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Normal Point and Tangent Cone of an Algebraic Variety

by

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Introduction.

In this paper we shall study the divisorial "holotomy group" of an algebraic variety.[1] When the variety is a cone, which is normal at its vertex, the local holotomy group can be described by the theory of "Picard variety." In the classical case this group is a direct sum of a compact dual group of the one dimensional local homology group at the vertex and a free abelian group. We then treat the general case by considering the "tangent cone" at the given point. When the tangent cone is normal at its vertex, so is the original variety. Moreover there exists a canonical homomorphism from the local holotomy group of the veriety into the same group of the tangent cone. Also there exists a canonical isomorphism from the local holotomy group of the variety into the similar group of its "sheet" at the given point. It seems that the nature of these homomorphisms depends on the "character" of the multiple point, about which the writer hopes to come back on another occasion.

Finally I wish to express my deepest appreciation to Prof. Akizuki for his constant interest in this paper.

1. Local holotomy group of the cone.

Let W^{d+1} be a cone in the affine $(n+1)$-space S^{n+1} over the "universal domain" $\boldsymbol{K}$.[2] Since the case $d=0$ is absurd, we shall

1) The holotomy groups were introduced together with some basic problems in the last part of Samuel's thesis, *La notion de multiplicité en algèbre et en géométrie algébrique*, Paris (1951). We cite this paper as S–T.

2) Cf. A. Weil, *Foundations of algebraic geometry*, Amer. Math. Soc. Colloquium Publications, Vol. 29 (1946).

assume $d \geqq 1$. Let $G(W)$ be the group of W-divisors which are generated by the "divisorial cones" on W. We note that every element of $G(W)$ is a local divisor of W at its vertex P.

Proposition 1. *The group $G(W)$ contains a representative of any local divisor of W at P with respect to holotomy.*

Proof. Let Z^d be a simple subvariety of W^{d+1} and let k be their common field of definition. Let $(\bar{x})$ and (x) be the generic points of Z and W over k and say $\bar{x}_0$ be a variable over k, then x_0 is also a variable over k and the point

$$M=(x_1/x_0, x_2/x_0, \cdots, x_n/x_0)$$

has a locus V^d over k. Let G_m be the multiplicative group of the universal domain and let T^{d+1} be the birational correspondence between W^{d+1} and the product variety $G_m \times V^d$ with the generic point $(x) \times ((1/x_0) \times M)$ over k. It is clear that T is biregular along Z; hence $T(Z)=pr_2(T\cdot(Z\times(G_m \times V)))$ is a $(G_m \times V)$-divisor. However since every G_m-divisor of degree zero is linearly equivalent to zero on G_m, every "correspondence" between G_m and V is of valence zero. In other words there exist a G_m-divisor $\mathfrak{a}$, a V-divisor X and a function φ on $G_m \times V$ such that

$$T(Z)=\mathfrak{a}\times V+G_m \times X+(\varphi).$$

Thereby T is biregular along every component of $\mathfrak{a} \times V$, $G_m \times X$ and (φ); hence Z is holotomic to the local divisor $T'(G_m \times X)$, which is surely an element of $G(W)$.

Let V^d be the projective variety with the representative cone W^{d+1}, then the following assertion is well known. *If W is normal at P, then W is everywhere locally normal and V is a projective normal variety; hence V is everywhere locally normal.* When Y^d is a representative cone of a variety X^{d-1} on V^d, we put

$$\Psi(Y)=X.$$

If we extend Ψ by linearity to $G(W)$, then Ψ gives a canonical isomorphism from $G(W)$ onto the group $\mathcal{G}_0(V)$ of all V-divisors. Thereby if Y is holotomic to zero on W at P, then $\Psi(Y)$ is holotomic to zero on V according to Chap. VI, § 4 in *S–T*. Since the converse is evident, Ψ induces an isomorphism from the local ho-

3) O. Zariski, *Some results in the arithmetic theory of algebraic varieties*, Amer. J. Math., vol. 61 (1939).

lotomy group $H_d(W;P)$ onto the global holotomy group $H_{d-1}(V)$.

On the other hand let $\mathcal{G}_l(V)$ be the group of V-divisors which are linearly equivalent to zero. Let C^{d-1} be the hyperplane section of V^d and let (C) be the free cyclic group which is generated by C, then a V-divisor X is holotomic to zero when and only when it is contained in $\mathcal{G}_l(V)+(C)$. Therefore we get the following theorem.

