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# Affinely Connected Spaces of Class One

By

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The purpose of this paper is to obtain a condition for space with affine connection to be of class one. In previous paper[1] for embedding of space with projective connection, we have defined the class number of the space after the example of Riemann space and got a necessary and sufficient condition for the space of class one. We can similarly define the class number of affinely connected space.

In order to solve the Gauss equations of hypersurface in affine space, we can utilize the method, which are used for the first Gauss equations of hypersurface in projective space, specially in case of unimodular affine connection ($R_{a\cdot ij}^{a}=0$). In the general case $R_{a\ ij}^{a}\neq 0$, there is a little different aspect; but the above method is also applicable after slight modifications.

There are many points (marked by [*]) in this paper, which are omitted to prove or are not discussed in details, as these points can be treated by similar way as in previous paper.

## § 1. Introduction.

Consider an $m$-dimensional space with affine connection $V_m$ where a current point $A$ is given by a system of coordinates $(y^1,\dots,y^m)$ and let $A_\lambda$ be linearly independent $m$ vectors at a point $A$. Then the connection is given by the following equations:

$$\left.\begin{aligned} dA &= A_\alpha dy^\alpha, \\ dA_\lambda &= \Gamma^\beta_{\lambda\alpha} A_\beta dy_\alpha\,; \end{aligned}\right\} \qquad \text{(I·I)}$$

where the functions $\Gamma^\beta_{\lambda\alpha}$ of $y$'s are called the components of affine connection of $V_m$ referring to the coordinates $y^\alpha$.

Let $V_n$ be a variety of $n$-dimensions in $V_m$ defined by the equations

$$y^\alpha = y^\alpha(x^1, \ldots, x^n) ;$$

where the functional matrix $\| \partial y^\alpha / \partial x^i \|$ is of rank $n$. When a current point $A$ displaces on $V_n$, we have

$$dA = A_\alpha B_i^{\cdot\alpha} dx^i ; \text{ where } \left( B_i^{\cdot\alpha} = \frac{\partial y^\alpha}{\partial x^i} \right).$$

Hence, if the quantities $A_i (i=1, \ldots, n)$ are defined by

$$A_i = A_\alpha B_i^{\cdot\alpha}, \tag{1·2}$$

we see that $A_i$ are linearly independent $n$ vectors on $V_n$ and obtain

$$dA = A_i dx^i. \tag{1·3}$$

Further we define

$$A_P = A_\alpha B_P^{\cdot\alpha} ; \tag{1·4}$$

where the determinant $| B_i^{\cdot\alpha}, B_P^{\cdot\alpha} |$ is not equal to zero; and it is to be seen that the quantities $A_i$, $A_P (i=1, \ldots, n ; P=n+1, \ldots, m)$ are linearly independent $m$ vectors of $V_m$. For the displacement on $V_n$ we put

$$dA_i = (\Gamma_{ij}^{\cdot k} A_k + H_{ij}^P A_P) dx^j, \tag{1·5}$$

$$dA_P = (H_{Pj}^k A_k + H_{Pj}^Q A_Q) dx^j. \tag{1·6}$$

Differentiating (1·2) and comparing to (1·5) give

$$\frac{\partial B_i^{\cdot\alpha}}{dx^j} = -\Gamma_{\lambda\mu}^{\cdot\alpha} B_i^{\cdot\lambda} B_j^{\cdot\mu} + \Gamma_{ij}^{\cdot k} B_k^{\cdot\alpha} + H_{ij}^P B_P^{\cdot\alpha}. \tag{1·7}$$

Similarly from (1·4) and (1·6) we obtain

$$\frac{\partial B_P^{\cdot\alpha}}{\partial x^j} = -\Gamma_{\lambda\mu}^{\cdot\alpha} B_P^{\cdot\lambda} B_j^{\cdot\mu} + H_{Pj}^k B_k^{\cdot\alpha} + H_{Pj}^Q B_Q^{\cdot\alpha}. \tag{1·8}$$

