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A Note on the Riemann-Roch-Weil's Theorem

By

Ryoichiro KAWAI

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The beautiful theory of hyperabelians functions through which A. Weil took the remarkable first step into the "non-abelian mathematics" is founded on the basis of the Riemann-Roch's theorem concerning with the generalized divisors which he introduced. He proved this theorem, using the abelian integrals of the 3rd kind, in a purely function-theoretical way. Under a remark of Mr. Igusa, that this theorem will be innerly related to the Riemann-Roch's theorem which E. Witt proved in the case of simple algebras over function-fields, in this note we shall show a relation between the above two theorems and prove the Weil's theorem in a purely algebraic way.

During my investigation I have received many kind advices from Mr. Igusa to whom I express my sincere gratitude.

§ 1. "Signature."

Let $K=k(x,y)$ be an algebraic function-field of one variable over an algebraically closed constant-field k, and let S be the ring of all matrices of degree m whose elements belong to K. We shall now construct a certain kind of Riemann-Roch's theorem in S. The letter P always denotes a prime divisor of K, and K_P, S_P denote the P-adic completion of K, S respectively.

We shall associate a positive integer $n=n(P)$ to each prime divisor P of K in the following way.

$$n(P)>1,\ (n,p)=1 \quad \text{for finite number of } P\neq P_\infty\text{'s},$$
$$n(P)=1 \quad \text{for the other prime divisors,}$$

where p is the characteristic of k. We shall call these integers $n(P)$ given in this way the "Signatures" of S (or of K).

For eachone of finite number of $P's$ for which $n(P)>1$, we choose a galois-extension Z_P such that $[Z_P : K_P]=n(P)$. Then the prime divisor P is completely ramified and therefore $P=\mathfrak{P}^n$

in Z_{P}. The ramification theorem of Hilbert shows that Z_{P}/K_{P} is cyclic as n is relatively prime to the characteristic p of K.

Lemma 1. *If $(n, p)=1$, there exists a number $\Pi // \boldsymbol{P}$ such that*

$$\Pi^{\sigma}=\zeta\Pi,$$

where σ is a generator of the galois-group of Z_{P} over K_{P}, and ζ is a primitive root of $x^{n}-1=0$.

Proof: Let Π be a number in $\boldsymbol{P}$ such that, $\Pi // \boldsymbol{P}$, then we have

$$\Pi^{\sigma^{i}}=\varepsilon_{i}\Pi \quad (i=1,2,\dots,n-1,\ \varepsilon_{0}=\varepsilon_{n}=1.)$$

with a unit ε_{i} of K_{P} and $\varepsilon_{i}=\varepsilon_{i-1}^{\sigma}\varepsilon_{1}$. Hence if we put

$$\varepsilon_{1}\equiv\eta \pmod{\boldsymbol{P}},$$

then we have $\qquad \varepsilon_{i}\equiv\eta^{i} \pmod{\boldsymbol{P}} \quad (i=1,2,\dots,n-1,n),$

therefore $\qquad \eta^{n}=1,$

that is $\qquad \eta=\zeta^{s},$

where ζ is a primitive root of $x^{n}-1=0$ and $1\leq s<n$.

Then a number

$$\bar{\Pi}=\sum_{i=0}^{n-1}\zeta^{-si}\Pi^{\sigma^{i}}=\Big(\sum_{i=0}^{n-1}\zeta^{-si}\varepsilon_{i}\Big)\Pi$$

satisfies all the conditions of the Lemma 1. For

$$\Big(\sum_{i=0}^{n-1}\zeta^{-si}\varepsilon_{i}\Big)\equiv n \pmod{\boldsymbol{P}},$$

this shows that $\bar{\Pi}//\boldsymbol{P}$, $\bar{\Pi}^{\sigma}=\zeta^{s}\bar{\Pi}$ and that $(s, n)=1$.

§ 2. Local divisors. (Canonical form.)

