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# On the Genus of Algebraic Curves.

By

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1. It is well known that the genus of any curve of an algebraic system is not greater than the genus of the generic curve of this system. Recently W. L. Chow has given the algebraic proof of the above theorem.[1] We shall give here the complementary result to this, i. e.

**Theorem 1.** *If the generic curve of an algebraic syctem has no multiple point, then any irreducble member in this system without singular points, has the genus not less than that of the generic curve of this system.*

Combinig these two results we have the following

**Theorem 2.** *Under the same hypothesis as in Theorem 1, any irreducible member in the algebraic system of algebraic curves has the same genus as that of the generic curve, provided the former has no singular point.*

2. Let $V^r$ be a projective model of an algebraic variety immersed in the projective space $L^N$, defined over $k$, and such that $V$ has no singular point. Let $\varphi_i(X)$ $(i=0, 1, \dots, n)$ be the homogeneous forms of same degree in $(X)=(X_0, X_1, \dots, X_N)$, then the forms $\varphi_i(X)$ determines a linar system $\sum$ on $V$. Let $P=(x_0, x_1, \dots, x_N)$ be the generic point of $V$ over $k$ and $Q$ be the point in $L^n$ whose homogeneous coordinates are $(\varphi_0(x), \varphi_1(x), \dots, \varphi_n(x))$. Then the point $P\times Q$ has the locus $W$ in $V\times L^n$. Tet $W'$ be the proiection of $W$ on $L^n$, then it can readily be seen that the variety $W'$ is not contained in any linear subvariety of $L^n$ if and only if the forns $\varphi_0(X), \varphi_1(X), \dots, \varphi_n(X)$ are linearly independent on $V$. In this case we shall say that the variety $W'$ *belongs* to the projective space $L^n$. Let $U^s (s<r)$ be any simple subvariety of $V$, defined over the field $K$(containing $k$), such that $U$ has no singular point and the projection from $W$ to $V$ is regular along U.

1) W. L. Chow, on the genus of curves of an algebraic system. Trans. Amer. Math. Soc. Vol 65 (1949), pp 137–140.

Then from F-IV, th. 15[2], there is one and only one subvariety $\tilde{U}$ of $W$ such that $[\tilde{U}:U]=1$. Let $U'$ be the projection of $U$ on $L^n$, then we have the

**Lemma.** *If $U'$ belongs to the linrar subvariety $L^t$ of $L^n$, then the dimension of the linear system on $U$ induced by the linear system $\sum$ is equal to $t$.*

proof. Let

$$\sum_{j=0}^{n} c_{ij}\tilde{X}_j=0 \quad (i=1, \ldots, n-t)$$

be the definig equations for $L^t$, and $M$ be a generic point of $U$ over $K(c)$, then we have

$$\sum_{j=0}^{n} c_{ij}\,\varphi_j(M)=0 \quad (i=1 \ldots n-t)$$

Since the rank of the matrix $(c_{ij})$ is $n-t$, we can suppose that we have det. $|c_{ij}| \neq 0$ $(i=1, \ldots, n-t\ j=0, 1, \ldots, n-t-1)$, hence we can express $\varphi_j(M)$ $(j=0, 1, \ldots, n-t-1)$ as the linear combinations of the remaining $\varphi_l(M)$ $(l=n-t, \ldots, n)$, with the coefficients in $K(c)$. Moreover the forms $\varphi_l(M)$ $(l=n-t, \ldots, n)$ are linearly independent on $V$, otherwise the variety $U'$ will be contained in a linear subvariety $L^{t'}$ with $t'<t$, which contradicts to our assumptions. Thus the induced linear system has exactly the dimension $t$.

We shall denote this number by $\chi(\sum, V;\ U)$.

