(X_{u\,\bullet,t}, u\,t)\ ,\ t \geq 0)$$

is a $(\mathcal{X}_t)$-martingale, where:

$$\mathcal{X}_t = \sigma\{X_{v,s}\ ;\ v \geq 0, s \leq t\}$$

Already, a large class of such functionals U may be obtained by taking (ψ denoting the characteristic exponent of the Lévy process L):

$$U(\varepsilon, s) = \exp\left(\mathrm{i}\int_0^1 h(v)\,\mathrm{d}\varepsilon(v) + s\int_0^1 \psi(h(v))\,\mathrm{d}v\right) \tag{1.1}$$

for, say, bounded Borel h's. Consequently, the process:

$$\begin{aligned} H_t &= \frac{1}{t}\int_0^t \exp\left(\mathrm{i}\int_0^1 h(v)\,\mathrm{d}_v L_{s\,v} + s\int_0^1 \psi(h(v))\,\mathrm{d}v\right)\,\mathrm{d}s \\ &= \frac{1}{t}\int_0^t \exp\left(\mathrm{i}\int_0^s h\left(\frac{v}{s}\right)\,\mathrm{d}L_v + \int_0^s \psi\left(h\left(\frac{v}{s}\right)\right)\,\mathrm{d}v\right)\,\mathrm{d}s \end{aligned} \tag{1.2}$$

is a 1-martingale.

To present real-valued variants of this construction, we assume that L is a subordinator τ, so that its Lévy-Khintchine representation is:

$$\mathbb{E}[\exp(-\lambda\,\tau_s)] = \exp(-s\,\phi(\lambda))\ , \quad s \geq 0,\ \lambda \geq 0$$

Then, we may modify the previous formula (1.2) as:

$$K_t = \frac{1}{t}\int_0^t \exp\left(-\int_0^s k\left(\frac{v}{s}\right)\,\mathrm{d}\tau_v + \int_0^s \phi\left(k\left(\frac{v}{s}\right)\right)\,\mathrm{d}v\right)\,\mathrm{d}s \tag{1.3}$$

for k a nonnegative bounded Borel function.

We believe that the reader who kept with us throughout this construction should not find the more general set-up for constructing 1-martingales, as it is developed below, either too difficult or too abstract.

1.4. A systematic construction of 1-martingales

We equip the Skorokhod space $\mathbb{D}_0$ with the law $\mathbb{P}$ of L.
In agreement with the preceding discussion in 1.3, we associate, in Sections 3 and 4, to a general functional $\Phi \in L^1(\mathbb{P})$ two processes, the first one being defined in terms of X, and the second one in terms of L: for $0 \leq t \leq 1$,

$$\Phi_t^m(X) = \Pi_{1-t}\Phi(X_{\bullet,t}) \quad \text{and} \quad \Phi_t^\sharp(L) = \Pi_{1-t}\Phi(L_{t\,\bullet})$$

where $(\Pi_s\ ,\ s \geq 0)$ is the semigroup of the $\mathbb{D}_0$-valued Lévy process $(X_{\bullet,t}\ ,\ t \geq 0)$. Two key properties of Φ^m and $\Phi^\sharp$ are:

i) $(\Phi_t^m\ ,\ t \le 1)$ is a $(\mathcal{X}_t)$-martingale;

ii) for any fixed $t \le 1$, $\quad \Phi_t^m \overset{\text{(law)}}{=} \Phi_t^\sharp$.

Consequently, $(\Phi_t^\sharp\ ,\ t \le 1)$ is a 1-martingale; hence, it is also increasing in the convex order.

1.5. Considering space-time harmonic functions for $(X_{\bullet,t})_{t\ge 0}$

Another manner to express the above property i) is to say that $\Pi_{1-t}\Phi(\varepsilon)$ is a *space-time harmonic function* of $(\varepsilon, t) \in \mathbb{D}_0 \times [0,1]$, for $(X_{\bullet,t})$.
It is then natural to look for some suitable extension of the discussion made in the previous subsection 1.4. This is easy indeed: start with a generic space-time harmonic function $F(\varepsilon, t)$ on $\mathbb{D}_0 \times \mathbb{R}_+$ and consider both processes:

$$F_t^m = F(X_{\bullet,t}, t) \quad \text{and} \quad F_t^\sharp = F(L_{t\,\bullet}, t)$$

They satisfy:

i') $(F_t^m\ ,\ t \ge 0)$ is a $(\mathcal{X}_t)$-martingale;

ii') for any fixed $t \ge 0$, $\quad F_t^m \overset{\text{(law)}}{=} F_t^\sharp$.

1.6. Further examples of 1-martingales

In Section 5, we exhibit examples of 1-martingales defined from stochastic integrals with respect to L. They are closely related to the extension of (G) we discussed in Subsection 1.3.

1.7. Extension to fractional Brownian and α-stable processes

Finally, in Section 6, we show that a slight variation of our method allows to prove that, if $(B_s^H\ ,\ s \ge 0)$ denotes the fractional Brownian motion with Hurst index H, then:

$$\frac{1}{t}\int_0^t \exp\left(\lambda\, B_s^H - \frac{\lambda^2}{2}\, s^{2H}\right) \mathrm{d}s\ , \quad t \ge 0$$

is a 1-martingale.
We also present extensions when B^H is replaced by any fractional α-stable process.

2. From a Lévy process L to its Lévy sheet X

In this section, we shall precise our framework and the notation.

2.1. The Lévy-Khintchine representation of L

We consider a real-valued Lévy process $(L_t\ ,\ t \ge 0)$ starting from 0. We denote by ψ its characteristic exponent:

$$\forall \lambda \in \mathbb{R},\ \forall t \ge 0, \quad \mathbb{E}\left[\exp(\mathrm{i}\,\lambda\, L_t)\right] = \exp(-t\,\psi(\lambda))$$

One has (*Lévy-Khintchine formula*):

$$\psi(\lambda) = \sigma^2 \frac{\lambda^2}{2} + \mathrm{i}\,\gamma\,\lambda + \int \left(1 - \mathrm{e}^{\mathrm{i}\lambda x} + \mathrm{i}\,\lambda\,x\,1_{|x|\le 1}\right)\,\nu(\mathrm{d}x)$$

with $\sigma, \gamma \in \mathbb{R}$ and ν a positive measure on $\mathbb{R} \setminus \{0\}$ such that

$$\int \frac{x^2}{1+x^2}\,\nu(\mathrm{d}x) < \infty$$

We refer e.g. to Bertoin [1] for a deep study of Lévy processes.

2.2. The Skorokhod space

We denote by $\mathbb{D}_0$ the Skorokhod space consisting of all càdlàg functions ε from $\mathbb{R}_+$ into $\mathbb{R}$ such that $\varepsilon(0) = 0$ (we refer, for example, to Jacod-Shiryaev [10, VI-1]). The space $\mathbb{D}_0$ is equipped with the probability $\mathbb{P}$ which is the law of L. We often identify L with the coordinate process on $\mathbb{D}_0$.
We denote by $(\mathcal{F}_t)$ the natural filtration of L on $(\mathbb{D}_0, \mathbb{P})$ and we set $\mathcal{F} = \mathcal{F}_\infty$. Thus, $\mathcal{F}$ is the Borel σ-field of $\mathbb{D}_0$ completed with respect to $\mathbb{P}$.

2.3. The X-integral of a rectangle R

If $(X_{s,t}\ ;\ s \ge 0, t \ge 0)$ is a real-valued two-parameter process and if

$$R = (s_1, s_2] \times (t_1, t_2]\,, \quad s_1 < s_2,\ t_1 < t_2$$

is a rectangle, we set

$$\Delta_R X = X_{s_2,t_2} - X_{s_1,t_2} - X_{s_2,t_1} + X_{s_1,t_1}$$

and we denote by $|R|$ the area of R:

$$|R| = (s_2 - s_1)(t_2 - t_1)$$

2.4. Defining the Lévy sheet X

The following results, for which we refer for example to Dalang-Walsh [4, Section 2], are essential for our purpose.

Theorem 2.1. *There exists a real-valued two-parameter process $X = (X_{s,t}\ ;\ s \ge 0, t \ge 0)$ satisfying the following properties*:

1)
$$\forall s,t \ge 0, \quad X_{s,0} = X_{0,t} = 0$$

2) *Almost surely, for any $s,t \ge 0$, $X_{s,\bullet}$ and $X_{\bullet,t}$ are càdlàg functions on $\mathbb{R}_+$.*

3) *For all finite sets of disjoint rectangles $R^1, \cdots, R^n$, the random variables $\Delta_{R^1} X, \cdots, \Delta_{R^n} X$ are independent.*

4) *For any rectangle R,*

$$\Delta_R X \overset{\text{(law)}}{=} L_{|R|}$$

The process X will be called *the Lévy sheet associated with L.*
Let, for $t \geq 0$,

$$\mathcal{X}_t = \sigma\{X_{u,v} \ ; \ u \geq 0 \ , \ 0 \leq v \leq t)$$

We summarize, in the following theorem, some straightforward consequences of Theorem 2.1, which we will need in the sequel.