Theorem 1. *If the cone W^{d+1} is normal at its vertex P, there exists a canonical isomorphism Ψ from $H_d(W;P)$ onto*

$$H_{l-1}(V)=\frac{\mathcal{G}_0(V)}{\mathcal{G}_l(V)+(C)}.$$

We note that the group $H_{d-1}(V)$ can be described by the theory of " Picard variety " according to which it is a direct sum of an abelian variety and a finitely generated abelian group.

2. Relation with the local homology group.

In this section we shall assume that K is the field of all complex numbers. We shall first consider the principal bundle $\mathfrak{F}$ with the circle T as its fibre and with compact manifold V as it base space. If we denote by $\mathcal{H}(\mathfrak{F})$ and $\mathcal{H}(V)$ the one dimensional homology groups of $\mathfrak{F}$ and V with integer coefficients, the projection p from $\mathfrak{F}$ to V induces a homomorphism from $\mathcal{H}(\mathfrak{F})$ onto $\mathcal{H}(V)$. On the other hand $\mathfrak{F}$ has two dimensional cohomology class Ω with integer coefficients as its " characteristic class ".

Now an element x in the free abelian group G generates a cyclic group of the form $a\cdot Z$ with respect to a suitable base of G, where Z means the additive group of integers. The integer a does not depend on the choice of the base in G, and will be called the *index* of x in G. If we consider the two dimensional cohomology " with division " in V, we get a free abelian group. Therefore Ω has an index $i(\Omega)$ in this group.

Lemma. *The kernel of p: $\mathcal{H}(\mathfrak{F})\to\mathcal{H}(V)$ is a cyclic group of order $i(\Omega)$.*

This can be proved readily by examining the nature of the obstruction cocycle. However if we apply the powerful tool of "exact sequence",[4] we proceed as follows. The kernel is isomorphic with the factor group of Z modulo its subgroup of Kronecker indices $I(\Omega \cap z)$ where z are the two-cycles in V. On the other hand by the meaning of the cap product, the $G.C.D.$ of $I(\Omega \cap z)$ is nothing but $i(\Omega)$.

When V^d is a compact complex manifold, we are sometimes led to the consideration of the principal bundle $\tilde{\mathfrak{F}}$ over V with the fibre G_m.[5] we note that G_m decomposes into the product $T \times R$ of T and the additive group R of real numbers. Therefore $\tilde{\mathfrak{F}}$ is reducible to the principal bundle $\mathfrak{F}$ with the fibre T. On the other hand since V^d is orientable in this case, the characteristic class Ω can be represented by some $2d-2$ dimensional homology class over Z according to Poincaré's duality.

Now let V^d be a non-singular variety in the complex projective space L^n and let W^{d+1} be its representative cone with the vertex P, then the abstract variety

$$\tilde{\mathfrak{F}}^{d+1} = W - P$$

is a fibre bundle of this type.

Proposition 2. *The characteristic class Ω of $\tilde{\mathfrak{F}}$ is represented by $-C^{d-1}$ when C is the hyperplane section of V.*

Proof. If we denote by $X_0, X_1, \cdots, X_n$ the homogeneous coordinates in L^n, then, after a suitable projective transformation if necessary, we may assume that the intersection C_0 of V and the coordinate plane with the equation $X_0 = 0$ is an irreducible variety without multiple point over some field. If we put

$$f(x) = a(x) = (ax_0, ax_1, \cdots, ax_n)$$

with

$$a = \frac{|x_0|}{x_0 \cdot (\sum_{j=0}^{n} |x_j|^2)^{1/2}}$$

for the point (x) in $V - C_0$, it is clear that f is a cross-section of $V - C_0$ in $\mathfrak{F}$. We then cover V by a sufficiently fine simplicial complex K such that C_0 is a $(2d-2)$-cycle in the dual complex of

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4) Cf. S. S. Chern and E. Spanier, *The homology structure of fibre bundles*, Proc. Nat. Acad. Sci. U. S. A., vol. 36 (1950).

5) Cf. A. Weil. *Fibre-spaces in algebraic geometry*, Conference on algebraic geometry and algebraic number theory, Chicago (1949).