Let $P\cap k(x)=\mathfrak{p}$, and let $\mathfrak{o}_{P}$ be the integral domain of K with respect to $k(x)_{\mathfrak{p}}$, then $I_{P}=(\mathfrak{o}_{P})_{m}$, which is the set of all matrices of degree m over $\mathfrak{o}_{P}$ is a "Maximalordnung" of S and the other "Maximalordnung" I'_{P} of S_{P} are represented as

$$I'_{P}=\rho^{-1}I_{P}\rho$$

with a regular element ρ of S_{P}. I_{P} has only one two-sided prime ideal (P) and the other two-sided ideal of it are powers of (P).

In the case $n(P)=1$, all the left-ideals $\mathfrak{A}_{P}$ of I_{P} are principal and are uniquely normalized as

$$\mathfrak{A}_P = I_P \theta_P,$$

where
$$\theta_P = \begin{pmatrix} \theta_{11} & \theta_{12} & \cdots & \theta_{1n} \\ 0 & \theta_{22} & \theta_{23} \cdots & \theta_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0, & \theta^{nn} \end{pmatrix}$$

and $\theta_{ik}(i<k)$ are determined uniquely modulo θ_{ii} (see Weil [1], Witt [2]). We shall call a left-ideal $\mathfrak{A}_P$, for which θ_P is regular, a *local leftdivisor* of S for $n(P)=1$. If we restrict the elements of I_P to the set of all P-adic units, we get a Weil's divisor $U_P\theta_P$.

For $n(P)>1$, if a left-ideal $\mathfrak{A}_{\boldsymbol{P}} = I_{\boldsymbol{P}}\theta_{\boldsymbol{P}}$ of $I_{\boldsymbol{P}}$ in $Z_{\boldsymbol{P}}$ satisfies the following two conditions, then we shall call it a *local left-divisor* of S.

$$\theta_{\boldsymbol{P}} \text{ is regular in } S, \tag{1}$$

$$\mathfrak{A}^\sigma_{\boldsymbol{P}} = \mathfrak{A}_{\boldsymbol{P}} \text{ for all } \sigma \text{ of the galois-group of } Z_P \text{ over } K_P.$$

We shall call this $\theta_{\boldsymbol{P}}$ the representative of $\mathfrak{A}_{\boldsymbol{P}}$.

Let θ be a representative of a local divisor and let $\theta^\sigma = V\theta$, then the other representative of the same divisor is given by $\theta' = U\theta$, where U is a modulo $\boldsymbol{P}$ unimodular matrix of $S_{\boldsymbol{P}}$, and θ' satsfies $\theta'^\sigma = V'\theta'$. Clearly $V' = U^\sigma V U^{-1}$. Now if we put

$$\begin{cases} V \equiv A \pmod{\boldsymbol{P}} \\ V' \equiv A' \pmod{\boldsymbol{P}} \\ U \equiv U_0 \pmod{\boldsymbol{P}}, \end{cases}$$

then we have $A' = U_0 A U_0^{-1}$. And if we assume $\theta^{\sigma^\nu} = V_\nu \theta$ ($\nu = 1, \dots, n$), then we have $V_\nu = V^\sigma_{\nu-1} V$, therefore $V_\nu \equiv A^\nu \pmod{\boldsymbol{P}}$ ($V_0 = E_m$). From $A^n = E_m$, there exists a regular constant matrix M such that

$$A = M^{-1} D M, \quad D = (\delta_{ij}\zeta^{d_i}),$$

where ζ is a primitive root of $x^n - 1 = 0$ of Lemma 1., and d's are uniquely determined by

$$n-1 \geqq d_1 \geqq \dots\dots \geqq d_k \geqq 0 > d_{k+1} \geqq \dots\dots \geqq d_m \geqq -(n-1), \, d_1 - d_m < n.$$

Clearly these d's are characteristic to θ. Replacing θ by $\theta' = M\theta$ we get a divisor θ satsfying (we write θ instead of θ')

$$\theta^\sigma = V\theta, \quad V \equiv D \pmod{\boldsymbol{P}}, \quad V_\nu \equiv D^\nu \pmod{\boldsymbol{P}}.$$