Theorem 2.2.

a) Let $0 \leq t_1 \leq t_2$. *Then the process* $(X_{s,t_2} - X_{s,t_1} \ , \ s \geq 0)$ *is a Lévy process starting from 0, independent of* $\mathcal{X}_{t_1}$*, and having the same law as* $(L_{(t_2-t_1)\,s} \ , \ s \geq 0)$.
In particular, for any fixed $t \geq 0$,

$$X_{\bullet,t} \overset{\text{(law)}}{=} L_{t\,\bullet}$$

Thus, $(X_{\bullet,t} \ , \ t \geq 0)$ *is a Lévy process taking values in* $\mathbb{D}_0$*, and having the same one-dimensional marginals as* $(L_{t\,\bullet} \ , \ t \geq 0)$.

b) There is the equality in law:

$$(X_{s,t} \ ; \ s,t \geq 0) \overset{\text{(law)}}{=} (X_{t,s} \ ; \ s,t \geq 0)$$

Thus, a) may be stated with the roles of s and t exchanged.

As the semigroups of both Lévy processes $(X_{\bullet,t} \ , \ t \geq 0)$ and $(L_t \ , \ t \geq 0)$ play some role in the sequel, the following definition may be helpful for the reader.

Definition 2.1. Let $\mathcal{M}_t(\mathrm{d}\eta)$ denote the law on $\mathbb{D}_0$ of $X_{\bullet,t}$ ($\overset{\text{(law)}}{=} L_{t\,\bullet}$) and $\mu_t(\mathrm{d}x)$ the law on $\mathbb{R}$ of L_t. The semigroups for the Lévy processes $(X_{\bullet,t} \ , \ t \geq 0)$ and $(L_t \ , \ t \geq 0)$ are, respectively, given by:

$$\Pi_t F(\varepsilon) = \mathbb{E}[F(\varepsilon + X_{\bullet,t})] = \int_{\mathbb{D}_0} F(\varepsilon + \eta)\, \mathcal{M}_t(\mathrm{d}\eta) \tag{2.1}$$

and

$$P_t f(x) = \mathbb{E}[f(x + L_t)] = \int_{\mathbb{R}} f(x+y)\, \mu_t(\mathrm{d}y) \tag{2.2}$$

2.5. Remark

In the sequel, the following elementary fact shall play some important role: Let $\widetilde{L}$ be an independent copy of L. Then, for any $A, B \geq 0$,

$$L_{A\,\bullet} + \widetilde{L}_{B\,\bullet} \overset{\text{(law)}}{=} L_{(A+B)\,\bullet}$$

In particular, this shows that the $\mathbb{D}_0$-valued random variable $L_\bullet$ is infinitely divisible. Theorem 2.2, a), then states that the Lévy sheet X may be understood as the $\mathbb{D}_0$-valued Lévy process $(X_{\bullet,t}\ ,\ t \geq 0)$ such that:

$$X_{\bullet,1} \overset{\text{(law)}}{=} L_\bullet$$

3. The processes $(\Phi_t^m(X)\ ,\ 0 \leq t \leq 1)$

3.1. Some equivalent formulae

To any $\Phi \in L^1(\mathbb{P})$, we associate a process $\Phi^m(X) = (\Phi_t^m(X)\ ,\ 0 \leq t \leq 1)$ by

$$\Phi_t^m(X) = \mathbb{E}\left[\Phi(X_{\bullet,1}) \mid \mathcal{X}_t\right]\ , \quad 0 \leq t \leq 1 \tag{3.1}$$

By definition, $\Phi^m(X)$ is thus a $(\mathcal{X}_t)$-martingale.
In what follows, we often will denote $\Phi_t^m(X)$ (resp. $\Phi^m(X)$) simply by Φ_t^m (resp. Φ^m).

Theorem 3.1. *For $0 \leq t \leq 1$, the following alternative formulae hold*:

$$\Phi_t^m = \mathbb{E}_{\widetilde{X}}\left[\Phi(X_{\bullet,t} + \widetilde{X}_{\bullet,1-t})\right] \tag{3.2}$$

$$\Phi_t^m = \mathbb{E}_{\widetilde{L}}\left[\Phi(X_{\bullet,t} + \widetilde{L}_{(1-t)\,\bullet})\right] \tag{3.3}$$

$$\Phi_t^m = \Pi_{1-t}\Phi(X_{\bullet,t}) \tag{3.4}$$

where

- *in* (3.2), $\widetilde{X}$ *is an independent copy of* X, *and* $\mathbb{E}_{\widetilde{X}}$ *means integrating with respect to* $\widetilde{X}$*;*
- *in* (3.3), $\widetilde{L}$ *is a copy of* L, *independent of* X, *and* $\mathbb{E}_{\widetilde{L}}$ *means integrating with respect to* $\widetilde{L}$*;*
- *in* (3.4), $(\Pi_t\ ,\ t \geq 0)$ *denotes the semigroup of* $(X_{\bullet,t}\ ,\ t \geq 0)$ *defined by* (2.1).

Proof. We have:

$$\Phi(X_{\bullet,1}) = \Phi(X_{\bullet,t} + (X_{\bullet,1} - X_{\bullet,t}))$$

Then, formulae (3.2) and (3.3) follow directly from Theorem 2.2.
We obtain formula (3.4) from (3.1) simply by the definition of the semigroup (Π_t). □

3.2. Interpretation in terms of space-time harmonic functions

There is another way to understand the previous definition (3.1), using the following notion of *space-time harmonic function.*

Definition 3.1. Let $I = [0, a]$ $(a > 0)$ or $I = [0, \infty)$.

(i) A function $F(\varepsilon, t)$ on $\mathbb{D}_0 \times I$ is called a *space-time harmonic function for* $(X_{\bullet,t}\ ,\ t \in I)$, if the process $(F(X_{\bullet,t}, t)\ ,\ t \in I)$ is a $(\mathcal{X}_t)$-martingale, or equivalently, if, for any $s, t \in I$ with $s < t$,

$$\Pi_{t-s} F^t = F^s \quad \mathcal{M}_s\text{-a.s.}$$

where, for $u \in I$, $F^u(\varepsilon) = F(\varepsilon, u)$.

(ii) A function $f(x, t)$ on $\mathbb{R} \times I$ is called a *space-time harmonic function for* $(L_t\ ,\ t \in I)$, if the process $(f(L_t, t)\ ,\ t \in I)$ is a $(\mathcal{F}_t)$-martingale, or equivalently, if, for any $s, t \in I$ with $s < t$,

$$P_{t-s} f^t = f^s \quad \mu_s\text{-a.s.}$$

where, for $u \in I$, $f^u(x) = f(x, u)$.

Let us mention that H. Föllmer [8] determined the nonnegative space-time harmonic functions for $(W_{\bullet,t}\ ,\ t \geq 0)$ where W is the Brownian sheet.

The definition (3.1) may be written as:

$$\Phi_t^m = F^{(\Phi)}(X_{\bullet,t}, t) \tag{3.5}$$

where $F^{(\Phi)}$, defined on $\mathbb{D}_0 \times [0, 1]$, is the space-time harmonic function for $(X_{\bullet,t}\ ,\ 0 \leq t \leq 1)$ such that $F^{(\Phi)}(\varepsilon, 1) = \Phi(\varepsilon)$.
We note that, from formulae (3.3), (3.4) and (3.5), one obtains:

$$F^{(\Phi)}(\varepsilon, t) = \mathbb{E}_{\widetilde{L}}[\Phi(\varepsilon + \widetilde{L}_{(1-t)\,\bullet})] = \Pi_{1-t}\Phi(\varepsilon) \tag{3.6}$$

where $\widetilde{L}$ is an independent copy of L.
We are then led to exhibit such space-time harmonic functions.

We first state some general results on space-time harmonic functions. I always denotes $[0, a]$ $(a > 0)$ or $[0, \infty)$.

Proposition 3.1. *Let $f(x, t)$ be a space-time harmonic function for $(L_t\ ,\ t \in I)$. Then the function*

$$F(\varepsilon, t) = f(\varepsilon(1), t)$$

is a space-time harmonic function for $(X_{\bullet,t}\ ,\ t \in I)$

Proof. We have, for $s, t \in I$, $s < t$,

$$\mathbb{E}[F(X_{\bullet,t}, t) \mid \mathcal{X}_s] = P_{t-s} f^t(X_{1,s}) = f(X_{1,s}, s) = F(X_{\bullet,s}, s)$$

□

Proposition 3.2. *Suppose that $F(\varepsilon, t)$ is a space-time harmonic function for $(X_{\bullet,t}\ ,\ t \in I)$. Then, for $r > 0$, the function*

$$F_r(\varepsilon, t) = F(\varepsilon(r\bullet)), rt)$$

is a space-time harmonic function for $(X_{\bullet,t}\ ,\ t \in r^{-1}I)$.

Proof. Let $s, t \in r^{-1}I$ with $s < t$. Since $X_{r\bullet,t} - X_{r\bullet,s}$ is independent of $\mathcal{X}_s$ and is identical in law to $X_{\bullet,r(t-s)}$, we have

$$\mathbb{E}[F(X_{r\bullet,t}, rt) \mid \mathcal{X}_s] = \Pi_{r(t-s)} F^{rt}(X_{r\bullet,s})$$

Therefore

$$\Pi_{t-s} F_r^t(X_{\bullet,s}) = F^{rs}(X_{r\bullet,s}) = F_r^s(X_{\bullet,s})$$

The proof is complete. □

We deduce directly from the above propositions the following corollaries.

Corollary 3.1. *Let $f(x, t)$ be a space-time harmonic function for $(L_t\ ,\ t \in I)$. Then, for $r > 0$, the function*

$$F_r(\varepsilon, t) = f(\varepsilon(r), rt)$$

is a space-time harmonic function for $(X_{\bullet,t}\ ,\ t \in r^{-1}I)$.