3. Let $U_1^s$, be another simple subvariety of $V$, such that the projection from $W$ to $V$ is regular along $U_1$. If $U_1$ is a specialisation of $U$ over $k$, then we shall show that we have $\chi(\sum, V; U_1) \leqq \chi(\sum, V; U)$. Let $\tilde{U}_1{}^*$ be the specialization of $\tilde{U}$ over the specialization $U \rightarrow U_1$ with reference to $k$. Then, since the specializations and the projections are commutable, the specialization of the projection of $\tilde{U}$ is equal to the projection of the specialization of $\tilde{U}$, i. e. $U_1=\mathrm{pr}_V\ \tilde{U}_1{}^*$. Let $\tilde{U}_1$ be the unique subvariety of $W$ such that $\mathrm{pr}_V\tilde{U}_1=U_1$, then since $\tilde{U}_1{}^*$ is contained in $W$ and $\mathrm{pr}_V\tilde{U}_1{}^*=U_1$, $\tilde{U}_1{}^*$ must be of the form $\tilde{U}_1{}^*=\tilde{U}_1+Y$, where $Y$ is a $W$-cycle such that $\mathrm{pr}_V\ Y=0$. From the argument in 2,

2) This means, the Theorem 15 of Chap. IV of "Foudations of Algebraic Geometry" by A. Weil.

$\chi(\Sigma, V; U_1)$ is eqnal to the dimension of the linear subvariety $L^{t'}$ of $L^n$, to which the projection $U'_1$ of $\tilde{U}_1$ on $L^n$ belongs, But since $\tilde{U}$ is csntained in $V\times L^t$, $\tilde{U}_1{}^*$, hence also $\tilde{U}_1$ must be contained in $V\times L^{t'}$ where $L^{t'}$ is the specialization of $L^t$ over $U\rightarrow U_1$ with reference to $k$, this means $t'\leq t$. Thus we get the required results.

4. Let $C$ be the generic member of the algebraic system of algebraic curves, defined over $k$, and $C'$ be any irreducible member of it. We shall suppose that they have no singular point. Let $\Sigma'$ be the linear system of all forms of degree $m$ in $L^N$, then it is well known that if $m$ is sufficiently large $\Sigma'$ containes a sublinear system $\Sigma$ which induces on both of $C$ and $C'$ the complete linear systems.[3] Then if we apply the argument of 2, and 3 to $L_N, C, C'$ and $\Sigma$ all the assumptions hold and these results are valid in our case, i. e. $\chi(\Sigma, L^N; C)\geqq\chi(\Sigma, L^N; C')$. If we denote by $h$ the degree of $C$ (hence also of $C'$) and $g, g'$ the genus of $C, C'$ respectively, then from the theorem of Riemann-Roch, we have if $m$ is sufficiently large

$$\chi(\Sigma, L^N; C)=\mathrm{mh}-g$$
$$\chi(\Sigma, L^N; C')=\mathrm{mh}-g'$$

Thus we have the desired result $g'\geqq g$.

5. Let $V^r$ be an algebraic variety immersed in a projective space $L^N$, $k$ a field of definition for $V$, and suppose that $V$ has no singular subvariety of dimension $r-1$. Then the generic 1-section $C$ of $V$ with generic $(N-r+1)$-linear subvariety $H$ is irreductble and has no singular point.[4] Then it is known that almost all 1-section is irreducible and has no singular point[5] Thus we have the following

**Theorem 3.** *Let $V^r$ be an algebraic variety, which has no sngular subvariety of dimeneion r-1, immersed in a projective space $L^N$. Then almost all l-section of V has the same genus as the genus of the generic l-section of V.*

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3) For the proof see e. g. W. L. Chow, loc. cit.

4) Y. Nakai, " Note on the intersection of an algebraic variety with the generic hʎperplɐne " Memoirs of the College of Science, University of Kyoto, series, A, Vol. XXVI, No. 2, 1951.

5) A. Seidenberg, " The hyperplane sections of normal varieties." Trans. Amer. Math. Soc. Vol. 69 (1950). Though his terminologies (following Zariski) are different from ours (following Weil) we can prove in the similar way that almost all 1-sections have no singular point.