MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO; SERIES A,
Vol. XXVI, Mathematics No. 1, 1950.

Note to Okamura's Last Paper

By

Sigeru Mizohata

(Received Jan. 11, 1950)

The manuscript left by late Prof. Okamura "On the surface integral and Gauss-Green's theorem" contains some points unsolved. In this paper I will consider about them.

In the first part, I shall give a proof of the subject mentioned in his footnote (6). It will be seen that Gauss-Green's theorem proved in Okamura's paper is valid even when we replace the jordanian surface S by a "jordanian manifold".[1] In the next part, I shall give an example of the jordanian surface of bounded $x{*}y$-variation having positive measure (in three dimensions). Here we remark the following fact: though it follows from the first part easily that a jordanian surface of bounded variation in the sense of Banach is also of bounded variation in the sense of Okamura, yet by our example we see that the latter is really a wider idea than the former.

I. We may state the first part to be proved as follows.

We fix the positive side of the jordanian surface S (or generally the jordanian manifold S) so that the order (Ordnung) of the interier domain with respect to S is -1. On the vertical line through the point P, we consider two points O_1, O_2, where O_1 is an interior point, O_2 is an exterior point. Then cutting S by a sufficiently narrow cylinder having the line O_1O_2 as its axis,[2] we denote by δ(an open set) the part of S contained in its interior and lying between O_1 and O_2.

We have the following alternatives:

I. If $\overrightarrow{O_1O_2}$ coincides with the positive sense of z-axis, then $A(P, \delta) = +1$,

II. If $\overrightarrow{O_1O_2}$ is opposite to the positive sense of z-axis, then $A(P, \delta) = -1$.

Proof. In the first place, we can choose a positive number ε_1

such that the order of O_1 with respect to the arbitrary "simpliziales Bild"(3)—we call this a net after Brouwer, which is of course "zweiseitig" and "geschlossen"—of a "Dichte" less than ε_1, is always -1, and that of O_2 is always zero.

Now, we describe a triangle $\varDelta$ on the xy-plane so that it may be entirely contained in the circle which is obtained as the projection of the surface of cylynder. Then by the definition of "Abbildungsgrad", we can choose $\varepsilon_2(>0)$ such that the whole of simplexes which are obtained by the projection of all the simplexes corresponding to δ for any arbitrary net of "Dichte" less than ε_2 covers the domain $\varDelta$ always $A(P, \delta)$ times (i. e. the difference of the number of positive coverings and that of negative coverings is always $A(P, \delta)$ at any point of $\varDelta$).

In the next place, denoting by $\varepsilon_3(>0)$ the distance between $S-\delta$ and $\overrightarrow{O_1O_2}$, we determine a net of "Dichte" less than Min $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ so that the simplexes corresponding to δ are all lying between O_1 and O_2 in height (it is always possible by definition). We translate $\overrightarrow{O_1O_2}$, if necessary, a little parallel to the xy-plane, and then we may suppose that $\overrightarrow{O_1O_2}$ does not meet the one-dimensional simplex of the net, and that the simplex which is penetrated by $\overrightarrow{O_1O_2}$ is all the one which corresponds to δ, and moreover the orders of O_1, O_2 with respect S are respectivelly $-1, 0$.

Then, by the relation between the order and the penetrating number (Kreuzungszahl), the penetrating number of $\overrightarrow{O_1O_2}$ with the net is evidently $+1$.

On the other hand, when a simplex is penetrated by $\overrightarrow{O_1O_2}$, its projection-simplex (in the xy-plane) covers P with the same sense, or with the opposite sense according the case I and II respectively (for example, in case I, if a simplex is positively penetrated, then its projection-simplex covers P positively). The converse holds also. Thus the proof is finished.

Remark. When both O_1 and O_2 are interior points, or exterior points, we consider as in the above a cylinder enough small, then for the correspondiog δ, we shall have $A(P, \delta)=0$.

The second part to prove is the formula: $A(P, \delta_i)=\sum_{l\{N_l\in\delta_i\}}(-1)^l$.