Corollary 3.2.
Suppose that $F(\varepsilon, t)$ is a space-time harmonic function for $(X_{\bullet,t}\ ,\ t \in I)$. Then $\widehat{F}$ defined by

$$\widehat{F}(\varepsilon, t) = \int_0^1 F_r(\varepsilon, t)\, \mathrm{d}r$$

also is a space-time harmonic function for $(X_{\bullet,t}\ ,\ t \in I)$.

Corollary 3.3.
Suppose that $f(x, t)$ is a space-time harmonic function for $(L_t\ ,\ t \in I)$. Then $\widehat{f}$ defined by

$$\widehat{f}(\varepsilon, t) = \int_0^1 f(\varepsilon(r), rt)\, \mathrm{d}r$$

is a space-time harmonic function for $(X_{\bullet,t}\ ,\ t \in I)$.

3.3. Some examples

Example 1. We recall that μ_t denotes the law of L_t. Let $u_0 = 0 < u_1 < \cdots < u_n$ and $u = (u_1, \cdots, u_n)$. We set:

$$\nu_t^u = \bigotimes_{j=1}^n \mu_{t(u_j - u_{j-1})}$$

We consider $h \in L^1(\nu_1^u)$ and

$$\Phi(\varepsilon) = h(\varepsilon(u_1), \varepsilon(u_2) - \varepsilon(u_1), \cdots, \varepsilon(u_n) - \varepsilon(u_{n-1}))$$

Then

$$F^{(\Phi)}(\varepsilon, t) \\ = \int_{\mathbb{R}^n} h(\varepsilon(u_1) + y_1, \varepsilon(u_2) - \varepsilon(u_1) + y_2, \cdots, \varepsilon(u_n) - \varepsilon(u_{n-1}) + y_n)\, \nu_{1-t}^u(\mathrm{d}y)$$

This is a straightforward consequence of formula (3.6), since ν_t^u is the law of

$$(L_{tu_1}, L_{tu_2} - L_{tu_1}, \cdots, L_{tu_n} - L_{tu_{n-1}})$$

Example 2. We recall that ψ denotes the characteristic exponent of L. Let $0 = u_0 < u_1 < \cdots < u_n$ and $\lambda_0, \cdots, \lambda_{n-1} \in \mathbb{R}$. Let, for $(\varepsilon, t) \in \mathbb{D}_0 \times \mathbb{R}_+$,

$$F(\varepsilon, t) = \exp\left[\sum_{j=0}^{n-1} [\mathrm{i}\, \lambda_j\, (\varepsilon(u_{j+1}) - \varepsilon(u_j)) + t\, \psi(\lambda_j)\, (u_{j+1} - u_j)]\right]$$

Then F is a space-time harmonic function for $(X_{\bullet,t}\ ,\ t \geq 0)$. As a consequence, if

$$\Phi(\varepsilon) = \exp\left[\sum_{j=0}^{n-1} [\mathrm{i}\, \lambda_j\, (\varepsilon(u_{j+1}) - \varepsilon(u_j))]\right]$$

then

$$F^{(\Phi)}(\varepsilon, t) = \exp\left[\sum_{j=0}^{n-1} [\mathrm{i}\, \lambda_j\, (\varepsilon(u_{j+1}) - \varepsilon(u_j)) + (t-1)\, \psi(\lambda_j)\, (u_{j+1} - u_j)]\right]$$

We note that the above space-time harmonic function F is a particular case of functions of the type

$$\exp\left[\mathrm{i} \int h(u)\, \mathrm{d}\varepsilon(u) + t \int \psi(h(u))\, \mathrm{d}u\right]$$

which can be well defined, for some classes of functions h (see, in particular, Section 5).

Example 3. Let $r > 0$ and $f \in L^1(\mu_r)$. We set, for $(x, t) \in \mathbb{R} \times [0, r]$,

$$\widetilde{f}(x, t) = P_{r-t} f(x)$$

Then $\widetilde{f}$ is a space-time harmonic function for $(L_t\ ,\ t \in [0, r])$. Therefore, by Corollary 3.1,

$$F(\varepsilon, t) := \widetilde{f}(\varepsilon(r), rt)$$

is a space-time harmonic function for $(X_{\bullet,t}\ ,\ t \in [0,1])$. In particular, if $\Phi(\varepsilon) = f(\varepsilon(r))$, then

$$F^{(\Phi)}(\varepsilon, t) = \widetilde{f}(\varepsilon(r), rt)$$

Moreover we have, for $0 \leq t \leq 1$,

$$\mathbb{E}[f(L_r) \mid \mathcal{F}_{rt}] = \widetilde{f}(L_{rt}, rt)$$

As a consequence, $(\widetilde{f}(L_{rt}, rt)\ ,\ t \in [0,1])$ is an $(\mathcal{F}_{rt})$-martingale.
On the other hand, by Corollary 3.3, the function

$$G(\varepsilon, t) := \int_0^r \widetilde{f}(\varepsilon(u), ut)\, \mathrm{d}u$$

is a space-time harmonic function for $(X_{\bullet,t}\ ,\ t \in [0,1])$. In particular, if

$$\Phi = \int_0^r \widetilde{f}(\varepsilon(u), u)\, \mathrm{d}u$$

then $F^{(\Phi)} = G$.

4. The processes $(\Phi_t^\sharp(L)\ ,\ 0 \leq t \leq 1)$

4.1. Definition and relation with $(\Phi_t^m(X)\ ,\ 0 \leq t \leq 1)$

To any $\Phi \in L^1(\mathbb{P})$, we now associate a process $\Phi^\sharp(L) = (\Phi_t^\sharp(L)\ ,\ 0 \leq t \leq 1)$ by

$$\Phi_t^\sharp(L) = \mathbb{E}_{\widetilde{L}}\left[\Phi(L_{t\,\bullet} + \widetilde{L}_{(1-t)\,\bullet})\right]\ ,\quad 0 \leq t \leq 1 \tag{4.1}$$

where $\widetilde{L}$ is an independent copy of L, and $\mathbb{E}_{\widetilde{L}}$ means integrating with respect to $\widetilde{L}$.
In what follows, we will often denote $\Phi_t^\sharp(L)$ (resp. $\Phi^\sharp(L)$) simply by $\Phi_t^\sharp$ (resp. $\Phi^\sharp$). Moreover, we identify in our notation, the coordinate process on $\mathbb{D}_0$ with the process L, that is we identify $\varepsilon(\bullet)$ with $L_\bullet$. We have:

$$\Phi_0^\sharp = \mathbb{E}(\Phi),\ \ \Phi_1^\sharp = \Phi,\ \ \mathbb{E}(\Phi_t^\sharp) = \mathbb{E}(\Phi)$$

Theorem 4.1. *We have, with the notation of Section* 3*,*

$$\Phi_t^\sharp = \Pi_{1-t}\Phi(L_{t\,\bullet}) = F^{(\Phi)}(L_{t\,\bullet}, t) \tag{4.2}$$

In particular, for any given $t \in [0,1]$*,*

$$\Phi_t^\sharp \overset{(\text{law})}{=} \Phi_t^m$$

Hence, $\Phi^\sharp$ *is a* 1*-martingale.*

Proof. Formula (4.2) follows directly from (3.3), (3.4), (3.6) and (4.1). Now, since $L_{t\,\bullet}$ has the same law as $X_{\bullet,t}$, we clearly have from (3.4) and (4.2): $\Phi_t^\sharp \overset{(\text{law})}{=} \Phi_t^m$. □

4.2. A chaos decomposition formula for $\Phi_t^\sharp$ in the Poisson case

We first recall that, among Lévy processes, only Brownian motion (with drift) and the Poisson process enjoy the chaos decomposition property. In the case $L = B$, we presented in Hirsch-Yor [9] a formula for $\Phi_t^\sharp(B)$, based on the chaos decomposition of Φ. We now derive such a formula when $L = N$ is the standard Poisson process, or rather (equivalently), when $(L_t = N_t - t\ ,\ t \geq 0)$ is the centered Poisson process. In this case, any $\Phi(L)$ which belongs to $L^2(\mathbb{P})$ may be written as:

$$\text{(4.3)}\quad \Phi(L) = \mathbb{E}(\Phi) + \sum_{n=1}^{\infty} \int_0^{\infty} \mathrm{d}L_{s_1} \int_0^{s_1-} \mathrm{d}L_{s_2} \cdots \int_0^{s_{n-1}-} \mathrm{d}L_{s_n}\ \varphi_n(s_1, \cdots, s_n)$$

with the sequence (φ_n) satisfying

$$\sum_{n=1}^{\infty} \int_0^{\infty} \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \cdots \int_0^{s_{n-1}} \mathrm{d}s_n\ \varphi_n^2(s_1, \cdots, s_n) < \infty$$

We may then deduce from formula (4.1) the following representation of $\Phi_t^\sharp(L)$.

Proposition 4.1. *Let $\Phi \in L^2(\mathbb{P})$ and $L = (N_t - t\ ,\ t \geq 0)$. Then, with the previous notation, the following formula holds for t fixed,* $0 \leq t \leq 1$:

$$\Phi_t^\sharp(L) = \mathbb{E}(\Phi) + \sum_{n=1}^{\infty} \int_0^{\infty} \mathrm{d}L_{t\,s_1} \int_0^{s_1-} \mathrm{d}L_{t\,s_2} \cdots \int_0^{s_{n-1}-} \mathrm{d}L_{t\,s_n}\ \varphi_n(s_1, \cdots, s_n)]$$

Proof. Combining formulae (4.1) and (4.3), we obtain the formula stated in the Proposition, since in formula (4.1), the expectation with respect to $\widetilde{L}$ of any stochastic integral involved is equal to 0, as $\widetilde{L}$ is a martingale.