K. Let σ be a two-simplex of K which meets with C_0 at its barycenter s^0. Then the boundary $\partial\sigma$ of σ has a parametric representation of the form $(x)=(x_0(t), x_1(t), \cdots, x_n(t))\,(0\leqq t\leqq 1)$, where $x_0(t)=|x_0(t)|\cdot\exp(2\pi it)$ does not vanish on $\partial\sigma$. In this case we have

$$f(x(t))=\frac{\exp(-2\pi it)\cdot(x(t))}{(\sum_{j=0}^{n}|x_j(t)|^2)^{1/2}}$$

on $\partial\sigma$, hence the mapping f: $\partial\sigma\rightarrow T$ has degree -1.

Corollary. *The integer $i(\Omega)$ is a divisor of deg(V)*

$$\deg(V)\equiv 0 \pmod{i(\Omega)}.$$

Thereby the equality hols surely when V is a curve.

Now we shall assume that W is normal at P, then we can prove the following remarkable

Theorem 2. *In the classical case the local holotomy group $H_d(W;P)$ of the cone W is a direct sum of the compact dual group of the one dimensional local homology group of W at P and a free abelian group.*

Proof. The local homology group of W at P is, by definition, the homology group of the intersection of W and a sufficiently small sphere with center P. Therefore it is the homology group of the principal bundle $\mathfrak{F}$. Moreover we conclude from Lefschetz-Hodge's theorem[6] that a V-divisor has the same index in the $2d-2$ dimensional homology group with division either topological or algebraic.

On the other hand if we denote by $\mathcal{G}(V)$ the group of V-divisors which are periodic modulo $\mathcal{G}_n(V)+(C)$, where $\mathcal{G}_n(V)$ is defined by the numerical equivalence, we see that the factor group of $\mathcal{G}(V)$ modulo $\mathcal{G}_n(V)+(C)$ is a finite cyclic group of order $i(\Omega)$. We now define a group multiplication between the compact group $\mathcal{G}(V)/\mathcal{G}_l(V)+(C)$ and the discrete group $\mathcal{H}(\mathfrak{F})$ as follows.[7]

6) W. V. D. Hodge, *The theory and applications of harmonic integrals*, Cambgidge University Press (1941).

7) Cf. J. Igusa, *On the Picard varieties attached to algebraic varieties*, Amer. J. Math., vol. 74 (1952).

Let $\boldsymbol{X}$ be any element in $\mathcal{G}(\boldsymbol{V})$, then we can find an element $\boldsymbol{Y}$ in $\mathcal{G}_n(\boldsymbol{V})$ and an integer a such that

$$i(\Omega)\cdot\boldsymbol{X}=\boldsymbol{Y}+a\cdot\boldsymbol{C}.$$

Since $\boldsymbol{Y}$ is in $\mathcal{G}_n(\boldsymbol{V})$, it is a divisor of some "multiplicative function" F on $\boldsymbol{V}$. If $\boldsymbol{C}$ is the section of $\boldsymbol{V}$ and the hyperplane with the equation $\sum_{i=0}^{n}u_iX_i=0$, we attach the function

$$\varphi_X(x)=[F(x)\cdot(\sum_{i=0}^{n}u_ix_i)^a]^{1/i(\Omega)}$$

on $\mathfrak{F}$ to $\boldsymbol{X}$. It is clear that φ_X is a multiplicative function on $\mathfrak{F}$ with the divisor $\Psi^{-1}(\boldsymbol{X})$ and the pairing

$$\left.\begin{array}{l}\mathcal{G}(\boldsymbol{V})\ni\boldsymbol{X}\\ \mathcal{H}(\mathfrak{F})\ni\gamma\end{array}\right)\boldsymbol{X}\cdot\gamma=\text{multiplicator of } \varphi_X \text{ along } \gamma$$

induces an orthogonal multiplication of $\mathcal{G}(\boldsymbol{V})/\mathcal{G}_t(\boldsymbol{V})+(\boldsymbol{C})$ and $\mathcal{H}(\mathfrak{F})$.

Finally if we denote by $\rho(\boldsymbol{V})$ the "Picard number" of $\boldsymbol{V}$, the factor group $\mathcal{G}_0(\boldsymbol{V})/\mathcal{G}(\boldsymbol{V})$ is a free abelian group with $\rho(\boldsymbol{V})-1$ generators. Therefore since W is normal at P, our theorem follows from the previous theorem.

We note that the local theory of the representive cone at its vertex is more comprehensive than the global theory of the base variety.

