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On a Theorem of Lueroth

By

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Let K be a field of degree of transcendency 1 over a field k, then the well-known theorem of Lüroth[1] asserts that K is a simple extension of k, when K is contained in such a field. Now we shall present three different proofs for a generalization of this theorem which are connected closely by the general theory of *Picard varieties*[2]. The present author interests more in the different methods of proof rather than the result itself, which can be stated as follows:

Let K be a field of degree of transcendency 1 *over a field k, then K is a simple extension of k, whenever K is contained in a purely transcendental extension of k.*

We assume thereby that k is a perfect field in order to assure the existence of a non-singular model for K over k; although the theorem is true for an arbitary field k, as we can see from another aspect.

Now let $(t)=(t_1,\dots,t_m)$ be a set of independent variables over k, then since K is an intermediary field of $k(t)$ and k, it can be generated over k by a finite set of quantities. Since we have assumed k as a perfect field, there exists a complete non-singular Curve $\boldsymbol{C}$ with a generic Point $\boldsymbol{P}$ over k such that

$$K=k(\boldsymbol{P}).$$

I was asked in a certain occasion to generalize Lüroth's theorem from Prof. Akizuki; and the publication of this note has been advised also by him. In this note we shall stick in results and terminologies to Weil's book: Foundations of algebraic geometry, Am. Math. Soc. Colloq., vol. 29 (1946).

1) Beweis eines Satzes über rationale Curven, Math. Ann. 9 (1876). See also B. L. v. d. Waerden, Moderne Algebra, § 63.

2) The first two proofs A and B concern clearly with this theory; the same is true for the proof C. See my papers, On the Picard varieties attached to algebraic varieties, to appear in the Amer. J. of Math.; Algebraic correspondences between algebraic varieties, to appear in the Jap. J. of Math.

On the other hand there exists a generic Point M over k of a projective space L^m or a Product E_m of m projective straight lines D such that $k(t)=k(M)$. There exists then a function f on L^m or on E_m with values in C defined over k by

$$f(M)=P.$$

Lemma 1. *The Curve C is rational.*

Proof A. Since for every integer s the two fields $k(P)$ and $k^{p^s}(P^{p^s})$ are isomorphic over the prime field of characteristic p, in order to prove our assertion, we may assume that P is not rational over $k(M^p)$. Let θ be a differential form of the first kind on C, then its inverse image $f^{-1}(\theta)$ by f is a similar form on E_m[3]. Moreover as P is not rational over $k(M^p)$, we have $f^{-1}(\theta)\neq 0$ unless $\theta=0$. However $f^{-1}(\theta)$ can be written as a sum of the differential forms of the first kind on D

$$f^{-1}(\theta)=\theta_1+\dots+\theta_m,$$

and we have $\theta_i=0$ $(1\leq i\leq m)$ since D is of genus 0.

Therefore C has no other differential form of the first kind other than 0; hence is of geuns 0.

Proof B. If C has a positive genus g, C is mapped birationally into its Jacobian Variety J^g by the canonical function φ on C.[4] Then the function $\varphi\circ f$ on L^m with values in J

$$L^m \xrightarrow{f} C \xrightarrow{\varphi} J^g$$

is not a constant, which is a contradiction.

Proof C. The graph Γ_f of f in the Product $L^m\times C$ is a correspondence with valence 0 between L^m and C, since every L^m-divisor which is continuously equivalent to 0 is linearly equivalent to 0. Therefore two Points of C are linearly equivalent, hence C is a rational Curve.

It does not follow from lemma 1 that $k(P)$ is a simple extension of k, even in the case of characteristic 0.

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3) See e. g. S. Koizumi, On the differential forms of the first kind on algebraic varieties, Jap. J. of Math., vol. 1 (1949).

4) A. Weil, Variétés Abeliennes et courbes algébriques, Act. Sc. et Ind. n° 1064 (1948).

Lemma 2. *The Curve C has at least one rational Point with reference to k.*

Proof. If the field k is infinite, since the coordinates of a representative of the Point $\boldsymbol{P}$ are rational expressions of the independent variables $t_1, \ldots, t_m$ over k with coefficients in k, we can readily find a rational Point on $\boldsymbol{C}$. On the other hand if k is a finite field, there exists a rational $\boldsymbol{C}$-divisor of degree 1 over k[5]. However since $\boldsymbol{C}$ is a rational Curve, there exists then a positive rational $\boldsymbol{C}$-divisor of degree 1 over k, which is nothing but a rational Point of $\boldsymbol{C}$ with reference to k.

Let $\boldsymbol{Q}$ be a rational Point of $\boldsymbol{C}$ with reference to k, then there exists a quantity x in $k(\boldsymbol{P})$ such that the function θ defined over k by $x=\theta(\boldsymbol{P})$ satisfies $(\theta)\succ -\boldsymbol{Q}$. In such a case $k(\boldsymbol{P})$ is generated over k by x

$$K=k(\boldsymbol{P})=k(x).$$

The above proof, it may be hoped, seems to reveal the true content of the theorem of Lüroth.

5) A. Weil, Courbes algébriques et les variétés qui s'en déduisent, Act. Sc. et Ind. n° 1041 (1948).