Note that we might also use Proposition 5.5 below, and reason by induction on the order of the chaos. □

4.3. Some examples

We now present examples corresponding to those given in Subsection 3.3.

Example 1. Let $u_0 = 0 < u_1 < \cdots < u_n$ and $u = (u_1, \cdots, u_n)$. We set:

$$\nu_t^u = \bigotimes_{j=1}^{n} \mu_{t(u_j - u_{j-1})}$$

We consider $h \in L^1(\nu_1^u)$ and

$$\Phi = h(L_{u_1}, L_{u_2} - L_{u_1}, \cdots, L_{u_n} - L_{u_{n-1}})$$

Then

$$\Phi_t^\sharp = \int_{\mathbb{R}^n} h(L_{tu_1} + y_1, L_{tu_2} - L_{tu_1} + y_2, \cdots, L_{tu_n} - L_{tu_{n-1}} + y_n)\, \nu_{1-t}^u(\mathrm{d}y)$$

Example 2. Let $0 = u_0 < u_1 < \cdots < u_n$ and $\lambda_0, \cdots, \lambda_{n-1} \in \mathbb{R}$. As a consequence of Example 2 in Subsection 3.3, the process

$$\exp\left[\sum_{j=0}^{n-1} [\mathrm{i}\, \lambda_j\, (L_{tu_{j+1}} - L_{tu_j}) + t\, \psi(\lambda_j)\, (u_{j+1} - u_j)]\right]\ ,\ t \geq 0$$

is a 1-martingale.

Set,

$$\Phi^{\lambda,u} = \exp\left[\sum_{j=0}^{n-1} [\mathrm{i}\, \lambda_j\, (L_{u_{j+1}} - L_{u_j}) + \psi(\lambda_j)\, (u_{j+1} - u_j)]\right]$$

Then

$$(\Phi^{\lambda,u})_t^\sharp = \Phi^{\lambda,tu} = \exp\left[\sum_{j=0}^{n-1} [\mathrm{i}\, \lambda_j\, (L_{tu_{j+1}} - L_{tu_j}) + t\, \psi(\lambda_j)\, (u_{j+1} - u_j)]\right]$$

Moreover, as we saw above, the process $(\Phi^{\lambda,tu}\ ,\ t \geq 0)$ is a 1-martingale.

Here again, we could consider more generally, as in Section 5,

$$\Phi = \exp\left[\mathrm{i} \int h(u)\, \mathrm{d}L_u + \int \psi(h(u))\, \mathrm{d}u\right]$$

for which we have

$$\Phi_t^\sharp = \exp\left[\mathrm{i} \int h(u/t)\, \mathrm{d}L_u + t \int \psi(h(u))\, \mathrm{d}u\right]$$

Example 3. We draw below, some direct consequences of the previous example.
Let, for $a > 0$,

$$\Delta_n^a = \{u = (u_1, \cdots, u_n)\ ;\ 0 < u_1 < \cdots < u_n \leq a\}$$

Then the process

$$\begin{aligned} t^{-n} &\int_{\Delta_n^{ta}} \Phi^{\lambda,u}\, \mathrm{d}u \\ &= t^{-n} \int_{\Delta_n^{ta}} \exp\left[\sum_{j=0}^{n-1} [\mathrm{i}\, \lambda_j\, (L_{u_{j+1}} - L_{u_j}) + \psi(\lambda_j)\, (u_{j+1} - u_j)]\right]\, \mathrm{d}u\ ,\ t \geq 0 \end{aligned}$$

is a 1-martingale. If we set

$$\Phi = \int_{\Delta_n^a} \Phi^{\lambda,u} \, \mathrm{d}u = \int_{\Delta_n^a} \exp\left[\sum_{j=0}^{n-1} [\mathrm{i}\, \lambda_j \, (L_{u_{j+1}} - L_{u_j}) + \psi(\lambda_j) \, (u_{j+1} - u_j)]\right] \, \mathrm{d}u$$

then, for $0 < t \leq 1$,

$$\begin{aligned} \Phi_t^\sharp &= t^{-n} \int_{\Delta_n^{ta}} \Phi^{\lambda,u} \, \mathrm{d}u \\ &= t^{-n} \int_{\Delta_n^{ta}} \exp\left[\sum_{j=0}^{n-1} [\mathrm{i}\, \lambda_j \, (L_{u_{j+1}} - L_{u_j}) + \psi(\lambda_j) \, (u_{j+1} - u_j)]\right] \, \mathrm{d}u \end{aligned}$$

In the particular case $n = 1$, we have the following extension of our guiding example in Subsection 1.2:
Let $\lambda \in \mathbb{R}$ and $a > 0$. If

$$\Phi = \int_0^a \exp\left(\mathrm{i}\, \lambda \, L_u + \psi(\lambda) \, u\right) \, \mathrm{d}u$$

then, for any $0 < t \leq 1$,

$$\Phi_t^\sharp = \frac{1}{t} \int_0^{at} \exp\left(\mathrm{i}\, \lambda \, L_u + \psi(\lambda) \, u\right) \, \mathrm{d}u$$

and $\Phi_0^\sharp = a$. Moreover, the process

$$\frac{1}{t} \int_0^{t} \exp\left(\mathrm{i}\, \lambda \, L_u + \psi(\lambda) \, u\right) \, \mathrm{d}u \, , \; t \geq 0$$

is a 1-martingale.

In some cases, it is possible to replace the above exponentials with purely imaginary arguments by real-valued exponentials.
Suppose that for some $r > 0$ and $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left[\exp(\lambda \, L_r)\right] < \infty$$

Then, there exists $\phi(\lambda) \in \mathbb{R}$ such that

$$\forall t \geq 0 \quad \mathbb{E}\left[\exp(\lambda \, L_t)\right] = \exp\left(t \, \phi(\lambda)\right)$$

Let, for $u > 0$,

$$\Phi^u = \exp(\lambda \, L_u - u \, \phi(\lambda))$$

Then, for $0 \leq t \leq 1$,

$$(\Phi^u)_t^\sharp = \exp(\lambda \, L_{t\,u} - t \, u \, \phi(\lambda))$$

As a consequence, if

$$\Phi = \int_0^a \exp(\lambda\, L_u - u\, \phi(\lambda))\, \mathrm{d}u$$

then, for $0 < t \leq 1$,

$$\Phi_t^\sharp = \frac{1}{t} \int_0^{at} \exp(\lambda\, L_u - u\, \phi(\lambda))\, \mathrm{d}u$$

Moreover, the process

$$\frac{1}{t} \int_0^t \exp(\lambda\, L_u - u\, \phi(\lambda))\, \mathrm{d}u \ , \ t \geq 0$$

is a 1-martingale.

Example 4. Let $r > 0$ and $f \in L^1(\mu_r)$. We set, for $(x,t) \in \mathbb{R} \times [0,r]$,

$$\widetilde{f}(x,t) = P_{r-t} f(x)$$

and $\Phi = f(L_r)$. Then

$$\Phi_t^\sharp = \widetilde{f}(L_{rt}, rt)$$

As we saw in Subsection 3.3, $(\Phi_t^\sharp \ , \ t \in [0,1])$ is an $(\mathcal{F}_{rt})$-martingale. Moreover,

$$\Phi_t^m = \widetilde{f}(X_{r,t}, rt)$$

Consequently, in this particular case, the processes Φ^m and $\Phi^\sharp$ have the same law.
On the other hand, if

$$\Phi = \int_0^r \widetilde{f}(L_u, u)\, \mathrm{d}u$$

then

$$\Phi_t^\sharp = \frac{1}{t} \int_0^{rt} \widetilde{f}(L_u, u)\, \mathrm{d}u$$

As a variant, we may consider a space-time harmonic function $f(x,t)$ for $(L_t \ , \ t \geq 0)$. Then, by Corollary 3.3, the process

$$\frac{1}{t} \int_0^t f(L_u, u)\, \mathrm{d}u \ , \ t \geq 0$$

is a 1-martingale. Taking, as a particular case, $f(x,t) = \exp(\mathrm{i}\, \lambda\, x + t\, \psi(\lambda))$, we recover the example mentioned in the previous paragraph.

Example 5. Here are two examples of application of the previous paragraph, for two particular cases of Lévy processes, namely $(\gamma_t\ ,\ t \geq 0)$ a standard Gamma process, and $(N_t\ ,\ t \geq 0)$ a standard Poisson process.
We recall (see, e.g., Schoutens [16]) the following generating function:

$$(1+\lambda)^u \exp(-\lambda\, g) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \widetilde{C}_n(u,g)\ , \quad u, g, \lambda \geq 0 \tag{4.4}$$

where $\{\widetilde{C}_n(u,g)\ ,\ n \in \mathbb{N}\}$ denotes the sequence of Charlier polynomials. Now, we note that, for every $\lambda \geq 0$,

$$((1+\lambda)^t \exp(-\lambda\,\gamma_t)\ ,\ t \geq 0) \text{ is a Gamma martingale,}$$
$$\text{and } ((1+\lambda)^{N_t} \exp(-\lambda\, t)\ ,\ t \geq 0) \text{ is a Poisson martingale.}$$

(Gamma (resp. Poisson) martingale means martingale with respect to the natural filtration of the process (γ_t) (resp. (N_t).) Consequently, from (4.4), for any $n \in \mathbb{N}$, $\{\widetilde{C}_n(t,\gamma_t)\ ,\ t \geq 0\}$ and $\{\widetilde{C}_n(N_t,t)\ ,\ t \geq 0\}$ are, respectively, a Gamma martingale, and a Poisson martingale. In other words, the function $\widetilde{C}_n(t,x)$ (resp. $\widetilde{C}_n(x,t)$) is a space-time harmonic function for $(\gamma_t\ ,\ t \geq 0)$ (resp. $(N_t\ ,\ t \geq 0)$). Finally, from the result at the end of the previous paragraph, we obtain that, for any $n \in \mathbb{N}$,

$$\left(\frac{1}{t}\int_0^t \widetilde{C}_n(s,\gamma_s)\,\mathrm{d}s\ ,\ t \geq 0\right) \quad\text{and}\quad \left(\frac{1}{t}\int_0^t \widetilde{C}_n(N_s,s)\,\mathrm{d}s\ ,\ t \geq 0\right)$$

are 1-martingales.