Then the divisor

$$\bar{\theta}=\sum_{\nu=0}^{n-1}D^{-\nu}\theta^{\sigma^\nu}=(\sum_{\nu=0}^{n-1}D^{-\nu}V_\nu)\,\theta$$

represents the same divisor as θ, since

$$\sum_{\nu=0}^{n-1}D^{-\nu}V_\nu\equiv n\,E_m \qquad (\text{mod } \boldsymbol{P})$$

is modulo $\boldsymbol{P}$ unimodular, and clearly we have $\bar{\theta}^\sigma=D\bar{\theta}$. From now on we always choose as a representative of a divisor such a θ that satisfies $\theta^\sigma=D\theta$. Then if we take a matrix $\Delta=(\delta_{ij}\Pi^{d_i})$, so the matrix $\theta_0=\Delta^{-1}\theta$ satisfies $\theta_0^\sigma=\theta_0$, i.e., is a divisor of K_P. Hence we have proved

Lemma 2. For $n(P)>1$, each local left-divisor $\theta_{\boldsymbol{P}}$ is uniquely normalized in the following form

$$\theta_{\boldsymbol{P}}=\Delta_{\boldsymbol{P}}\theta_{0\boldsymbol{P}},$$

where

$$\Delta_{\boldsymbol{P}}=(\delta_{ij}\Pi^{d_i}),\ n-1\geqq d_1\geqq\cdots\cdots\geqq d_m\geqq-(n-1),\ d_1-d_m<n.$$

and $\theta_{0\boldsymbol{P}}$ is a local left-divisor of K_P.

§ 3. Divisors and their ideals.

If we were given a left-divisor $\mathfrak{A}=\prod_{n(P)=1}\mathfrak{A}_P\prod_{n(P)>1}\mathfrak{A}_{\boldsymbol{P}}$, where $\mathfrak{A}_P$ and $\mathfrak{A}_{\boldsymbol{P}}$ are all equal to E_m but a finite number of P, the set of the numbers of S

$$\alpha=\prod_{n(P)=1}\alpha_P\prod_{n(P)>1}\alpha_{\boldsymbol{P}},$$

which satisfy the conditions

$$\alpha_P\in\mathfrak{A}_P \qquad \text{for all } P\neq P_\infty,\ n(P)=1,$$

and

$$\mathfrak{A}_{\boldsymbol{P}}\in\mathfrak{A}_{\boldsymbol{P}} \qquad \text{for all } \boldsymbol{P},\ n(P)>1,$$

form an I-ideal $(\mathfrak{A})$. For (1°) if $\alpha,\beta\in\mathfrak{A}$, then it follows $\alpha_P\in\mathfrak{A}_P$, $\beta_P\in\mathfrak{A}_P$ for all $P\neq P_\infty$'s, $n(P)=1$ and $\alpha_{\boldsymbol{P}}\in\mathfrak{A}_{\boldsymbol{P}}$, $\beta_{\boldsymbol{P}}\in\mathfrak{A}_{\boldsymbol{P}}$ for all $\boldsymbol{P}$, $n(P)>1$, therefore $(\alpha\pm\beta)_P=\alpha_P\pm\beta_P\in\mathfrak{A}_P$ and $(\alpha\pm\beta)_{\boldsymbol{P}}=\alpha_{\boldsymbol{P}}\pm\beta_{\boldsymbol{P}}$ *i. e.* $\alpha\pm\beta\in(\mathfrak{A})$. (2°) If $\alpha\in(\mathfrak{A})$, $o\in I$, it follows that $(o\alpha)_P=o\ \alpha_P\subset I_P\alpha_P\subset\mathfrak{A}_P$ and $(o\alpha)_{\boldsymbol{P}}=o_{\boldsymbol{P}}\alpha_{\boldsymbol{P}}\subset I_{\boldsymbol{P}}\alpha_{\boldsymbol{P}}\subset\mathfrak{A}_{\boldsymbol{P}}$ *i. e.* $o\alpha\in(\mathfrak{A})$. (3°) Because $\mathfrak{A}_P$ is an I_P-ideal, there is a number μ_P such that $\mu_P\mathfrak{A}_P\subset I_P$ for each P, $n(P)=1$ and $\mu_{\boldsymbol{P}}\mathfrak{A}_{\boldsymbol{P}}\subset I_{\boldsymbol{P}}$ for $n(P)>1$. But μ_P (or $\mu_{\boldsymbol{P}}$) $=E_m$ all but a finite number of P(or $\boldsymbol{P}$). Let