As the quantities $B_i^{\cdot\alpha}$ must satisfy the relations

$$\frac{\partial B_i^{\cdot\alpha}}{\partial x^j}=\frac{\partial B_j^{\cdot\alpha}}{\partial x^i},$$

which are the integrability condtion of a system of equations

$$\frac{\partial y^\alpha}{\partial x^i}=B_i^{\cdot\alpha}, \tag{1·9}$$

so we get from (1·7)

$$B_i^{\cdot\lambda}B_j^{\cdot\mu}S_{\lambda\mu}^\alpha=B_k^{\cdot\alpha}S_{ij}^k+B_P^{\cdot\alpha}(H_{ij}^P-H_{ji}^P)\ ; \tag{1·10}$$

where we put

$$S_{\lambda\mu}^\alpha=\Gamma_{\lambda\mu}^{\prime\alpha}-\Gamma_{\mu\lambda}^{\prime\alpha}, \tag{1·11}$$

$$S_{ij}^k=\Gamma_{ij}^k-\Gamma_{ji}^k. \tag{1·12}$$

The integrability condition of (1·7), i. e.

$$\frac{\partial^2 B_i^{\cdot\alpha}}{\partial x^j\partial x^k}=\frac{\partial^2 B_i^{\cdot\alpha}}{\partial x^k\partial x^j}$$

is, making use of (1·10)

$$B_i^{\cdot\lambda}B_j^{\cdot\mu}B_k^{\cdot\nu}R_{\lambda\cdot\mu\nu}^{\ \alpha}=B_l^{\cdot\alpha}(R_{i\cdot jk}^{\ l}+H_{i[j}^PH_{|P|k]}^{\ l})$$
$$+B_P^{\cdot\alpha}(H_{i[j,k]}^P+H_{i[j}^QH_{|Q|k]}^{\ P}+H_{i\alpha}^PS_{jk}^\alpha)\ ; \tag{1·13}$$

where we put

$$R_{\lambda\cdot\mu\nu}^{\ \alpha}=\frac{\partial\Gamma_{\lambda[\mu}^{\prime\alpha}}{\partial y^{\nu]}}+\Gamma_{\lambda[\mu}^{\prime\sigma}\Gamma_{|\sigma|\nu]}^{\prime\alpha}, \tag{1·14}$$

$$R_{i\cdot jk}^{\ l}=\frac{\partial\Gamma_{i[j}^l}{\partial x^{k]}}+\Gamma_{i[j}^a\Gamma_{|a|k]}^l. \tag{1·15}$$

The tensor $S_{\lambda\mu}^\alpha$ and $R_{\lambda\cdot\mu\nu}^{\ \alpha}$ are called the torsion and curvature tensors of $V_m$. The integrability condition of (1·8) is similarly

$$B_P^{\cdot\lambda}B_j^{\cdot\mu}B_k^{\cdot\nu}R_{\lambda\cdot\mu\nu}^{\ \alpha}=B_l^{\cdot\alpha}(H_{P[j,k]}^{\ l}+H_{P[j}^QH_{|Q|k]}^{\ l}+H_{P\alpha}^{\ l}S_{jk}^\alpha)$$
$$+B_Q^{\cdot\alpha}(H_{P[j,k]}^Q+H_{P[j}^RH_{|R|k]}^Q+H_{P[j}^lH_{|l|k]}^Q+H_{P\alpha}^QS_{jk}^\alpha). \tag{1·16}$$

When we transform the coordinates $y^\alpha$ of $V_m$ to the another $\bar{y}^\alpha$, we see easily that the functions $\Gamma_{ij}^k$, $H_{ij}^P$, $H_{Pj}^{\ i}$ and $H_{Pi}^Q$ are all invariant. The other hand, for the transformation of the coordinates $x^i$ of $V_n$ to the another $\bar{x}^i$, we see that those functions enjoy the transformations

$$\bar{\Gamma}^k_{ij}=\frac{\partial\bar{x}^k}{\partial x^a}\left(\frac{\partial^2 x^a}{\partial\bar{x}^i\partial\bar{x}^j}+\Gamma^a_{bc}\frac{\partial x^b}{\partial\bar{x}^i}\frac{\partial x^c}{\partial\bar{x}^j}\right),\qquad \bar{H}^Q_{Pi}=H^Q_{Pa}\frac{\partial x^a}{\partial\bar{x}^i},$$

$$\bar{H}^P_{ij}=H^P_{ab}\frac{\partial x^a}{\partial\bar{x}^i}\frac{\partial x^b}{\partial\bar{x}^j},\qquad \bar{H}^i_{Pj}=H^a_{Pb}\frac{\partial\bar{x}^i}{\partial x^a}\frac{\partial x^b}{\partial\bar{x}^j}.$$

Therefore the functions $\Gamma^k_{ij}$ enjoy the transformations analogous to the components of affine connection, so that we shall call $\Gamma^k_{ij}$ *the components of affine connection and* $S^k_{ij}$, $R_{i\cdot jk}^{\ l}$ *the torsion and curvature tensors of* $V_n$ *induced from* $V_m$ *with reference to* $A_P$.