4.4. Some properties of the map: $\Phi \longrightarrow \Phi^\sharp$

In this subsection, we present some general results concerning the map: $\Phi \longrightarrow \Phi^\sharp$. They extend those obtained by Hirsch-Yor [9] in the Brownian setting, that is when $L = B$ is a Brownian motion.

In what follows, we denote, for $1 \leq p < \infty$, simply by L^p the L^p-space with respect to $\mathbb{P}$. For $0 \leq r \leq \infty$, $L^p(\mathcal{F}_r)$ denotes the subspace of L^p consisting of those functions which are $\mathcal{F}_r$-measurable. $\|\ \|_p$ denotes the L^p-norm.

Theorem 4.2. *Let $1 \leq p < \infty$, $r \in [0,\infty]$ and $\Phi \in L^p(\mathcal{F}_r)$. Then,*

1) *for any $t \in [0,1]$, $\Phi_t^\sharp \in L^p(\mathcal{F}_{t\,r})$ (with the usual convention: $t\,\infty = \infty$ if $t \neq 0$ and $0\,\infty = 0$);*

2) *for any $t \in [0,1]$, $\|\,\Phi_t^\sharp\,\|_p \leq \|\,\Phi\,\|_p$;*

3) *the map:*

$$t \in [0,1] \longrightarrow \Phi_t^\sharp \in L^p$$

is continuous;

4) *for any* $t, s \in [0,1]$,

$$(\Phi^\sharp_t)^\sharp_s = \Phi^\sharp_{t\,s}$$

Proof. Property 1) is clear by definition of $\Phi^\sharp$ (formula (4.1)).
We also have by (4.1):

$$|\Phi^\sharp_t|^p \leq \mathbb{E}_{\widetilde{L}} \left[|\Phi|^p (L_{t\,\bullet} + \widetilde{L}_{(1-t)\,\bullet}) \right]$$

Therefore,

$$\| \Phi^\sharp_t \|^p_p \leq \mathbb{E}_{(L,\widetilde{L})} \left[|\Phi|^p (L_{t\,\bullet} + \widetilde{L}_{(1-t)\,\bullet}) \right]$$

By the remark in Subsection 2.5, the right hand side is equal to $\| \Phi \|^p_p$.

Let now $h_1, \cdots, h_n$ be functions with compact support in $[0,\infty)$, integrable with respect to the Lebesgue measure, and let φ be a bounded continuous function on $\mathbb{R}^n$. We consider

$$\Phi = \varphi \left(\int_0^\infty L_u\, h_1(u)\, \mathrm{d}u, \cdots, \int_0^\infty L_u\, h_n(u)\, \mathrm{d}u \right)$$

Then

$$\Phi^\sharp_t = \mathbb{E}_{\widetilde{L}} \left[\varphi \left(\int_0^\infty L_{tu}\, h_1(u)\, \mathrm{d}u + \int_0^\infty \widetilde{L}_{(1-t)u}\, h_1(u)\, \mathrm{d}u, \cdots \right) \right]$$

Since the paths of L and $\widetilde{L}$ have a countable set of discontinuities, a dominated convergence argument yields:

$$\forall t_0 \geq 0 \quad \lim_{t \to t_0} \Phi^\sharp_t = \Phi^\sharp_{t_0} \quad \text{a.s.}$$

and, therefore, L^p-continuity also holds. Now, the functions Φ of the above kind are dense in L^p, which, thanks to Property 2), implies Property 3) .

By formula (4.1) again,

$$(\Phi^\sharp_t)^\sharp_s = \mathbb{E}_{(\widehat{L},\widetilde{L})} \left[\Phi(L_{ts\bullet} + \widehat{L}_{t(1-s)\bullet} + \widetilde{L}_{(1-t)\bullet}) \right]$$

where $(L, \widehat{L}, \widetilde{L})$ are three independent copies. According to Subsection 2.5, $\widehat{L}_{t(1-s)\bullet} + \widetilde{L}_{(1-t)\bullet}$ has the same law as $L_{(1-ts)\bullet}$ and is independent of L. Property 4) therefore follows from (4.1). □

4.5. Some further examples

4.5.1 The following example is the analogue of the third example in Hirsch-Yor [9, Subsection 3.4].

Proposition 4.2. *Let* $F \in L^1(\mathbb{P})$ *and* ℓ *be a Borel function on* $[0,1]$.

1) *Assume*

$$\int_0^1 |\ell(u)|\,\mathrm{d}u < \infty \tag{4.5}$$

We consider

$$\Phi = \int_0^1 F_u^\sharp\,\ell(u)\,\mathrm{d}u$$

Then, for $0 < t \leq 1$,

$$\Phi_t^\sharp = \frac{1}{t}\int_0^t F_u^\sharp\,\ell\left(\frac{u}{t}\right)\,\mathrm{d}u \tag{4.6}$$

2) *Assume that* ℓ *is absolutely continuous on* $]0,1]$ *and*

$$\int_0^1 |\ell'(u)|\,u\,\mathrm{d}u < \infty \tag{4.7}$$

Then (4.5) *holds and* $\Phi^\sharp$ *defined by* (4.6) *is a process with finite variation on* $(0,1]$.

Proof. We have, by Property 4) in Theorem 4.2:

$$\Phi_t^\sharp = \int_0^1 F_{u\,t}^\sharp\,\ell(u)\,\mathrm{d}u$$

and therefore, formula (4.6) holds.

The proof of Property 2) is similar to the proof of Proposition 3.7 in Hirsch-Yor [9]. We actually have, on $(0,1]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^\sharp = \frac{1}{t}\left[F_t^\sharp\,\ell(1) - \int_0^1 F_{ut}^\sharp\,[\ell(u) + u\,\ell'(u)]\,\mathrm{d}u\right]$$

In particular, for $0 < a < 1$,

$$\int_a^1 \|\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^\sharp\|_1\,\mathrm{d}t < \infty$$

□

4.5.2 We extend, in this paragraph, the first part of Proposition 3.11 in Hirsch-Yor [9].

We assume:

$$\int_{\mathbb{R}} \frac{1}{1 + \Re\psi(z)}\,\mathrm{d}z < \infty \tag{4.8}$$

Let $a \in \mathbb{R}$ and $r > 0$. We take as functional Φ, the local time of L at level a and time r (see Bertoin [1, Chapter V], where condition (4.8) is shown to ensure the existence of these local times).

Proposition 4.3. *We have, for $t \in [0,1)$,*

$$\Phi_t^\sharp = \frac{1}{2\pi} \int_0^r \mathrm{d}s \int_{-\infty}^{+\infty} \mathrm{d}z \, \exp\left[\mathrm{i}\,(L_{st} - a)z - s(1-t)\psi(z)\right]$$

Proof. Denote, for $\epsilon > 0$, by φ_ϵ the indicator function of the interval $[-\epsilon, +\epsilon]$. We set

$$\Phi_\epsilon = \frac{1}{2\epsilon} \int_0^r \varphi_\epsilon(L_s - a)\, \mathrm{d}s$$

In view of Example 4 in Subsection 4.3, we have for $0 \le t < 1$,

$$(\Phi_\epsilon)_t^\sharp = \frac{1}{2\epsilon} \int_0^r \mathrm{d}s \int \varphi_\epsilon(L_{st} - a + y)\, \mu_{s(1-t)}(\mathrm{d}y)$$

A computation by Fourier transform, taking into account the assumption (4.8), yields:

$$(4.9) \qquad (\Phi_\epsilon)_t^\sharp = \frac{1}{2\pi} \int_0^r \mathrm{d}s \int_{-\infty}^{+\infty} \mathrm{d}z \, \exp\left[\mathrm{i}\,(L_{st} - a)z - s(1-t)\psi(z)\right] \frac{\sin(\epsilon z)}{\epsilon z}$$

We know (Bertoin [1, Chapter V]) that

$$\lim_{\epsilon \to 0} \Phi_\epsilon = \Phi \quad \text{in } L^2$$

Therefore,

$$\Phi_t^\sharp = \lim_{\epsilon \to 0} (\Phi_\epsilon)_t^\sharp \quad \text{in } L^2$$

Consequently, taking the limit as ϵ tends to 0 in (4.9), we obtain by dominated convergence, thanks to (4.8), the announced result. □

4.6. A Markov semigroup related to the processes $\Phi^\sharp$

We can interpret the family of maps: $\Phi \longrightarrow \Phi_t^\sharp$, indexed by $t \le 1$, as defined in Theorem 4.2, in terms of a Markovian semigroup (Q_h), where t and h are related by: $t = \mathrm{e}^{-h}$.

