MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A,
Vol. XXVI, Mathematics No. 2. 1950.

Specialization of Cycles on a Projective Model[1]

By

Teruhisa MATSUSAKA

(Received April 3, 1950)

A. Weil has proposed in his "Foundations of Algebraic Geometry"[2] to define the notion of specialization of cycles of arbitrary dimensions. We attempt to give a solution to this problem on a projective model.

I wish here to express my sincere gratitude to Prof. Y. Akizuki and to Mr. J. Igusa for their kind advices throughout this work.

1. Following A. Weil, we fix once for all the universal domain Ω of the given characteristic p. We use the same terminologies and conventions which have been used in his book.

Let V^n be a Subvariety of a projective Space L^N, which we assume to have no multiple Point. Let k be a field of definition for V, and fix k hereafter, to which we refer as the basic field. All fields we shall consider are assumed to contain k.

By a generic linear Variety over a field K, we shall understand such a linear Variety whose coefficients of the defining linear equations are algebraically independent over K.

Let $L_1 \times \ldots\ldots \times L_h$ be the Product of h projective Spaces of dimensions n_i respectively. Then it is well-known that there is the everywhere biregular birational correspondence T between $\prod_i L_i$ and the Subvariety R of a certain projective Space L^f. By F-VI, th. 10, all problems concerning intersection-theories on $\prod L_i$ can be reduced by T, to similar problems on R. If X is a $\prod_i L_i$-cycle, we denote by X^T the transform of X by T. By the "associated-form" of a positive $\prod_i L_i$-cycle X, we understand such of X^T.

---

(1) Prof. A. Weil kindly communicated to me that the same problem has been solved by Dr. P. Samuel. I wish to express here my sincere gratitude to him for his kind information and encouragement.

(2). We denote this book by F-.........

Let $X=X_1-X_2, X_i\succ 0$ be an arbitrary L^N-cycle and let (x_1), (x_2) be the coefficients of the associated-forms of X_1, X_2. Then (x_i) determines uniquely the points $x_i (i=1, 2)$ on projective Spaces of suitable dimensions. We shall say that $x_1\times x_2$ is the *Chow-Point* of X, following J. Igusa. For the detailed discussions about the associated-form, see v. d. Waerden's book.[3]

Following two lemmas follow immediately from the definition of the associated-forms and from their fundamental properties.

LEMMA 1. *Let X be a positive L-cycle and x the Chow-Point of X. Then X is rational over the field $k(x)$*

LEMMA 2. *Let x be the Chow-Point of a posititve L-cycle X. Then if x' is a specialization of x over k, x' determines uniquely the positive L-cycle X' in such a way that the Chow-Point of X' is x'. If X is a V-cycle, so is X'.*

2. Let $Z=X-Y$, $X\succ 0$, $Y\succ 0$ be a V-cycle and x, y the Chow-Points of X, Y. If $x'\times y'$ is a specialization of $x\times y$ over a field K, x' and y' determines on V the V-cycles X' and Y'. Putting

$$Z'=X'-Y'$$

we shall say that Z' is a specialization of Z over K.

Thus we have reduced the specialization of cycles to those of the Points and hence ordinary properties of specializations of Points can immediately be extended to those of cycles. Dimensions and degrees are not altered by specializations. A specialization of a positive cycle is also positive and a generic specialization of a set of cycles over K is the transform of that set by the isomorphism of the suitable field K_1 containing K over K. (It is easy to see, using lemma 1, that we may take as K_1 the field over which all the cycles in the set are rational). Moreover, it is easy to see, from the properties of the associated-forms, that the notion of specialization of cycles is compatible with the operation of addition of cycles.