## § 2. The fundamental theorem of embedding.

Let $\Gamma^k_{ij}$ be components of connection of a given $n$-dimensional space $V_n$ with affine connection. We shall call that $V_n$ can be embedded in an affine space $S_m$ of $m$-dimensions, if there exists an $n$-dimensional subspace $S_n$, whose components of affine connection induced from $S_m$ for suitable choice of $A_P$ are equal to the given $\Gamma^k_{ij}$. The space $V_n$ is called to be *of class* $p$, if $V_n$ can be embedded in an affine space of $(n+p)$-dimensions but not of $(n+q)$-dimensions $(p>q\geqq 0)$.

The torsion and curvature tensors $S^{\alpha}_{\lambda\mu}$, $R_{\lambda\cdot\mu\nu}^{\ \alpha}$ of flat $S_m$ vanish and hence from (1·10), (1·13) and (1·16) we have

$$S^k_{ij}=\Gamma^k_{ij}-\Gamma^k_{ji}=0, \tag{2·1}$$

$$H^P_{ij}-H^P_{ji}=0, \tag{2·2}$$

$$R_{i\cdot jk}^{\ l}=H^P_{i[k}H^{\ l}_{|P|j]}, \tag{2·3}$$

$$H_{i[j,k]}^{\ P}+H^Q_{i[j}H^{\ P}_{|Q|k]}=0, \tag{2·4}$$

$$H_{P[j,k]}^{\ l}+H^Q_{P[j}H^{\ l}_{|Q|k]}=0, \tag{2·5}$$

$$H_{P[j,k]}^{Q}+H^l_{P[j}H^Q_{k]l}+H^R_{P[j}H^{\ Q}_{|R|k]}=0\,; \tag{2·6}$$

where we call equations (2·3), (2·4), (2·5) and (2·6) *the Gauss, the first and second Codazzi and the Ricci equations* respectively. First we have from (2·1) the

**Theorem 1**: *If an n-dimensional space with affine connection* $V_n$ *can be embedded in affine space, the components of affine connec-*

*tion of $V_n$ is necessarily symmetric.*

Now, by means of the fact that the system of equations (2·1), (2·2), (2·3), (2·4), (2·5) and (2·6) gives the integrability condition of (1·7), (1·8) and (1·9), we obtain the fundamental theorem of embedding problem as follows:

**Theorem 2:** *A space with affine connection of n-dimensions can be embedded in an $(n+p)$-dimensional affine space if, and only if, the connection is symmetric and there exist three systems of functions $H^P_{ij}(=H^P_{ji})$, $H^i_{Pj}$ and $H^Q_{Pj}(i,j=1,\dots,n\ \ P,Q=n+1,\dots,n+p)$ satisfying the Gauss, the first and second Codazzi and the Ricci equations* (2·3), (2·4), (2·5) *and* (2·6).

In our particular case of class one we put

$$H_i{}^{n+1}{}_j\equiv H_{ij},\quad H_{n+1}{}^i{}_j\equiv H^i_j,\quad H_{n+1}{}^{n+1}{}_i\equiv H_i.$$

Then, as the fundamental equations of class one, we get

$$R_{i\cdot jk}^{\ \ l}=H_{ik}H^l_j-H_{ij}H^l_k, \tag{2·7}$$

$$H_{ij,k}-H_{ik,j}+H_{ij}H_k-H_{ik}H_j=0, \tag{2·8}$$

$$H^i_{j,k}-H^i_{k,j}+H_jH^i_k-H_kH^i_j=0, \tag{2·9}$$

$$H_{j,k}-H_{k,j}+H_{kl}H^l_j-H_{jl}H^l_k=0. \tag{2·10}$$

## § 3. The second Codazzi and Ricci equations as consequences of the Gauss and first Codazzi equations.

As in the case of Riemann spaces[2)] and projectively connected spaces of class one[3)], so in this case of affinely connected spaces, we can prove that the second Codazzi and Ricci equations are automatically satisfied, if the Gauss and first Codazzi equations are satisfied. In fact, differentiating (2·7) covariantly with respect to $x^m$ and summing three equations obtained from the first by cyclic permutation of the indices $j, k$ and $m$, and making use of the Bianchi identities