Let $1 \le p < \infty$. We set, for $\Phi \in L^p(\mathbb{P})$ and $h \ge 0$,

$$Q_h \Phi = \Phi_{\mathrm{e}^{-h}}^\sharp$$

In other words, we have with the notation of Subsection 4.1,

$$Q_h \Phi = \mathbb{E}_{\widetilde{L}}\left[\Phi(L_{\mathrm{e}^{-h}\,\bullet} + \widetilde{L}_{(1-\mathrm{e}^{-h})\,\bullet})\right], \quad h \ge 0$$

We now may state Theorem 4.2 in the following way.

Proposition 4.4. $Q = (Q_h\ ,\ h \ge 0)$ *is a Markovian strongly continuous semigroup on* L^p*. Moreover,*

$$\forall h \ge 0,\ \forall \Phi \in L^p, \quad \| Q_h \Phi \|_p \le \| \Phi \|_p$$

and

$$\lim_{h \to \infty} Q_h \Phi = \mathbb{E}(\Phi) \quad \textit{in } L^p$$

A Markov process can now be associated with the semigroup Q.

Theorem 4.3. *We define a* $\mathbb{D}_0$*-valued process*: $(Y^h\ ,\ h \geq 0)$, *by*

$$Y^h_\bullet = X_{\mathrm{e}^{-h}\bullet,\mathrm{e}^h}$$

Then, if $\Phi \in L^1(\mathbb{P})$, *for all* $h, k \geq 0$,

$$\mathbb{E}\left[\Phi(Y^{h+k}) \mid \mathcal{X}_{\mathrm{e}^h}\right] = Q_k \Phi(Y^h)$$

Proof. We write $\Phi(Y^{h+k})$ as:

$$\Phi\left(X_{\mathrm{e}^{-(h+k)}\bullet,\mathrm{e}^h} + X_{\mathrm{e}^{-(h+k)}\bullet,\mathrm{e}^{h+k}} - X_{\mathrm{e}^{-(h+k)}\bullet,\mathrm{e}^h}\right)$$

Therefore, by Theorem 2.2,

$$\begin{aligned}\mathbb{E}\left[\Phi(Y^{h+k}) \mid \mathcal{X}_{\mathrm{e}^h}\right] &= \mathbb{E}_{\widetilde{L}}\left[\Phi(Y^h_{\mathrm{e}^{-k}\bullet} + \widetilde{L}_{(1-\mathrm{e}^{-k})\bullet})\right] \\ &= Q_k \Phi(Y^h)\end{aligned}$$

□

5. Examples of processes Φ^m and $\Phi^\sharp$ obtained from stochastic integrals with respect to L

5.1. Definition of the stochastic integrals with respect to L

It is well-known that any Lévy process L can be written as the sum of three independent Lévy processes $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ where:

- $L^{(1)}$ is a Brownian motion with drift,
- $L^{(2)}$ is a compound Poisson process with jumps of size at least 1,
- $L^{(3)}$ is a pure-jump martingale having only jumps of size less than 1.

We are interested in this section in functionals Φ defined from stochastic integrals $\int_0^\infty H_s\, \mathrm{d}L_s$. (We identify, as previously, L with the coordinate process on $\mathbb{D}_0$.)

If L is Brownian, these functionals were studied in Hirsch-Yor [9].

If L is a compound Poisson process, stochastic integrals are ordinary Stieltjes integrals and the study is rather simple.

Consequently, we concentrate our attention, in this section, on the third part of L. A little more generally, we assume in the rest of this section, that the characteristic exponent ψ of L is

$$\psi(\lambda) = \int \left(1 - \mathrm{e}^{\mathrm{i}\lambda x} + \mathrm{i}\,\lambda\, x\right)\ \nu(\mathrm{d}x)$$

with ν a positive measure on $\mathbb{R} \setminus \{0\}$ satisfying

$$0 < a := \int x^2\ \nu(\mathrm{d}x) < \infty$$

As a consequence,

$$\mathbb{E}(L_t) = 0 \quad \text{and} \quad \mathbb{E}(L_t^2) = a\,t$$

We denote in what follows, by $\mathcal{H}$ the space of $(\mathcal{F}_s)$-predictable processes $H = (H_s\ ,\ s \geq 0)$ such that

$$\int_0^\infty \mathbb{E}(H_s^2)\ \mathrm{d}s < \infty$$

We then obtain easily:

Proposition 5.1. *Let $H \in \mathcal{H}$. Then, the stochastic integral*

$$\int_0^\infty H_s\ \mathrm{d}L_s$$

is well defined as an element of L^2 and

$$\mathbb{E}\left[\left(\int_0^\infty H_s\ \mathrm{d}L_s\right)^2\right] = a \int_0^\infty \mathbb{E}(H_s^2)\ \mathrm{d}s \tag{5.1}$$

Moreover,

$$\left(\int_0^t H_s\ \mathrm{d}L_s\ ,\ t \geq 0\right)$$

is an $(\mathcal{F}_t)$-martingale.

5.2. Examples of processes Φ^m and $\Phi^\sharp$ defined from Wiener type integrals with respect to L

We study in this subsection, functionals of Wiener type integrals $\int_0^\infty h(s)\ \mathrm{d}L_s$ with $h \in L^2(\mathbb{R}_+)$.

Proposition 5.2. *Let $h \in L^2(\mathbb{R}_+)$. Then,*

$$\left(\int_0^\infty h(s)\ \mathrm{d}_s X_{s,t}\ ,\ t \geq 0\right)$$

is a Lévy process whose characteristic exponent is

$$\psi^h(\lambda) = \int_0^\infty \psi(\lambda\, h(s))\ \mathrm{d}s$$

Proof. We first remark:

$$|\psi(\lambda)| \leq \frac{a}{2}\,\lambda^2$$

The integral $\int_0^\infty \psi(\lambda\, h(s))\ \mathrm{d}s$ is therefore convergent.

By approximation of h by simple functions, we get:

$$\mathbb{E}\left[\exp\left(\mathrm{i}\,\lambda\int_0^\infty h(s)\,\mathrm{d}_s X_{s,t}\right)\right] = \mathbb{E}\left[\exp\left(\mathrm{i}\,\lambda\int_0^\infty h(s)\,\mathrm{d}_s L_{t\,s}\right)\right]$$
$$= \exp\left(-t\int_0^\infty \psi(\lambda\,h(s))\,\mathrm{d}s\right)$$

The rest of the proposition follows from the properties of the Lévy sheet X (Theorem 2.2). □

We denote in the sequel, by μ_t^h the law of $\int_0^\infty h(s)\,\mathrm{d}_s X_{s,t}$. The following extension of Example 4 in Subsection 4.3 holds.

Proposition 5.3. *Let $h \in L^2(\mathbb{R}_+)$ and $f \in L^1(\mu_1^h)$. We set, for $(x,t) \in \mathbb{R}\times[0,1]$,*

$$\widetilde{f}(x,t) = \int_{\mathbb{R}} f(x+y)\,\mu_{1-t}^h(\mathrm{d}y)$$

Let

$$\Phi = f\left(\int_0^\infty h(s)\,\mathrm{d}L_s\right)$$

Then, for $0 \le t \le 1$,

$$\Phi_t^\sharp = \widetilde{f}\left(\int_0^\infty h(s)\,\mathrm{d}_s L_{t\,s}\,,t\right) \quad \text{and} \quad \Phi_t^m = \widetilde{f}\left(\int_0^\infty h(s)\,\mathrm{d}_s X_{s,t}\,,t\right)$$

In particular, the function:

$$(L,t) \in \mathbb{D}_0\times[0,1] \longrightarrow \widetilde{f}\left(\int_0^\infty h(s)\,\mathrm{d}L_s\,,t\right)$$

is a space-time harmonic function for $(X_{\bullet,t}\,,\ 0\le t\le 1)$.

Proof. By formulae (3.3) and (4.1),

$$\Phi_t^m = \mathbb{E}_{\widetilde{L}}\left[f\left(\int_0^\infty h(s)\,\mathrm{d}_s X_{s,t} + \int_0^\infty h(s)\,\mathrm{d}_s\widetilde{L}_{(1-t)\,s}\right)\right]$$

and

$$\Phi_t^\sharp = \mathbb{E}_{\widetilde{L}}\left[f\left(\int_0^\infty h(s)\,\mathrm{d}_s L_{t\,s} + \int_0^\infty h(s)\,\mathrm{d}_s\widetilde{L}_{(1-t)\,s}\right)\right]$$

Now, as $\widetilde{L}_{(1-t)\,\bullet} \overset{(\mathrm{law})}{=} X_{\bullet,1-t}$, we have the announced result. □

Corollary 5.1. *Let $h \in L^2(\mathbb{R}_+)$ and*

$$\Phi = \exp\left(\mathrm{i}\int_0^\infty h(s)\,\mathrm{d}L_s + \int_0^\infty \psi(h(s))\,\mathrm{d}s\right)$$

Then, for $0 \le t \le 1$,

$$\Phi_t^m = \exp\left(\mathrm{i}\int_0^\infty h(s)\,\mathrm{d}_s X_{s,t} + t\int_0^\infty \psi(h(s))\,\mathrm{d}s\right)$$

and

$$\Phi_t^\sharp = \exp\left(\mathrm{i}\int_0^\infty h(s)\,\mathrm{d}_s L_{t\,s} + t\int_0^\infty \psi(h(s))\,\mathrm{d}s\right)$$

The above corollary leads to the example presented in Subsection 1.3. Likewise, Example 2 in the subsections 3.3 and 4.3 appear as a particular case of Corollary 5.1, taking

$$h(s) = \sum_{j=0}^{n-1} \lambda_j\, 1_{(u_j,u_{j+1}]}(s)$$

Finally, by the same kind of arguments as previously, we obtain the following extension of Example 2 in Subsection 3.3.