PROPOSITION 1. *Let $X^s=X_1-X_2$, $X_i\succ 0$ be a V-cycle, X' a non generic specialization of X over K, and $x_1\times x_2$, $x_1'\times x_2'$, the Chow-Points of X and X'. Then there is a Curve Γ defined over a field $K_1\supset K$, a rational $\Gamma\times V$-cycle Y over K_1, every compo-*

(3) v. d. Waerden. "Einführung in die Algebraische Geometrie'.

nent of which having the projection Γ on Γ, a generic Point $\bar{M}=\bar{x}_1 \times \bar{x}_2 \times \bar{y}$ of Γ over K_1 and a simple Point $M'=x_1' \times x' \times y'$ such that we have

$$(\bar{M} \times V) \cdot Y = \bar{M} \times \bar{X}$$
$$(M' \times V) \cdot Y = M' \times X'$$

where $\bar{x}_1 \times \bar{x}_2$ is a generic specialization of $x_1 \times x_2$ over K and $\bar{X}$ is a generic specialization of X over the generic specialization $x_1 \times x_2 \rightarrow \bar{x}_1 \times \bar{x}_2$ with respect to K.

Proof. The existence of Γ, $\bar{M}$, M' and K_1 follow immediately from prop. 7 of F-Appendix II. Let $\bar{X}$ be the generic specialization of X over the generic specialization $x_1 \times x_2 \rightarrow \bar{x}_1 \times \bar{x}_2$ with respect to K. Then, by lemma 1 and 2, $\bar{X}$ is rational over $K_1(\bar{M})$, and there is the uniquely determined $\Gamma \times V$-cycle Y, rational over K_1 such that every component of Y has the projection Γ on Γ and that

$$(\bar{M} \times V) \cdot Y = \bar{M} \times Y(\bar{M}) = \bar{M} \times \bar{X}.$$

(Cf, F-VII, th. 12, (iii)). Put

$$(M' \times V) \cdot Y = M' \times Y(M').$$

Let H^{N-s} be a generic linear Variety over $K_1(\bar{M}, M')$ and put $H \cdot V^n = W^{n-s}$. As $(\bar{M} \times Y(\bar{M})) \cdot (\Gamma \times W)$ is defined on $\Gamma \times V$, $\bar{M} \times V$, Y, $\Gamma \times W$ intersect properly on $\Gamma \times V$ (cf. F-VII, pr. 16). Moreover $Y \cdot (\Gamma \times W)$ is defined on $\Gamma \times V$. Hence we have

$$((\bar{M} \times V) \cdot Y)(\Gamma \times W) = (\bar{M} \times V) \cdot (Y \cdot (\Gamma \times W))$$
$$= \bar{M} \times \bar{X} \cdot H.$$

(cf, F-VII, th. 10 (v)), as $(W \cdot \bar{X})_V = (H \cdot \bar{X})_L$. In the same way, we have

$$((M' \times V) \cdot Y) \cdot (\Gamma \times W) = (M' \times V) \cdot (Y \cdot (\Gamma \times W))$$
$$= M' \times Y(M') \cdot H.$$

Since it holds $((\Gamma \times W) \cdot Y)_{\Gamma \times V} = ((\Gamma \times H) \cdot Y)_{\Gamma \times L}$(cf. F-VII, cor. th. 18), every component of $(\Gamma \times W) \cdot Y$ has the projection Γ on Γ. Then, $Y(M') \cdot H$ is the uniquely determined specialization of

$\bar{X}\cdot H$ over $\bar{M}\rightarrow M'$ with respect to K_1 by th. 12 of F-VI. But it is easily seen that $X'\cdot H$ is the uniquely determined specialization of $\bar{X}\cdot H$ over K_1 since X' is a specialization of $\bar{X}$ over K_1. This shows that $Y(M')\cdot H=X'\cdot H$. As H is generic over $K_1(\bar{M}, M')$, over which X' and $Y(M')$ are rational, we conclude that $Y(M')=X'$.

PROPOSITION 2. *Let W be an abstract Variety, X a $W\times V$-cycle and M' a simple Point of W such that $(M'\times V)\cdot X$ is defined. Let K be a common field of definition for W, V over which X is rational and M a generic Point of W over K. Then $X(M')$ is the uniquely determined specialization of $X(M)$ over $M\rightarrow M'$ with reference to K.*

We omit the proof. This can be proved by the same idea as above.