3. Homomorphisms Γ and Δ.

Let K be an arbitrary field and let $\mathcal{O}$ be the ring of polynomials in $n+1$ letters $X_0, X_1,\cdots, X_n$ with coefficients in K. The elements of $\mathcal{O}$ with no constant terms form a maximal ideal $\mathcal{X}$ in $\mathcal{O}$, by which we can construct the quotient-ring $\mathcal{O}_X$. After Krull[8] we shall attach to every element $F(X)$ in $\mathcal{O}_X$ its "beginning form" $\bar{F}(X)$ and to every ideal $\mathfrak{A}$ in $\mathcal{O}_X$ its "direction ideal" $\bar{\mathfrak{A}}$. We note that $\bar{\mathfrak{A}}$ is the module which is generated by the beginning forms of the elements in $\mathfrak{A}$. On the other hand we can consider the "graded-ring" $\mathcal{G}(\mathcal{O}_X/\mathfrak{A})$ of the factor-ring $\mathcal{O}_X/\mathfrak{A}$ and we get the relation

8) W. Krull, *Dimensionstheorie in Stellenringen*, J. Reine Angew. Math., vol. 179 (1938)

$$\mathcal{G}(\mathcal{O}_X/\mathfrak{a})=\sum_{p=0}^{\infty}\frac{\mathfrak{k}^{p}\mathcal{O}_X+\mathfrak{a}}{\mathfrak{k}^{p+1}\mathcal{O}_X+\mathfrak{a}}=\mathcal{O}/\bar{\mathfrak{a}}.$$

Since the local rings

$$\mathfrak{o}=\mathcal{O}_X/\mathfrak{a} \text{ and } \bar{\mathfrak{o}}=\mathcal{O}_X/\bar{\mathfrak{a}}\mathcal{O}_X$$

have the same graded-ring, they have the same dimension and the same multiplicity. The first assertion is known as the "dimension theorem" of Krull; the following lemmas are due also to him. (1) *When two ideals $\mathfrak{a}_1\subset\mathfrak{a}_2$ in $\mathcal{O}_X$ have the same direction ideal, then $\mathfrak{a}_1=\mathfrak{a}_2$.* This is a simple consequence of the fact that every ideal in a local ring is closed with respect to its natural topology. A combination of this lemma and the dimension theorem leads to (2) *Let $\mathfrak{a}$ be an ideal in $\mathcal{O}_X$ such that every component of $\bar{\mathfrak{a}}$ has the same dimension, then every component of $\mathfrak{a}$ has the same dimension.* The following lemma is elementary and goes back to Lasker. (3) *Let $\mathfrak{a}$ be an ideal and $F(X)$ an element in $\mathcal{O}_X$ such that $\bar{F}(X)$ is prime to $\bar{\mathfrak{a}}$, then we have $\overline{\mathfrak{a}+F\mathcal{O}_X}=\bar{\mathfrak{a}}+\bar{F}\mathcal{O}$.*

These lemmas are valid also when $\mathcal{O}_X$ is replaced by its completion $\mathcal{O}^*$. We shall now prove the following

Proposition 3. *Let $\mathfrak{a}$ be an ideal in $\mathcal{O}_X$ such that $\bar{\mathfrak{a}}$ belongs to a variety W^{d+1}, then $\mathfrak{a}$ also belongs to a variety U^{d+1} of the same dimension which is analytically irreducible at the vertex P of the cone W.*

Proof. We first understand the word variety as an irreducible variety over K. Let $\mathfrak{p}$ be a minimal prime of $\mathfrak{a}$, then $\bar{\mathfrak{p}}$ contains $\bar{\mathfrak{a}}$. However since $\bar{\mathfrak{p}}$ and $\bar{\mathfrak{a}}$ have the same dimension and since $\bar{\mathfrak{a}}$ is prime, we have $\bar{\mathfrak{p}}=\bar{\mathfrak{a}}$. Therefore by lemma 1, we get $\mathfrak{p}=\mathfrak{a}$ and $\mathfrak{a}$ is prime. Moreover since a local ring and its completion have the same graded-ring, we conclude readily that $\mathfrak{a}$ remains prime in the completion $\mathcal{O}^*$ of $\mathcal{O}_X$.

On the other hand if W is an absolutely irreducible variety, $\bar{\mathfrak{a}}$ is absolutely prime. However since the field extension and the transition to the direction ideal are compatible, $\mathfrak{a}$ is also absolutely irreducible.

The following theorem is a natural generalization of Zariski's result according to which every simple point is a normal point.[9]

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9) O. Zariski, *Algebraic varieties over the ground field of characteristic zero,* Amer. J. Math., vol. 62 (1940).