$$\mu_P=(\mu_{ij}^{(P)}) \text{ and } \mu_{\boldsymbol{P}}=(\mu_{ij}^{(\boldsymbol{P})}) \text{ for } P=P_1,\dots,P_l,\ \boldsymbol{P}=\boldsymbol{P}_1,\dots,\boldsymbol{P}_l'$$

and $\mu_{ij}^{(P)}=\pi^{\nu_{ij}(P)}\varepsilon_{ij}^{(P)}$ and $\mu_{ij}^{(\boldsymbol{P})}=\Pi^{\nu_{ij}(\boldsymbol{P})}\varepsilon_{ij}^{(\boldsymbol{P})}$ $(\pi//P)$,

then there exists a matrix of S such that

$$\mu=(\mu_{ij}),\ \mu_{ij}=\pi^{\nu_{ij}}\varepsilon_{ij};\ \nu_{ij}\geqq\nu_{ij}^{(P)} \text{ and } \nu_{ij}\geqq\nu_{ij}^{(\boldsymbol{P})}$$

and clearly this μ satisfies $\mu(\mathfrak{A})\subset I$.

From the above we can conclude that every left-divisor uniquely determins a left-ideal of I, and that, if $\mathfrak{A}_P$ and $\mathfrak{A}_{\boldsymbol{P}}$ are normal, $(\mathfrak{A})$ is also normal and vice versa.

The above all things which we have proved about left-divisors and left-ideals are also true for any right-divisors and right-ideals. (See [2], [3]). If we are given a left-divisor $\mathfrak{A}=\Pi\ \mathfrak{A}_P\ \Pi\ \mathfrak{A}_{\boldsymbol{P}}$, then the problem of finding an element of S which satisfies the conditions

$$\mathfrak{A}_P\Phi\in I_P \text{ and } \mathfrak{A}_{\boldsymbol{P}}\Phi\in I_{\boldsymbol{P}} \text{ for all } P \text{ and } \boldsymbol{P},$$

is reduced to the problem of finding an element (of S) from the right-ideal $(\mathfrak{A}^{-1})$ such that

$$\Phi\in I_P \text{ for all } P_\infty\text{'s}$$

because of $\mathfrak{A}_P\mathfrak{A}_P^{-1}=I_P$ and $\mathfrak{A}_{\boldsymbol{P}}\mathfrak{A}_{\boldsymbol{P}}^{-1}=I_{\boldsymbol{P}}$ for all P and $\boldsymbol{P}$ (Cf. [2], [3], [4]). The number of linearly independent Φ satisfying (3), we shall call dim $\mathfrak{A}$. Let $\Phi=(\varphi_{ij})$ $(i,j=1,2,\dots,m)$ and assume that the given divisor $\mathfrak{A}=\prod_{n(P)=1}\theta_P\prod_{n(P)>1}\theta_{\boldsymbol{P}}$ is normalized such that $\theta_{\boldsymbol{P}}=\varDelta_{\boldsymbol{P}}\theta_{0P}$, $\varDelta$ and θ_{0P} means as before the fractional and integral part of the local divisor $\theta_{\boldsymbol{P}}$, then the second condition of (3) is transformed as follows:

If we put $\theta_0\Phi=\Psi$

in $$\varDelta\theta_0\cdot\Phi\in I_{\boldsymbol{P}},$$

then we have $\Psi\in I_P$ and $\theta_0\Phi\in I_P$

therefore Φ must lie in the ideal (θ_0^{-1}). And if we put $\Psi=(\psi_{ij})$, then we have

$$\varDelta\Psi=(\psi_{ij}\Pi^{d_i})\ \ (d_1\geqq d_2\geqq\dots\geqq d_m,\ n-1\geqq d_i\geqq-(n-1)),$$

and the condition $\varDelta\Psi\in I_{\boldsymbol{P}}$ insists that

$$\psi_{k+i,j}\equiv 0(\text{mod } P)\ \ \begin{pmatrix} i=1,2,\dots,m-k \\ j=1,2,\dots,m \end{pmatrix}.$$

Therefore Ψ must satisfy the above $m[m-k(P)]$ conditions and

$$\dim \mathfrak{A} = \dim \tilde{\mathfrak{A}} - m \sum_{n(P)>1} [m-k(P)] \qquad (4)$$

where $\tilde{\mathfrak{A}}$ denotes K-divisor

$$\tilde{\mathfrak{A}} = \prod_{n(P)=1} \theta_P \prod_{n(P)>1} \theta_{0P}.$$

§ 4. Riemann-Roch-Witt's theorem for given "Signatures".