Proposition 5.4. *Let* ρ *be a bounded signed measure on* $L^2(\mathbb{R}_+)$ *such that*

$$\int_{L^2(\mathbb{R}_+)} \exp\left(\lambda\int_0^\infty h^2(s)\,\mathrm{d}s\right)\,|\rho(\mathrm{d}h)| < \infty$$

for every $\lambda > 0$. *Let* F *be defined (almost surely) on* $\mathbb{D}_0 \times \mathbb{R}_+$ *by*

$$F(L,t) = \int_{L^2(\mathbb{R}_+)} \exp\left(\mathrm{i}\int_0^\infty h(s)\,\mathrm{d}L_s + t\int_0^\infty \psi(h(s))\,\mathrm{d}s\right)\,\rho(\mathrm{d}h)$$

Then F *is a space-time harmonic function for* $(X_{\bullet,t}\ ,\ t \ge 0)$. *In particular, setting for* $t \ge 0$:

$$F_t^\sharp = \int_{L^2(\mathbb{R}_+)} \exp\left(\mathrm{i}\int_0^\infty h(s)\,\mathrm{d}_s L_{t\,s} + t\int_0^\infty \psi(h(s))\,\mathrm{d}s\right)\,\rho(\mathrm{d}h)$$

the process $(F_t^\sharp\ ,\ t \ge 0)$ *is a* 1*-martingale.*

5.3. Processes $\Phi^\sharp$ for Φ a stochastic integral with respect to L

Proposition 5.5. *Let* $H \in \mathcal{H}$ *and*

$$\Phi = \int_0^\infty H_s\,\mathrm{d}L_s$$

Then, for $0 \le t \le 1$,

$$\Phi_t^\sharp = \int_0^\infty (H_s)_t^\sharp\,\mathrm{d}_s L_{t\,s} \tag{5.2}$$

As a consequence,

$$\|\,\Phi_t^\sharp\,\|_2 \le \sqrt{t}\,\|\,\Phi\,\|_2 \tag{5.3}$$

Proof. We first remark that $((H_s)^\sharp_t\ ,\ s \geq 0)$ is $(\mathcal{F}_{ts})_{s\geq 0}$-predictable: it is indeed enough to consider left-continuous, bounded, adapted processes H, for which the property is clear by the definition (formula (4.1)). Then, (5.2) also follows easily from the definition. Now, by formula (5.1) and Property 2) in Theorem 4.2,

$$\| \Phi^\sharp_t \|_2^2 = t\, a \int_0^\infty \mathbb{E}\left[((H_s)^\sharp_t)^2\right]\ \mathrm{d}s \leq t\, a \int_0^\infty \mathbb{E}(H_s^2)\ \mathrm{d}s = t\, \| \Phi \|_2^2$$

which yields (5.3). □

The following corollary improves, in some cases, upon Proposition 4.2.

Corollary 5.2. *Let $H \in \mathcal{H}$ and $F = \int_0^\infty H_s\, \mathrm{d}L_s$. Let ℓ be an absolutely continuous function on $(0,1]$ such that*

$$\int_0^1 |\ell'(u)|\, u^{3/2}\ \mathrm{d}u < \infty \tag{5.4}$$

and let

$$\Phi = \int_0^1 F^\sharp_u\, \ell(u)\ \mathrm{d}u$$

Then $\Phi^\sharp$ is an absolutely continuous process on $[0,1]$ and its variation belongs to L^2.

Proof. The proof is similar to that of Proposition 3.9 in [9]. By Proposition 5.5, formula (5.3),

$$\| F^\sharp_t \|_2 \leq \sqrt{t}\, \| F \|_2 \tag{5.5}$$

Therefore, hypothesis (5.4) is sufficient to entail, as in Proposition 4.2, that $\Phi^\sharp$ is absolutely continuous on $(0,1]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^\sharp_t = \frac{1}{t}\ \left[F^\sharp_t\, \ell(1) - \int_0^1 F^\sharp_{ut}\, [\ell(u) + u\, \ell'(u)]\ \mathrm{d}u\right]$$

Consequently, by (5.5),

$$\| \frac{\mathrm{d}}{\mathrm{d}t}\Phi^\sharp_t \|_2 \leq \frac{\| F \|_2}{\sqrt{t}}\ \left[|\ell(1)| + \int_0^1 |u^{1/2}\ell(u) + u^{3/2}\ell'(u)|\ \mathrm{d}u\right]$$

and hence the announced result follows. □

The following result is an extension of Hirsch-Yor [9, Theorem 5.1]. The proof is quite similar and will be omitted.

Theorem 5.1. *Let $H \in \mathcal{H}$ and $\Phi = \int_0^1 H_s\ \mathrm{d}L_s$. Then $\Phi^\sharp$ is an $(\mathcal{F}_t)$-martingale if and only if the following condition is fulfilled:*

There exists a version of H which is L^2-continuous on $[0,1)$ and satisfies:

$$\forall u \in [0,1),\ \forall t \in [0,1], \quad (H_u)^\sharp_t = H_{u\,t}$$

In particular, if $F \in L^2(\mathcal{F}_1)$ and $\Phi = \int_0^1 F^\sharp_s\ \mathrm{d}L_s$, then $\Phi^\sharp$ is an $(\mathcal{F}_t)$-martingale.

6. Examples related to fractional α-stable processes

6.1. Fractional stable processes

In this section, we fix $\alpha \in (0,2]$, and we consider a $\mathbb{R}^2$-valued Lévy process: $L = (L^1, L^2)$, where L^1 and L^2 are two independent copies of a strictly α-stable $\mathbb{R}$-valued Lévy process with characteristic exponent:

$$\psi(\lambda) = c\,|\lambda|^\alpha (1 + \mathrm{i}\, b\, \mathrm{sign}(\lambda)), \quad c > 0 \text{ and } b \in \mathbb{R}$$

with, if $\alpha \neq 1$, $b = \beta \tan(\pi\,\alpha/2)$ for some $\beta \in [-1,1]$. In particular, for $\alpha = 2$, $\psi(\lambda) = c\,\lambda^2$, and L is a multiple of Brownian motion. For $\alpha = 1$, $\psi(\lambda) = c\,|\lambda| + \mathrm{i}\,\lambda\, d$, so that L is a symmetric Cauchy process with drift.

Let $H \in (0,1)$ and $\gamma = H - \frac{1}{\alpha}$. Following for example Samorodnitsky-Taqqu [15, Definition 7.4.1], we define a fractional stable motion $L^{\alpha,H}$, setting, for $t \geq 0$,

$$L_t^{\alpha,H} = \int_0^t (t-s)^\gamma \, \mathrm{d}L_s^1 + \int_0^\infty [(t+s)^\gamma - s^\gamma] \, \mathrm{d}L_s^2$$

(Here, for simplicity, we consider the process as defined only on $\mathbb{R}_+$, and not on $\mathbb{R}$ as usual.)
It is easy to see that this process has the scaling property of index H^{-1}, which means that, for any $k > 0$,

$$L_{k\bullet}^{\alpha,H} \overset{\text{(law)}}{=} k^H \, L_\bullet^{\alpha,H} \tag{6.1}$$

As a particular case, if $\alpha = 2$ (and hence $b = 0$) and

$$c = c_H := \left[H^{-1} + 2 \int_0^\infty [(1+s)^{H-1/2} - s^{H-1/2}]^2 \, \mathrm{d}s \right]^{-1}$$

then $L^{\alpha,H}$ is the classical fractional Brownian motion B^H with Hurst index H. In this case, L^1 and L^2 have the same law as $\sqrt{2\,c_H}\,B$, where B denotes a standard Brownian motion.
We set, for $\epsilon = \pm 1$,

$$d_\epsilon = c \left[\frac{1 + \mathrm{i}\,\epsilon\, b}{\alpha\, H} + (1 + \mathrm{i}\,\epsilon\, b\, \mathrm{sign}(\gamma)) \int_0^\infty |(1+s)^\gamma - s^\gamma|^\alpha \, \mathrm{d}s \right]$$

(for the fractional Brownian motion, $d_\epsilon = 1/2$). The following proposition, which is well-known (see [15]), can also be seen as a consequence of Proposition 5.2.

Proposition 6.1. *We have, for $\lambda \in \mathbb{R}$ and $t \geq 0$,*

$$\mathbb{E}\left[\exp(\mathrm{i}\,\lambda\, L_t^{\alpha,H}) \right] = \exp(-d_{\mathrm{sign}(\lambda)} |\lambda|^\alpha t^{\alpha\, H})$$

Proof. Though the hypotheses of Section 5 are not satisfied, we may extend Proposition 5.2 by approximation of the Lévy measure. Therefore,

$$\mathbb{E}\left[\exp(\mathrm{i}\,\lambda\, L_t^{\alpha,H})\right] =$$

$$\exp\left(-\left[\int_0^t \psi(\lambda\,(t-s)^\gamma)\,\mathrm{d}s + \int_0^\infty \psi(\lambda\,[(t+s)^\gamma - s^\gamma])\,\mathrm{d}s\right]\right)$$

The result follows from the expression of ψ. □

We denote, in the sequel, by $\mu_t^{\alpha,H}$ the law of $L_t^{\alpha,H}$.

6.2. Definition of the processes $\Phi^\sharp$ and Φ^m

We now introduce processes $\Phi^\sharp$ and Φ^m which are slightly different from those which were defined before, but better adapted to the present framework.