Theorem 3. *If the tangent cone W is normal at P, then the variety U is also normal at P.*

Proof. Let $F_\lambda(X)$ be a set of polynomias in $\mathfrak{A}$ such that $\bar{F}_\lambda(X)$ form a base of $\bar{\mathfrak{A}}$, then $F_\lambda(X)$ form a base of $\mathfrak{A}$. Let ∂_β be a base of all derivations in K, then we denote by $F^{\partial_\beta}(X)$ the polynomial which is obtained by applying ∂_β to all coefficients in $F(X)$.

When W^{d+1} is an irreducible variety over K, then the "mixed Jacobian matrix"

$$\left(\frac{\partial \bar{F}_\lambda}{\partial X_\alpha}\quad F_\lambda^{\partial_\beta}(X)\right)$$

has rank $n-d$ on W except for the singular locus $\bar{\mathfrak{M}}$ of W with reference to K.[10] Therefore if we denote by $\bar{D}_\mu(X)$ the minors of order $n-d$ extracted from the above matrix, $\bar{\mathfrak{M}}$ is the common zero of $\bar{F}_\lambda(X)$ and $\bar{D}_\mu(X)$. On the other hand if $D_\mu(X)$ is constructed from $F_\lambda(X)$ as $\bar{D}_\mu(X)$ from $\bar{F}_\lambda(X)$, then, by the basic property of the determinant, $\bar{D}_\mu(X)$ is the beginning form of $D_\mu(X)$. Therefore according to the dimension theorem, the local common zero $\mathfrak{M}$ of $F_\lambda(X)$ and $D_\mu(X)$ has dimension at most equal to that of $\bar{\mathfrak{M}}$. Naturally $\mathfrak{M}$ is the singular locus of U at P with reference to K. Hence if W^{d+1} is is normal at P with reference to K, we have dim $\bar{\mathfrak{M}} \leqq d-1$; and a fortiori dim $\mathfrak{M} \leqq d-1$.

Moreover let $f\mathfrak{o}$ be a principal ideal in $\mathfrak{o}$, then it can be written in the form $f\mathfrak{o}=\mathfrak{A}+F\mathcal{O}_X/\mathfrak{A}$ with some polynomial $F(X)$. Thereby if $\bar{F}(X)$ is contained in $\bar{\mathfrak{A}}$, we can find a polynomial $A_0(X)$ in $\mathfrak{A}$ such that $\bar{F}(X)=\bar{A}_0(X)$. If we take $F_1(X)=F(X)-A_0(X)$ instead of $F(X)$ and repeat the same process, either we get a polynomial $F_p(X)$ such that $\bar{F}_p(X)$ is not contained in $\bar{\mathfrak{A}}$ or the process can be continued indefinitely. However in the latter case $F(X)$ would be contained in $\mathfrak{A}+\mathcal{K}^p\mathcal{O}_X$ for every p, hence $F(X)$ would be in $\mathfrak{A}$; a contradiction. Therefore by taking $F_p(X)$ instead of $F(X)$, we may assume that $\bar{F}(X)$ is not contained in $\bar{\mathfrak{A}}$. Then we have $\overline{\mathfrak{A}+F\mathcal{O}_X}=\bar{\mathfrak{A}}+\bar{F}\bar{\mathcal{O}}$ according to lemma 3. However since W is normal at P with reference to K, we see

10) O. Zariski, *The concept of a simple point of an abstract algebraic variety,* Trans. Amer. Math. Soc., vol. 62 (1947).

readily that every component of $\bar{\mathcal{A}}+\bar{F}\mathcal{O}$ has dimension d; and so is also $\mathcal{A}+F\mathcal{O}_x$ according to lemma 2. Thus the principal ideal $f\mathfrak{o}$ is unmixed and $\mathfrak{o}$ is integrally closed.

On the other hand if P is an absolutely normal point of an absolutely irreducible variety W, by the compatibility of the field extension and the transition to the direction ideal, U is also absolutely normal at P.

We have also proved the following

Supplement 1. *Let $\mathfrak{M}$ be the singular locus of U at P and let $\bar{\mathfrak{M}}$ be the singular locus of W, then it holds*

$$\dim \mathfrak{M} \leqq \dim \bar{\mathfrak{M}}.$$

Moreover the following supplement can be proved in the same way[11]

Supplement 2. *If W is normal at P, then the unique sheet U^* of U at P is normal.*

Now we shall assume that U is normal at P, hence is also U^* with reference to K. We define the local holotomy group $H_d(U^*;P)$ in an obvious manner, which is canonically isomorphic with the "class-group" in the completion $\mathfrak{o}^*$ of $\mathfrak{o}$. We shall denote this isomorphism by ϕ^*.