3. Now we state the theorems and sketch the proof.

THEOREM 1 *Let W^m be a non-singular projective Model in L^M, $X^r\times Y^s$ a $W\times V$-cycle and (X', Y') a specialization of (X, Y) over a common field K of definition for W, V. Then $X'\times Y'$ is the uniquely determined specialization of $X\times Y$ over that specialization with respect to K.*

Proof. It is sufficient to prove this when $X\times Y$ is a Subvariety $A\times B$ of $W\times V$. By prop. 1, there are a Curve Γ, defined over $K_1\supset K$, a rational $\Gamma\times W\times V$-cycle Y over K_1, a generic Point $\bar{M}$ over K_1 and a simple Point M' of Γ which is a specialization of $\bar{M}$ over $(\bar{A}, \bar{B})\rightarrow(A', B')$ with reference to K_1 such that

$$(\bar{M}\times W\times V)\cdot Y=\bar{M}\times\bar{A}\times\bar{B}$$
$$(M'\times W\times V)\cdot Y=M'\times C$$

where $\bar{A}\times\bar{B}$ is a generic specialization of $A\times B$ over K. We show that $C=A'\times B'$.

Let H_1^{M-r}, H_2^{N-s} be independent generic linear Varieties over $K_1(\bar{M}, M')$ in L^M and in L^N. Then we have

$$(\bar{M}\times W\times V)\cdot(Y\cdot(\Gamma\times W\cdot H_1\times V\cdot H_2))=\bar{M}\times\bar{A}\cdot H_1\times\bar{B}\cdot H_2. \quad (1)$$

Moreover, we can show that $C\cdot(H_1\times H_2)$ is not empty. Hence C is of the form $U_1\times U_2$ and we have

$$(M'\times W\times V)\cdot(Y\cdot(\Gamma\times W\cdot H_1\times V\cdot H_2))$$
$$=M'\times U_1\cdot H_1\times V_2\cdot H_2 \quad (2)$$

Form (1) and (2), using th. 12 of F-VI, we conclude that $U_1 \cdot H_1 = A' \cdot H_1$, $U_2 \cdot H_2 = B' \cdot H_2$. This shows that $A' = U_1$, $B' = U_2$. q. e. d.

Now we shall prove the compatibility with the operation of intersection product.

THEOREM 2 *Let X^r, Z^s be two V^n cycles and (X', Z') a specialization of (X, Z) over a field K. If $X \cdot Z$ and $X' \cdot Z'$ are defined on V, $X' \cdot Z'$ is the uniquely determined specialization of $X \cdot Z$ over $(X, Z) \rightarrow (X', Z')$ with reference to K.*

Proof. It is sufficient to prove this when X and Z are Varieties and $s = n - r$. Using the same notations as in prop. 1, there is a Curve Γ defined over $K_1 \supset K$, a rational $\Gamma \times V \times V$-cycle Y over K_1, a generic Point M and a simple Point M' of Γ such that

$$(\bar{M} \times V \times V) \cdot Y = \bar{M} \times \bar{X} \times \bar{Z}$$

$$(M' \times V \times V) \cdot Y = M' \times X' \times Z'. \quad \text{(cf. th. 1)}$$

where $\bar{X} \times \bar{Z}$ is a generic specialization of $X \times Z$ over K. This shows, in particular, that Y is a Variety. From this we can easily derive

$$(\Gamma \times \Delta_V) \cdot ((\bar{M} \times V \times V) \cdot Y) = \bar{M} \times (\bar{X} \times \bar{Z}) \cdot \Delta_V$$

$$(\Gamma \times \Delta_V) \cdot ((M' \times V \times V) \cdot Y) = M' \times (X' \times Z') \cdot \Delta_V.$$