$$R_{i\cdot(jk,m)}^{\ \ l}=0,$$

we have

$$D_{i(jk}H^l_{m)}=D^l_{(jk}H_{m)i}\,; \qquad (3\cdot 1)$$

where we put

$$D_{ijk}=H_{i[j,k]}+H_{i[j}H_{k]},$$

$$D^l_{jk}=H^l_{[j,k]}+H_{[j}H^l_{k]}.$$

And so if the first Codazzi equations are satisfied, i. e. $D_{ijk}=0$, we have from $(3\cdot 1)$

$$H_{im}D^l_{jk}+H_{ij}D^l_{km}+H_{ik}D^l_{mj}=0.$$

Form these relations we see easily that $D^l_{jk}=0$ $(l, j, k=1,\dots, n)$, i. e. the second Codazzi equations are satisfied, if the matrix $\| H_{ij} \|$ is of rank $\geqq 3$ [*].

Next, differentiating $(2\cdot 8)$ covariantly with respect to $x^l$ and summing three equations obtained by the above process and making use of the equations $(2\cdot 7)$, $(2\cdot 8)$, $R_{(j\cdot kl)}^{\ i}=0$ and

$$H_{ij,\,kl}-H_{ij,lk}=-H_{aj}R_{i\cdot kl}^{\ a}-H_{ia}R_{j\cdot kl}^{\ a},$$

we have

$$H_{ij}D_{kl}+H_{ik}D_{lj}+H_{il}D_{jk}=0\,; \qquad (3\cdot 2)$$

where we put

$$D_{kl}=H_{[k,l]}+H^a_{[k}H_{l]a}.$$

From these relations we get $D_{kl}=0 (k, l=1,\dots, n)$, i. e. the Ricci equations, if the matrix $\| H_{ij} \|$ is of rank $\geqq 3$ [*].

Thus we have find the similar circumstances as in the case of projective connection.

## § 4. The case of unimodular affine connection.

First, we consider the unimodular affine conncetion, that is, the contracted curvature tensor $R_{a\cdot ij}^{\ a}$ is identically equal to zero. In this case we can have $H_i=0$ for suitable choice of vector $A_{n+1}$[(4)], and hence the first Codazzi and Ricci equations have the forms

$$H_{ij,k}-H_{ik,j}=0, \qquad (4\cdot 1)$$

$$H_{lk}H^l_j - H_{lj}H^l_k (= R_{l\cdot jk}^{\ \ l}) = 0, \tag{4·2}$$

respectively. If we put

$$K_{h\cdot ij}^{\ k} = H_{hi}H^k_j, \tag{4·3}$$

and its contraction

$$K_{ij} = K_{h\cdot ij}^{\ h} = H_{hi}H^h_j, \tag{4·4}$$

the tensor $K_{ij}$ so defined is symmetric on account of (4·2). Therefore we can applicate the methods, by which we have dealed with the first Gauss equntions of hypersurface in projective space.

That is, from (2·7), (4·3) and (4·4) we have

$$K_{a\cdot ij}^{\ b}K_{kl} - K_{a\cdot ik}^{\ b}K_{jl} = H_{ai}H^c_l R_{c\cdot jk}^{\ b}, \tag{4·5}$$

and further from (4·5)

$$R_{a\cdot li}^{\ c} R_{c\cdot jk}^{\ b} = K_{a\cdot i[j}^{\ b}K_{k]l} - K_{a\cdot l\ j}^{\ b}K_{k\ i}, \tag{I}$$

and finally contracting (I) with respect to $a$ and $b$ we get

$$M_{ijkl} = K_{ik}K_{jl} - K_{il}K_{jk}\,; \tag{4·6}$$

where we put

$$M_{ijkl} = -\frac{1}{2} R_{a\cdot ij}^{\ b} R_{b\cdot kl}^{\ a}.$$

This intrinsic tensor $M_{ijkl}$ satisfies the identities

$$M_{ijkl} = -M_{jikl} = M_{klij},$$

and that it is necessary to satisfy the identity $M_{i(jkl)} = 0$ from (4·6). This identity is result from

$$R_{a\cdot m(h}^{\ b} R_{|l|\cdot ij)}^{\ k} + R_{a\cdot (ij}^{\ b} R_{|l\cdot m|h)}^{\ k} + R_{l\cdot m(h}^{\ b} R_{|a|\cdot ij)}^{\ k} + R_{l\cdot (ij}^{\ b} R_{|a\cdot m|h)}^{\ k} = 0, \tag{II}$$

by contraction of $a$ and $k$, $b$ and $l$ [*].