The following theorem is a slight generalization of Zariski's theorem according to which the quotient-ring of a simple point is a unique factorization domain.[12]

Theorem 4. *There exists a canonical isomorphism Γ from the local holotomy group $H_d(U;P)$ into $H_d(U^*;P)$.*

Proof. Let $\mathfrak{p}_\alpha$ be the highest prime in $\mathfrak{o}$ belonging to a local variety Z_α^d of U^{d+1} at P, then we put

$$\phi(\sum_\alpha m_\alpha Z_\alpha) = \bigcap_\alpha \mathfrak{p}_\alpha^{(m_\alpha)}.$$

In view of prop. 1 of Chap. VII, § 4 in *S-T*, the mapping ϕ induces a canonical isomorphism between $H_d(U;P)$ and the class-group in $\mathfrak{o}$. Moreover we put

$$\gamma(\bigcap_\alpha \mathfrak{p}_\alpha^{(m_\alpha)}) = (\bigcap_\alpha \mathfrak{p}_\alpha^{(m_\alpha)})\cdot\mathfrak{o}^*,$$

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11) Recently O. Zariski suceeded to prove that the unique sheet of any locally normal variety is also normal. This theorem is more general than our supplement 2.

12) O. Zariski, loc. cit. 10.

then γ gives a homomorphism from the class-group in $\mathfrak{o}$ into the class-group in $\mathfrak{o}^*$. However it holds

$$(\cap_{\alpha} \mathfrak{p}_\alpha^{(m_\alpha)}) \cdot \mathfrak{o}^* = \cap_{\alpha} \mathfrak{p}_\alpha^{(m_\alpha)} \mathfrak{o}^* = \cap_{\alpha\beta} \mathfrak{p}^*{}_{\alpha\beta}{}^{(m_\alpha)},$$

when $\mathfrak{p}_\alpha \mathfrak{o}^* = \cap_{\beta} \mathfrak{p}^*_{\alpha\beta}$ is a minimal representation.[13] Therefore if it is "quasi-equal" to a principal ideal, it must coincide with the principal ideal. On the other hand if the completion of any ideal $\mathfrak{z}$ in $\mathfrak{o}$ is a principal ideal $f^*\mathfrak{o}^*$, then it is a principal ideal in $\mathfrak{o}$.

In fact[14] let $a_1, a_2, \cdots, a_s$ be a base of $\mathfrak{z}$, then we can find s elements $b_1^*, b_2^*, \cdots, b_s^*$ in $\mathfrak{o}^*$ such that

$$a_i = f^* b_i^* \quad (i=1, 2, \cdots, s).$$

Thereby at least one of the b_i^* must be a unit accosding to Krull's theorem. However if $b_{i_0}^*$ is a unit, we have

$$\mathfrak{z} = a_{i_0} \mathfrak{o}^* \cap \mathfrak{o} = a_{i_0} \mathfrak{o}$$

as asserted.

Therefore if we put $\Gamma = \phi^{*-1} \circ \gamma \circ \phi$, Γ gives an isomorphism from $H_d(U; P)$ into $H_d(U^*; P)$.

If we assume that W is normal at P, then U and U^* are also normal at P and it holds the

Theorem 5. *Th re exists a canonical homomorphism Δ from the local holotomy group $H_d(U^*; P)$ into $H_d(W; P)$.*

We shall use the following lemma which follows immediately from the definition.

Lemma. *Let $\mathfrak{o}_i$ $(i=1,2)$ be two local rings of the same dimension such that $\mathfrak{o}_1$ is homomorphic to $\mathfrak{o}_2$, then we have*

$$e(\mathfrak{o}_1) \leqq e(\mathfrak{o}_2),$$

where $e(\mathfrak{o}_i)$ is the absolute multiplicity of $\mathfrak{o}_i (i=1,2)$.