For this it suffices to consider the following fact. Each of intersections of S with a vertical line belong to some one of $\delta_1, \delta_2, \ldots, \delta_n$.

We denote by E_i, the (linear) set of points which belong to δ_i, then $E_iE_j=0$ $(i\neq j)$ and all E_i are closed. In this case, we may cover E_1, with a finite number of (linear) disjoint closed intervals $\omega_1,\ldots,\omega_p$ i.e., $\mathcal{E}=\omega_1+\ldots+\omega_p\supset E_1$, such that the intervals have no common point with $E_2,\ldots,E_n$, hence $\mathcal{E}\cdot E_i=0$ $(i=2,3,\ldots,n)$.

2. An example of bounded $x*y$-variation and of positive measure.

We describe on the xy-plane a jordanian closed curve C having (two dimensional) positive Lebesgue measure. Then, we consider the cylinder generated by the segment $0\leqq z\leqq 1$, perpendicular to the xy-plane and guided by the curve C. Adding the top and the bottom areas to the cylinder we obtain a jordanian closed surface..

In the next place, we extract from the interval $0\leqq z\leqq 1$ of the generator of the cylinder a sequence of open intervals as follows: at first the interval of length $\frac{1}{4}$ having $\frac{1}{2}$ as its midpoint and subsequently the two intervals of length $\left(\frac{1}{4}\right)^2$ having the midpoints of the remaining two intervals as their midpoint respectively, and so forth.

We obtain a sequence of intervals whose measure is

$$\frac{1}{4}+2\left(\frac{1}{4}\right)^2+4\left(\frac{1}{4}\right)^3+\cdots \qquad <1.$$

Now cutting off the parts of the cylinder corresponding to these intervals, we substitute them by the other peaces of surface.

For this purpose we choose $\varepsilon(>0)$ such that

$$\varepsilon_1+2\varepsilon_2+4\varepsilon_3+\cdots \qquad =A<+\infty,$$

and denoting by D the jordanian domain determined by C, we map the closed domain D conformally on a unit circle. Now, letting $\varepsilon(>0)$ any positive number, we consider, in the unit circle, a concentric cercle of the diameter sufficiently near to unity such that denoting by j the primitive image (a jordan curve), we shall have

$$m[D-(\text{the jordanian domain determined by } j)]<\varepsilon.$$

Now we denote by $j_i(i=i,2,\ldots)$ the jordanian curves determined as above for $\varepsilon=\varepsilon_i(i=1,2,\ldots)$ respectively.

We define for the first extracted part $\frac{1}{2}-\frac{1}{8}<z<\frac{1}{2}+\frac{1}{8}$, a surface as follows: we consider the cylinder generated by the vertical line through j_1, from which we cut off the part $\frac{1}{2}-\frac{1}{8}\leqq z \leqq\frac{1}{2}+\frac{1}{8}$ then we have a zone. We connect the upper and the lower curves of the zone by the horizontal areas to the original cylinder respectively.

Next, for the two extracted intervals each of length $\left(\frac{1}{4}\right)^2$, we construct two zones through j_2 and add the horizontal parts in the same manner as the first, and so forth.

It will be clear that this surface is a jordanian closed surface, and we easily see that this surface fulfils the condition (B_1) of the bounded variation in Okamura's paper.

We have obviously,

$$\int_D \alpha(P)\,dm(P) < 2A+2|D|,$$

and moreover, when $P\epsilon C$, we have $\alpha(P)=0$. Therefore it is a surface of bounded x-y-variation; nevertheless it has positive measure.

This reserch has been done by the kind guidance of Prof. T. Matsumoto and by a co-operation of my friend M. Yamaguchi, to both of them, I express my sincere thanks.

1) We call after Brouwer " Jordanische Mannigfaltigkeit " in the n-dimensinal space a topological image of a closed $(n-1)$-dimensional manifold.

2) precisely speaking, we make the diameter so small that the corresponding $\bar{\delta}$ may be lying between O_1 and O_2 in height.

3) It may be seen as a polyhedron inscribed on the surface.