Now, we define the type number of space as follows:

**Definion:** *A unimodular hypersurface S in an affine space will be said to be of type one if the rank of the matrix* $\| K_{ij} \|$ *is zero or one. It will be said to be of type* $\tau$ *where* $\tau$ *is an integer of the set* $2, \ldots, n$, *if the rank of the above matrix is* $\tau$.

It is easily seen [*] that $S$ is of type one if, and only if, the tensor $M_{ijkl}$ is identically equal to zero; and the type number $\tau$ $(\geqq 2)$ of $S$ is equal to the rank of the matrix

$$\left\| \begin{matrix} M_{abc1} \; M_{abc2} \cdots\cdots\cdots\cdots\cdots M_{abcn} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ M_{ijk1} \; M_{ijk2} \cdots\cdots\cdots\cdots\cdots M_{ijkn} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ M_{pqr1} \; M_{pqr2} \cdots\cdots\cdots\cdots\cdots M_{pqrn} \end{matrix} \right\|$$

Hence the type number is determined by intrinsic properties of $S$.

Next we can prove that, if $S$ is of type $\tau(\geqq 3)$, the system of functions $K_{ij}$ satisfying the equations $(4\cdot 6)$ is uniquely determined to within algebraic sign and this solution will be real if, and only if, the condition

$$\begin{vmatrix} M_{abij} & M_{abjk} & M_{abki} \\ M_{bcij} & M_{bcjk} & M_{bcki} \\ M_{caij} & M_{cajk} & M_{caki} \end{vmatrix} \geqq 0, \tag{III}$$

is satisfied, when $S$ is of type $(\geqq 3)$ [*]. Next let us write $(4\cdot 6)$ in the homogeneous form

$$t^2 \cdot M_{ijkl} = K_{ik}K_{jl} - K_{il}K_{jk}, \tag{4·6′}$$

and we obtain easily from $(4\cdot 6)$

$$K_{h[i}M_{m]ljk} - K_{l[j}M_{k]him} = 0. \tag{4·7}$$

Represent the resultant system of $(4\cdot 6')$ and $(4\cdot 7)$ by $R_n(M)$. We can prove that the equations $(4\cdot 6)$ will have a real solution if, and only if, the inequalities (III) and

$$\sum_{a,b,c,i,j,k} \begin{vmatrix} M_{abij} & M_{abjk} & M_{abki} \\ M_{bcij} & M_{bcjk} & M_{bcki} \\ M_{caij} & M_{bajk} & M_{caki} \end{vmatrix} > 0, \tag{IV}$$

and that the equations

$$R_n(M) = 0, \tag{V}$$

are satisfied [*]. Further we see easily $\sigma \geqq \tau$ [*]; where $\sigma$ is the rank of the matrix $\| H_{ij} \|$ and from (I) we have a tensor $K_{h\cdot ij}^{\ k}$ intrinsically for $\tau \geqq 3$, which must satisfy the following equations

$$R_{h\cdot ij}^{\ k} = K_{h\cdot ji}^{\ k} - K_{h\cdot ij}^{\ k}, \tag{VI}$$

$$K_{i\cdot jk}^{\ h} = K_{j\cdot ik}^{\ h} \tag{VII}$$

$$\begin{vmatrix} K_{a\cdot bj}^{\ i} & K_{a\cdot bl}^{\ k} \\ K_{c\cdot dj}^{\ i} & K_{c\cdot dl}^{\ k} \end{vmatrix} = 0, \tag{VII}$$

and (I) [*]. Finally we put

$$L_{ij} \equiv K_{i\cdot ja}^{\ a}, \tag{4·8}$$

and shall confine our consideration such a domain in $V_n$ that $L_{ij}$ does not vanish. We put

$$H_{ij} = e^{\rho} L_{ij}, \tag{4·9}$$

and substituting this expression in (4·1) gives

$$\rho_k L_{ij} - \rho_j L_{ik} + L_{ijk} = 0\,; \tag{4·10}$$

where we put

$$L_{ijk} = L_{ij,k} - L_{ik,j}, \tag{4·11}$$

$$\rho_i = \frac{\partial \log \rho}{\partial x^j}. \tag{4·12}$$

Let us write (4·10) in the homogeneous form

$$\rho_k L_{ij} - \rho_j L_{ik} + t \cdot L_{ijk} = 0, \tag{4·10$'$}$$

and represent the resultant system of (4·10′) by $Q_n(L)$, it follows that the equations

$$Q_n(L) = 0 \tag{IX}$$

are necessary and sufficient for (4·10) to have a solution and then the solution is uniquely determined [*].