Now we shall prove the theorem. Let $\mathfrak{z}^*$ be any ideal in $\mathfrak{o}^*$, then it can be written in the form $\mathfrak{z}^* = \mathfrak{Z}^*/\mathfrak{A}\mathcal{O}^*$ with some ideal $\mathfrak{Z}^*$ in $\mathcal{O}^*$. We put $\bar{\mathfrak{z}}^* = \bar{\mathfrak{Z}}^*/\bar{\mathfrak{A}}\mathcal{O}$, then the mapping

$$\delta(\mathfrak{z}^*) = \bar{\mathfrak{z}}^* \bar{\mathfrak{o}}$$

13) O. Zariski, *Analytical irreducibility of normal varieties*, Ann. of Math., vol. 49 (1948).

14) This part was proved independently by Y. Mori and Nagata.

induces a homomorphism from the class-group in $\mathfrak{o}^*$ into the class-group in $\bar{\mathfrak{o}}$.

We first note the identity

$$e[\mathfrak{o}^*/\bigcap_\alpha \mathfrak{p}_\alpha^{*(m_\alpha)}]=\sum_\alpha m_\alpha\cdot e(\mathfrak{o}^*/\mathfrak{p}_\alpha^*)$$

which follows from the definition. Moreover for any $\bigcap_\alpha \mathfrak{p}_\alpha^{*(m_\alpha)}$ we can find $\bigcap_{\alpha'} \mathfrak{p}_{\alpha'}^{*(m_{\alpha'})}$ which is prime to $\bigcap_\alpha \mathfrak{p}_\alpha^{*(m_\alpha)}$ such that their intersection $\bigcap_{\lambda=\alpha,\alpha'} \mathfrak{p}_\lambda^{*(m_\lambda)}$ is a principal ideal $f^*\mathfrak{o}^*$. Then as in the proof of theorem 3, we see that $\delta(f^*\mathfrak{o}^*)$ is a principal ideal $\bar{f}\bar{\mathfrak{o}}$ in $\bar{\mathfrak{o}}$. Since the local rings $\mathfrak{o}^*/f^*\mathfrak{o}^*$ and $\bar{\mathfrak{o}}/\bar{f}\bar{\mathfrak{o}}$ have the same graded-ring, it holds $e(\mathfrak{o}^*/f^*\mathfrak{o}^*)=e(\bar{\mathfrak{o}}/\bar{f}\bar{\mathfrak{o}})$ and similarly $e(\mathfrak{o}^*/\mathfrak{p}_\lambda^*)=e(\bar{\mathfrak{o}}/\delta(\mathfrak{p}_\lambda^*))$ $(\lambda=\alpha,\alpha')$. On the other hand if $\delta(\mathfrak{p}^*_\lambda)$ is quasi-equal to $\bigcap_\beta \tilde{\mathfrak{p}}_\beta^{(n_{\lambda\beta})}$, we get

$$\bigcap_\beta \tilde{\mathfrak{p}}_\beta^{(\Sigma\, m_\lambda n_{\lambda\beta})}\subset \bar{f}\bar{\mathfrak{o}}.$$

Therefore by the previous lemma it holds the following inequality

$$e(\bar{\mathfrak{o}}/\bar{f}\bar{\mathfrak{o}})\leqq e[\bar{\mathfrak{o}}/\bigcap_\beta \tilde{\mathfrak{p}}_\beta^{(\Sigma\, m_\lambda n_{\lambda\beta})}]\leqq e(\mathfrak{o}^*/f^*\mathfrak{o}^*).$$

However since the both sides are equal, the strict inequality can not hold. Since $\bar{f}\bar{\mathfrak{o}}$ is an intersection of the symbolic powers, we conclude the identity

$$\bigcap_\beta \tilde{\mathfrak{p}}_\beta^{(\Sigma\, m_\lambda n_{\lambda\beta})}=\bar{f}\bar{\mathfrak{o}}.$$

Therefore we see that $\delta(\bigcap_\alpha \mathfrak{p}_\alpha^{*(m_\alpha)})$ is quasi-equal to $\prod_\alpha \delta(\mathfrak{p}_\alpha^*)^{m_\alpha}$, hence δ induces a homomorphism from the class-group in $\mathfrak{o}^*$ into the class-group in $\bar{\mathfrak{o}}$.

Finally since there exists a canonical isomorphism $\bar{\phi}$ between $H_d(W;P)$ and the class-group in $\bar{\mathfrak{o}}$, the mapping $\Delta=\bar{\phi}^{-1}\circ\delta\circ\phi^*$ gives a homomorphism from $H_d(U^*;P)$ into $H_d(W;P)$.