The Lévy process which is henceforth considered is $L = (L^1, L^2)$ presented in the previous subsection. The associated Lévy sheet is obviously $X = (X^1, X^2)$, where X^1 and X^2 are two independent Lévy sheets associated with L^1 and L^2. We consider the space $\mathbb{D}_0 \times \mathbb{D}_0$ equipped with the probability $\mathbb{P}\times\mathbb{P}$, which is the law of L. Here again, we identify the coordinate process $\varepsilon = (\varepsilon^1, \varepsilon^2)$ on $\mathbb{D}_0 \times \mathbb{D}_0$ with the process L.

As L has the scaling property of index α^{-1}, for any $t > 0$,

$$t^\gamma L_{t\,\bullet} \overset{\text{(law)}}{=} L_{t^{\alpha H}\,\bullet}$$

Therefore, if $\widetilde{L}$ is an independent copy of L,

$$t^\gamma L_{t\,\bullet} + \widetilde{L}_{(1-t^{\alpha H})\,\bullet} \overset{\text{(law)}}{=} L_\bullet$$

Now, if $\Phi \in L^1(\mathbb{P}\times\mathbb{P})$, we set, for $0 < t \le 1$,

$$\Phi_t^\sharp = \mathbb{E}_{\widetilde{L}}\left[\Phi(t^\gamma L_{t\,\bullet} + \widetilde{L}_{(1-t^{\alpha H})\,\bullet})\right] \text{ and } \Phi_t^m = \mathbb{E}_{\widetilde{L}}\left[\Phi(X_{\bullet, t^{\alpha H}} + \widetilde{L}_{(1-t^{\alpha H})\,\bullet})\right]$$

where, in the last equality, $\widetilde{L}$ is assumed independent of X.
We also set $\Phi_0^\sharp = \Phi_0^m = \mathbb{E}(\Phi)$.

The processes $(\Phi_t^\sharp\ ,\ 0 \le t \le 1)$ and $(\Phi_t^m\ ,\ 0 \le t \le 1)$ have quite similar properties as before (Sections 3, 4, 5). In particular:

Theorem 6.1. *The process* $(\Phi_t^m\ ,\ 0 \le t \le 1)$ *is a* $(\mathcal{X}_{t^{\alpha H}})$*-martingale and, for* $0 \le t \le 1$,

$$\Phi_t^\sharp \overset{\text{(law)}}{=} \Phi_t^m$$

As a consequence, $\Phi^\sharp$ *is a* 1*-martingale.*

6.3. Some examples

In the sequel, we consider some simple functionals of $L^{\alpha,H}$.

Proposition 6.2. *Let $r > 0$ and $f \in L^1(\mu_r^{\alpha,H})$. We consider $\Phi = f(L_r^{\alpha,H})$. Then, for $0 \le t \le 1$,*

$$\Phi_t^\sharp = \int f(L_{t\,r}^{\alpha,H} + (1-t^{\alpha H})^{1/\alpha}\, y)\, \mu_r^{\alpha,H}(\mathrm{d}y)$$

Proof. Let $a > 0$. Then we obtain by change of variable:

$$\forall t \ge 0, \quad \int_0^t (t-s)^\gamma\, \mathrm{d}L^1_{a\,s} + \int_0^\infty [(t+s)^\gamma - s^\gamma]\, \mathrm{d}L^2_{a\,s} = a^{-\gamma}\, L^{\alpha,H}_{a\,t}$$

Therefore, by the definition of $\Phi_t^\sharp$:

$$\Phi_t^\sharp = \mathbb{E}_{\widetilde{L}^{\alpha,H}}\left[f\left(L_{t\,r}^{\alpha,H} + (1-t^{\alpha H})^{-\gamma}\, \widetilde{L}^{\alpha,H}_{(1-t^{\alpha H})\,r}\right)\right]$$

where $\widetilde{L}^{\alpha,H}$ denotes an independent copy of $L^{\alpha,H}$. Now, by the scaling property (6.1),

$$(1-t^{\alpha H})^{-\gamma}\, \widetilde{L}^{\alpha,H}_{(1-t^{\alpha H})\,r} \overset{\text{(law)}}{=} (1-t^{\alpha H})^{1/\alpha}\, \widetilde{L}^{\alpha,H}_r$$

which gives the announced result. □

As a straightforward consequence of Propositions 6.1 and 6.2, we obtain the following important example.

Proposition 6.3. *We set, for $\lambda \in \mathbb{R}$ and $u \ge 0$,*

$$\Phi^{\lambda,u} = \exp\left(\mathrm{i}\,\lambda\, L_u^{\alpha,H} + d_{\mathrm{sign}(\lambda)}\, |\lambda|^\alpha\, u^{\alpha H}\right)$$

Then, for $0 \le t \le 1$,

$$(\Phi^{\lambda,u})_t^\sharp = \Phi^{\lambda,t\,u}$$

Corollary 6.1. *For every $\lambda \in \mathbb{R}$ and $r > 0$, the process*

$$\left(\frac{1}{t}\int_0^t \exp\left(\mathrm{i}\,\lambda\, L_u^{\alpha,H} + d_{\mathrm{sign}(\lambda)}\, |\lambda|^\alpha\, u^{\alpha H}\right)\, \mathrm{d}u\ ,\ 0 \le t \le r\right)$$

is a 1-martingale.

Proof. Let

$$\Phi = \int_0^r \Phi^{\lambda,u}\, \mathrm{d}u$$

Then, by Proposition 6.3, we have for $0 \le t \le 1$,

$$\Phi_t^\sharp = \int_0^r \Phi^{\lambda,t\,u}\, \mathrm{d}u = \frac{1}{t}\int_0^{rt} \Phi^{\lambda,u}\, \mathrm{d}u$$

Therefore, the process

$$\left(\frac{1}{rt} \int_0^{rt} \exp\left(\mathrm{i}\,\lambda\, L_u^{\alpha,H} + d_{\mathrm{sign}(\lambda)}\, |\lambda|^\alpha\, u^{\alpha H}\right)\, \mathrm{d}u\ ,\ 0 \le t \le 1 \right)$$

is a 1-martingale, which, after replacing rt by t, is the announced result. □

In the case of the fractional Brownian motion with Hurst index H ($\alpha = 2$, $d_\epsilon = 1/2$), the previous Proposition 6.3 and Corollary 6.1 can be extended to any $\lambda \in \mathbb{C}$. In particular, we have the following extension of our guiding example of Subsection 1.2:

Proposition 6.4. *For every $\lambda \in \mathbb{R}$ and $r > 0$, the process*

$$\left(\frac{1}{t} \int_0^{t} \exp\left(\lambda\, B_u^H - \frac{\lambda^2}{2}\, u^{2H}\right)\, \mathrm{d}u\ ,\ 0 \le t \le r \right)$$

is a 1-martingale.

Finally, we remark that Corollary 6.1 and Proposition 6.4 extend to $t \ge 0$ instead of $0 \le t \le r$:

Proposition 6.5.

1) *For every $\lambda \in \mathbb{R}$, the process*

$$\left(\frac{1}{t} \int_0^{t} \exp\left(\mathrm{i}\,\lambda\, L_u^{\alpha,H} + d_{\mathrm{sign}(\lambda)}\, |\lambda|^\alpha\, u^{\alpha H}\right)\, \mathrm{d}u\ ,\ t \ge 0 \right)$$

is a 1-martingale.

2) *For every $\lambda \in \mathbb{R}$, the process*

$$\left(\frac{1}{t} \int_0^{t} \exp\left(\lambda\, B_u^H - \frac{\lambda^2}{2}\, u^{2H}\right)\, \mathrm{d}u\ ,\ t \ge 0 \right)$$

is a 1-martingale.

Proof. We prove, for example, property 1).
Let, for $\lambda \in \mathbb{R}$ and $t \ge 0$,

$$M_t = \int_0^1 \exp\left(\mathrm{i}\,\lambda\, L_u^{\alpha,H}(X_{\bullet,t^{\alpha H}}) + d_{\mathrm{sign}(\lambda)}\, |\lambda|^\alpha\, (t\,u)^{\alpha H}\right)\, \mathrm{d}u$$

where $L_u^{\alpha,H}(X_{\bullet,t^{\alpha H}})$ means that, in the expression of $L_u^{\alpha,H}$, one has replaced the process $L_\bullet$ by the process $X_{\bullet,t^{\alpha H}}$. Then $(M_t\ ,\ t \ge 0)$ is a $(\mathcal{X}_{t^{\alpha H}})$-martingale: this comes, by Proposition 6.1, from the fact that, if $0 \le s \le t$,

$$X_{\bullet,t^{\alpha H}} - X_{\bullet,s^{\alpha H}} \overset{\text{(law)}}{=} (t^{\alpha H} - s^{\alpha H})^{1/\alpha}\, L_\bullet$$

Now, by the scaling properties, for $t \geq 0$ fixed,

$$\left(L^{\alpha,H}_{t\,u}\ ,\ 0 \leq u \leq 1\right) \overset{\text{(law)}}{=} \left(L^{\alpha,H}_{u}(X_{\bullet,t^{\alpha H}})\ ,\ 0 \leq u \leq 1\right)$$

and the result follows easily. □

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Laboratoire d'Analyse et Probabilités
Université d'Évry - Val d'Essonne, Boulevard F. Mitterrand
F-91025 Évry Cedex
France
e-mail: francis.hirsch@univ-evry.fr

Laboratoire de Probabilités et Modèles Aléatoires
Université Paris VI et VII, 4 Place Jussieu - Case 188
F-75252 Paris Cedex 05
France
e-mail: deaproba@proba.jussieu.fr
and Institut Universitaire de France

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