This intrinsic tensor $\rho_j$ so determined must satisfy the equations

$$\rho_{i,j} - \rho_{j,i} = 0 \tag{X}$$

from (4·12) ; and then we define a system of functions $H_{ij}(i, j=1,\ldots, n)$ by (4·9) ; and we prove easily that the functions $H_{ij}$ satisfy the equations (4·1) and further we have a system of fuctions $H^i_j(i, j=1,\ldots, n)$ satisfying the Gauss equations (2·7) [*]. Consequently we have the

**Theorem 3 :** *If an $n$ ($\geqq 3$)-dimensional space $V_n$ with symmetric unimodular affine connection is of type $\tau(\geqq 3)$ and the tensor $L_{ij}$ does not vanish, $V_n$ is of class one if, and only if, the inequalities* (III) *and* (IV), *and the equations* (I), (II), (V), (VI), (VII), (VIII), (IX), *and* (X) *are satisfield.*

## § 5. The general case.

Now we consider the general case when the tensor $R_{a\cdot ij}^{\cdot a}$ does not identically vanish. In this case, though we define also the tensor $K_{h\cdot ij}^{\cdot k}$ and $K_{ij}$ by (4·3) and (4·4) respectively, $K_{ij}$ is never symmetric and yet we have

$$R_{a\cdot li}^{\cdot c}R_{c\cdot jk}^{\cdot b}=K_{a\cdot i[j}^{\cdot b}K_{k]l}-K_{a\cdot l[j}^{\cdot b}K_{k]i}, \tag{I'}$$

that is analogous to (I). We put

$$P_{a\cdot ij}^{\cdot b}=\frac{1}{2}(K_{a\cdot ij}^{\cdot b}+K_{a\cdot ji}^{\cdot b}), \tag{5·1}$$

and then we obtain from (4·3) and (2·7)

$$K_{a\cdot ij}^{\cdot b}=P_{a\cdot ij}^{\cdot b}-\frac{1}{2}R_{a\cdot ij}^{\cdot b}, \tag{5·2}$$

and hence $P_{a\cdot ij}^{\cdot b}$ and $-R_{a\cdot ij}^{\cdot b}/2$ are symmetric and skew-symmetric parts of $K_{a\cdot ij}^{\cdot b}$ with respect to $i$ and $j$ respectively. Contracting (5·2) with respect to $a$ and $b$ gives

$$K_{ij}=P_{ij}+Q_{ij}\,; \tag{5·3}$$

where we put

$$P_{ij}=P_{a\cdot ij}^{\cdot a}, \tag{5·4}$$

$$Q_{ij}=-\frac{1}{2}R_{a\cdot ij}^{\cdot a} \tag{5·5}$$

and hence $P_{ij}$ and $Q_{ij}$ are symmetric and skew-symmetric parts of $K_{ij}$ respectively, and that $Q_{ij}$ is intrinsic tensor. Substituting (5·2) and (5·3) in (I′) and summing the equation obtained from (I′) by interchanging $i, k$ into $j, l$ respectively, give

$$N_{a\cdot iljk}^{\ \ b}=P_{a\cdot i}^{\ \ q}{}_{j}P_{k]l}-P_{a\cdot l[j}^{\ \ b}P_{k]i}\,; \tag{XI}$$

where we put

$$2N_{a\cdot iljk}^{\ \ b}=R_{a\cdot li}^{\ \ c}R_{c\cdot jk}^{\ \ b}+R_{a\cdot kj}^{\ \ c}R_{c\cdot il}^{\ \ b}+R_{a\cdot i[j}^{\ \ b}Q_{k]l}-R_{a\cdot l[j}^{\ \ b}Q_{k]i}. \tag{5·6}$$

Contracting (XI) with respect to $a$ and $b$ gives

$$N_{ijkl}=P_{ik}P_{jl}-P_{il}P_{jk}\,; \tag{5·7}$$

where we put

$$N_{ijkl}=\frac{1}{2}N_{a\cdot ijkl}^{\ \ a}=-R_{a\cdot ij}^{\ \ b}R_{b\cdot kl}^{\ \ a}-Q_{ik}Q_{jl}+Q_{il}Q_{jk}. \tag{5·8}$$

The intrinsic tensor $N_{ijkl}$ so defined has the identities

$$N_{ijkl}=-N_{jikl}=N_{klij}. \tag{5·9}$$

Moreover this tensor must have the identity $N_{i(jkl)}=0$ from (5·7) and this is equivalent to (II), which is easily seen by contraction.