We have defined the following three homomorphisms

$$H_d(U;P)\underset{\Gamma}{\to}H_d(U^*;P)\underset{\Delta}{\to}H_d(W;P)\underset{\Psi}{\to}H_{d-1}(V).$$

Thereby if we define directly a homorphism from $H_d(U;P)$ into $H_d(W;P)$ in the same way as Δ, we get $\Delta\circ\Gamma$. Also if we define the multiplicity function in $H_d(U;P)$ as

$$m(\sum_\alpha m_\alpha Z_\alpha^d)=\sum_\alpha m_\alpha\cdot e(\mathfrak{o}/\mathfrak{p}_\alpha)$$

and similarly in the others, it is preserved by Γ and Δ and coincides with deg. in $H_{d-1}(V)$.

Appendix

In the following we shall explain why we prefers the " direction ideal " to the " leading ideal " for the " Leitideal " of Krull. Let K be the universal domain and for the sake of simplicity let $\mathfrak{a}$ be a prime ideal in $\mathfrak{O}_X$. Then $\mathfrak{a}$ determines a variety U^{d+1} in S^{n+1} and if we denote by P the origin of the ambient space, $\mathfrak{O}_X/\mathfrak{a}$ is the " quotient-ring " of U at P with respect to the universal domain

$$Q_U(P)=\mathfrak{O}_X/\mathfrak{a}.$$

We shall denote by $\mathfrak{W}$ the bunch in S^{n+1} attached to $\bar{\mathfrak{a}}$ and *we shall show that* $\mathfrak{W}$ *consists of the " exceptional directions" D through P such that either $e[Q_U(P)]<i(P; U\cdot D)$ or $D\subset U$.*

Let k be a common field of definition of U and W over which the direction (a) of the ray D is rational. We may assume, say, $a_{j_0}=1$ and we take a set (u_{ij}) $(0\leqq i\leqq d,\ j\neq j_0)$ of $(d+1)n$ independent variables over k; we put

$$u_{ij_0}=-\sum_{j\neq j_0}u_{ij}a_j,\ K=k(u).$$

Then the set of equations $\sum_{j=0}^{n}u_{ij}X_j=0$ $(0\leqq i\leqq d)$ defines a generic linear variety L^{n-d} through D and, by definition, we get $i(P;\ U\cdot D)=i(P; U\cdot L)$ when one of them is defined. On the other hand let (x) be a generic point of U over K and $Q'_U(P)$ be the specialization-ring of P in $K(x)$. Then if D does not belong to $\mathfrak{W}$, we conclud without difficulties from the content of Chap. II in S–T that the ideal

$$\sum_{i=0}^{d}(\sum_{j=0}^{n}u_{ij}x_j)\cdot Q'_U(P)$$

in $Q'_U(P)$ has the same multiplicity as its maximal ideal. Therefore the symbol $i(P; U\cdot L)$ is defined and is equal to $e[Q_U(P)]$, hence D is not exceptional.

On the other hand let (U_{ij}) $(0\leqq i\leqq d+1, 0\leqq j\leqq n)$ be a set of $(d+2)(n+1)$ letters and put

$$Y_i=\sum_{j=0}^{n}U_{ij}X_j\ \ (0\leqq i\leqq d+1),$$

then the polynomials in $\mathfrak{a}$ which can be written without (X) under this substitution are of the form

$$\frac{P(U, Y)}{Q(U)} \cdot H(U, Y)$$

with some $P(U, Y)$ and $Q(U)$ and with a fixed $H(U, Y)$ in $\boldsymbol{K}[U, Y]$. We write $H(U, Y) = \sum_\lambda P_\lambda(U) \cdot H_\lambda(X)$ with $H_\lambda(X)$ in $\mathfrak{O}$ and with linearly independent $P_\lambda(U)$ in $\boldsymbol{K}[U]$. It is then clear that $H_\lambda(X)$ are polynomials in $\mathfrak{a}$, hence $\bar{H}_\lambda(X)$ are contained in $\bar{\mathfrak{a}}$. Moreover when the direction (a) of D is a common zero of $\bar{H}_\lambda(X)$, we conclude from the results of Chap. V, § 3 in *S–T* that D is exceptional.

The above consideration shows also that *the minimal primes of the direction ideal* $\bar{\mathfrak{a}}$ *have the same dimension*. Also we could formulate in geometric terms a suffiicient condition such that the homomorphism $\Delta \circ \Gamma$ is an isomorphism. Moreover in the classical case the variety U and the cone W have the same local homology group.