By the associativity of intersections (cf. F-VI, th. 5), there is at least a proper component in $\Gamma \times \Delta_V \cap Y$ on $\Gamma \times V \times V$. Let $C_i (i = 1, \ldots\ldots m)$ be all the proper components in that intersection of multiplicties c_i respectively, having the projection Γ on Γ. Then we can show, from the associativity that, if we put $\bar{Y} = \sum c_i C_i$,

$$(\bar{M} \times V \times V) \cdot \bar{Y} = \bar{M} \times (\bar{X} \times \bar{Z}) \cdot \Delta_V$$

$$(M' \times V \times V) \cdot \bar{Y} = M' \times (X' \times Z') \cdot \Delta_V$$

From this, we conclude, using prop. 2, that $X' \cdot Z'$ is the uniquely determined specialization of $\bar{X} \cdot \bar{Z}$ over $(\bar{X}, \bar{Z}) \rightarrow (X', Z')$ with reference to K. q. e. d.

The following theorem states the compatibility with the operation of algebraic projection.

THEOREM 3 *Let V and W be two non-singular projective Varieties defined over K, X a $V \times W$-cycle, and X' its specialization over K. Then $pr_V X'$ is the uniquely determined specialization of $pr_V X$ over $X \rightarrow X'$ with reference to K.*

Proof. It is sufficient to prove this, when X is a Variety. We can show that if $pr_V X=0$, then $pr_V X'=0$ and in this case our theorem holds true. By prop. 1, there is a Curve Γ defined over $K_1 \supset K$, a Subvariety Y of $\Gamma \times V \times W$ defined over K_1, a generic Point M over K_1 and a simple Point M' of Γ such that

$$(M \times V \times W) \cdot Y = M \times \bar{X}$$
$$(M' \times V \times W) \cdot Y = M' \times X'$$

where $\bar{X}$ is a generic specialization of X over K. From this we can easily see that, if $pr_V \bar{X} \neq 0$, then $pr_V X' \neq 0$. Let H^{N-t} be a generic linear Variety over $K_1(M, M', \bar{x}, x')$, where $\bar{x}$, x' denote the Chow-Points of $pr_V \bar{X}$, $(pr_V \bar{X})'$. ($(pr_V \bar{X})'$ means the specialization of $(pr_V \bar{X})$ over our specialization with reference to K_1.) Then we have

$$pr_V((H \cdot V \times W) \cdot \bar{X}) = (pr_V \bar{X}) \cdot H$$
$$pr_V((H \cdot V \times W) \cdot X') = (pr_V X') \cdot H.$$

This shows that $(pr_V X') \cdot H$ is the uniquely determined specialization of $(pr_V \bar{X}) \cdot H$ over $(\bar{X}, H) \rightarrow (X', H)$ with reference to K_1. (cf. th. 2) As $(pr_V \bar{X})' \cdot H$ is such of $(pr_V \bar{X}) \cdot H$, we have

$$(pr_V \cdot \bar{X})' \cdot H = (pr_V \cdot X') \cdot H.$$

From this we conclude that $pr_V X' = (pr_V \bar{X})'$. q. e. d.

COROLLARY *Let V_1 and V_2 be non-singular projective Varieties such that they are birationally equivalent by the birational correspondence T. Let K be a common field of definition for V_1, V_2, T and X_1 or X_2 be a V_1- or V_2-cycle such that*

$$pr_{V_2}((X_1 \times V_2) \cdot T) = X_2.$$

Let (X_1', X_2') be a specialization of (X_1, X_2) over K and A_1 and A_1 the biregularly corresponding Subvarieties of V_1 and V_2 by T. Then we have

$$r_{A_1}(X_1') = r_{A_2}(X_2').$$

Our notion of specialization coincides with that of A. Weil, when the cycle is zero-dimensional. Hence we have proved that all the requirements of Weil on the notion of specialization is fulfiled by our definition.

ADDENDUM. I have received from P. Samuel, a brief account of his result on this problem (Comptes rendus, 228. "Multiplicités des composantes singulières d'intersection". 1949. p. 158). He communicated to me that the same problem was also solved by Chow and Barsotti.

Revised Sep. 10, 1950.