Now, if the conditions (II) are satisfied, the tensor $N_{ijkl}$ has the same properties with the tensor $M_{ijkl}$ in the last section. Hence, we can define the *type number* of space with non-unimodular connection, in terms of the symmetric part $P_{ij}$ of $K_{ij}$ instead of $K_{ij}$ itself. Consequently we have the

**Lemma:** *If a space with symmetric non-unimodular affine connection of dimensions $n(\geqq 3)$ is of type $\tau(\geqq 3)$, the equations (5·7) have s ch a real solution $P_{ij}(=P_{ji})$, that is uniqely determined to within algebraic sign if, and only if, the inequalities*

$$\begin{vmatrix} N_{abij} & N_{abjk} & N_{abki} \\ N_{bcij} & N_{bcjk} & N_{bcki} \\ N_{caij} & N_{cajk} & N_{caki} \end{vmatrix} \geqq 0, \tag{III′}$$

*and*

$$\sum_{a,b,c,i,j,k} \begin{vmatrix} N_{abij} & N_{abjk} & N_{abki} \\ N_{bcij} & N_{bcjk} & N_{bcki} \\ N_{caij} & N_{cajk} & N_{caki} \end{vmatrix} > 0, \tag{IV'}$$

*and the equations* (II) *and further*

$$R_n(N) = 0, \tag{V'}$$

*are satisfied; where the system of polynomials* $R_n(N)$ *of* $N_{ijkl}$ *are analogous to* (V).

Next, making use of $P_{ij}$ and (XI), we obtain the functions $P_{a\cdot ij}^{\cdot b}$ intrinsically [*], and, from (5·2), $K_{a\cdot ij}^{\cdot b}$. It is to be remarked here, that we have two kinds of solutions $P_{ij}$ and $\bar{P}_{ij}(=-P_{ij})$ satisfying (5·7) and hence, according to them, two kinds of functions $P_{a\cdot ij}^{\cdot b}$ and $\bar{P}_{a\cdot ij}^{\cdot b}$ $(=-P_{a\cdot ij}^{\cdot b})$ from (XI), and from (5·2) and (5·3) we get

$$\begin{aligned} K_{a\cdot ij}^{\cdot b} &= P_{a\cdot ij}^{\cdot b} - \frac{1}{2} R_{a\cdot ij}^{\cdot b}, \\ K_{ij} &= P_{ij} + Q_{ij}, \end{aligned} \tag{5·10}$$

and

$$\begin{aligned} \bar{K}_{a\cdot ij}^{\cdot b} &= -P_{a\cdot ij}^{\cdot b} - \frac{1}{2} R_{a\cdot ij}^{\cdot b}, \\ \bar{K}_{ij} &= -P_{ij} + Q_{ij}. \end{aligned} \tag{5·10'}$$

Now we impose the conditions (I′), which must be satisfied two intrinsic tensors $K_{a\cdot ij}^{\cdot b}$ and $K_{ij}$, and also $\bar{K}_{a\cdot ij}^{\cdot b}$ and $\bar{K}_{ij}$. Then we get

$$K_{a\cdot i[j}^{\cdot b} K_{k]l} - K_{a\cdot l[j}^{\cdot b} K_{k]i} = R_{a\cdot li}^{\cdot c} R_{c\cdot jk}^{\cdot b},$$

$$\bar{K}_{a\cdot i[j}^{\cdot b} \bar{K}_{k]l} - \bar{K}_{a\cdot l[j}^{\cdot b} \bar{K}_{k]i} = R_{a\cdot li}^{\cdot c} R_{c\cdot jk}^{\cdot b}.$$

Subtracting the above two equations and making use of (5·10) and (510′) give

$$R_{a\cdot i[j}^{\cdot b} P_{k]l} - R_{a\cdot l[j}^{\cdot b} P_{k]i} + 2P_{a\cdot i[j}^{\cdot b} Q_{k]l} - 2P_{a\cdot l[j}^{\cdot b} Q_{k]i} = 0,$$

and contracting with respect to $a$ and $b$ we have

$$P_{i[j} Q_{k]l} - P_{l[j} Q_{k]i} = 0, \tag{5·11}$$

By the similar methods, which we find the functions $K_{a\cdot ij}^{\cdot b}$ from (I), we obtain $Q_{ij}=0(i,j=1,\ldots,n)$ for $\tau\geqq 3$ on accont of (5·11), contradicting to hypothesis of $V_n$ to be non-unimodular. Therefore, if the conditions (I′) are imposed, we can not adopt both systems $P_{ij}, K_{ij}$ and $K_{a\cdot ij}^{\cdot b}$; and hence those functions are uniquely determined.

Now the functions $K_{a\cdot ij}^{\cdot b}$ so determined must satisfy the conditions (VII′) and (VIII′), which are formally analogous to (VII) and (VIII) respectively, on account of the definition of $K_{a\cdot ij}^{\cdot b}$ (4·3). The other hand, the condtion (VI) are satisfied naturally by means of (5·2) in this case. Making use of $K_{a\cdot ij}^{\cdot b}$ and (VII′), (VIII′) gives two systems of functions $H_{ij}(=H_{ij})$ and $H_j^i(i,j=,\ldots,n)$ satisfying (4·3) and therefore the Gauss equations (4·7) [*]. Consequently we have the

**Theorem 4.** ...... *If a space with symmetric non-unimodular affine connection of $n(\geqq 3)$-dimensions is of type $\tau(\geqq 3)$, there exist two systems of functions $H_{ij}(=H_{ij})$ and $H_j^i(i,j=1,\ldots,n)$ satisfying the Gauss equations* (2.7) *if, and only if, the inequalties* (III′) *and* (IV′), *and the equations* (I′), (II), (V′), (VII′), (VIII′) *and* (XI) *are satisfied.*

Next the Ricci equations (2·10) can be written in

$$H_{i,j}-H_{j,i}=2Q_{ij}, \qquad (5\cdot 12)$$

from which the functions $H_i$ can be always found. Because the integrability condition[5] of a system of partial differential equations (5·12) is

$$Q_{(ij,k)}=0,$$

that is

$$R_{a\cdot(ij,k)}^{\cdot a}=0,$$

i.e. the Bianchi identities. Thus we obtain the functions $H_i$ $(i=1,\ldots,n)$ satisfying (5·12) but not uniquely. Howeuer, this fact is not hindered in the following discussions.

Finally, in the similar manner with the case of unimodular connection we shall confine our consideration such a domain in $V_n$ that the functions $L_{ij}=K_{i\cdot ja}^{\cdot a}$ does not vanish. Then we put (4·9) and substitute in the first Codazzi equations (2·8) and have

$$\sigma_k L_{ij} - \sigma_j L_{ik} + L_{ijk} = 0 \tag{5·13}$$

analogous to (4·10) ; where we put

$$\sigma_j = \rho_j + H_j. \tag{5·14}$$

Hence we have

$$Q_n(L) = 0, \tag{IX'}$$

as a necessary and sufficient condition for (5·14) to have a solution $\sigma_j$ ; where the system of equations (IX′) is analogous to (IX), that is, the system of polynomials of the curvature tensor and its covariant derivatives. Then a solution $\sigma_j$ is uniquely determined and is expressed intrinsically [*].

The integrability condition of

$$\frac{\partial \log \rho}{\partial x^i} = \rho_i$$

is equivalent to the equation

$$\sigma_{j,k} - \sigma_{k,j} = 2Q_{jk}, \tag{X'}$$

from (5·14) and (5·12) ; where (X′) is a system of equations involving only the curvature tensor and its first and second covariant derivatives. Conversely if (X′) are satisfied by the solution $\sigma_j$ of (5·13), we obtain a function $\rho$ and from (4, 9) $H_{ij}(=H_{ji})$ $(i, j=1, \ldots, n)$ satisfying (2·9). Consequently we have the

**Theorem 5 :** *If a space $V_n$ with symmetric non-unimodular affine connection of $n(\geq 3)$-dimensions is of type $\tau(\geq 3)$, and the tensor $L_{ij}$ does not vanish, $V_n$ is of class one if, and only if, the inequalities* (III′) *and* (IV′), *and the equations* (I′), (II), (V′), (VII′), (VIII′), (IX′), (X′) *and* (XI) *are satisfied.*

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