Lemma 3. (*Riemann-Roch-Witt's theorem*).

$$\dim \tilde{\mathfrak{A}}_{12} = \deg \tilde{\mathfrak{A}}_{12} - G + 1 + \dim \tilde{\mathfrak{A}}^{21},$$

where $\tilde{\mathfrak{A}}_{12} \tilde{\mathfrak{A}}^{21} = \mathfrak{k}$ *and* $\mathfrak{k}$ *denotes the canonical divisor of* K, *and* G *the genus of* S, *and we assume that* $I_1 = I$.

The proof is well known, so we shall not write it down (see [2]). A. Well introduced a symbol $I(\theta)$ by

$$I(\theta) = \sum_{n(P)=1} I(\theta \ ) + \sum_{n(P)>1} I(\theta_{\mathbf{P}})$$

where $I(\theta_P)$ and $I(\theta_P)$ is defined for each P and $\mathbf{P}$ by

$$\det \theta_P = P^{I(\theta_P)} \text{ and } \det \theta_{\mathbf{P}} = \mathbf{P}^{I(\theta_{\mathbf{P}})}$$

The theorem 6 of Deuring's "Algebren" in VI § 4 (P. 82) (see [5]) shows that, if we put $P \cap k(x) = \mathfrak{p}$,

$$(\mathfrak{p}^{I(\theta_{0P})})^m = \mathfrak{p}^{\deg \theta_{0P}} \quad (\theta_{0P} \in S_P),$$

therefore we have

$$\deg \theta_{0P} = m\, I(\theta_{0P}). \qquad (5)$$

Hence
$$\deg \theta_0 = \sum_P \deg \theta_{0P} = m \sum_P I(\theta_P) = m\, I(\theta_0),$$

therefore in lemma 3 we have

$$\deg \tilde{\mathfrak{A}}_{12} = m \sum_{n(P)=1} I(\theta_P) + m \sum_{n(P)>1} I(\theta_{0P}).$$

According to the Weil's definition, if we put

$$\deg \mathfrak{A}_{12} = m[\sum_{n(P)=1} I(\theta_P) + \sum_{n(P)>1} I(\theta_{\mathbf{P}})],$$

so we have

$$\deg \mathfrak{A}_{12} = m \sum_{n(P)1} I(\theta_P) + m \sum_{n(P)>1}^{m} [I(\theta_{0P}) + \sum_{i=1} \frac{d_i}{n(P)}].$$

From Lemma 3, we have

$$\dim \mathfrak{A}_{12}=\deg \tilde{\mathfrak{A}}_{12}-G+1+\dim \tilde{\mathfrak{A}}^{21}-m\sum_{n(P)>1}[m-k(P)]$$

$$=\deg \mathfrak{A}_{12}-G+1+\dim \mathfrak{A}^{21}-m\sum_{n(P)>1}[\sum_{i=1}^{m}\frac{d_i}{n(P)}+m-k(P)]$$

$$=\deg \mathfrak{A}_{12}-G+1+\dim \tilde{\mathfrak{A}}^{21}-m\sum_{n(P)>1}[\sum_{i=1}^{k(P)}\frac{d_i}{n(P)}+\sum_{i=k+1}^{m}(1+\frac{d_i}{n(P)})]$$

$$=\deg \mathfrak{A}_{12}-G+1+\dim \tilde{\mathfrak{A}}^{21}-m\sum_{P}\sum_{i=1}^{m}\langle\frac{d_i}{n(P)}\rangle.$$

In this formula $\langle * \rangle$ denotes the fractional part of $*$, and dim $\mathfrak{A}$ denotes also the rank of the modul generated by the differntial matrices $d\Phi$ (Cf. [1]) satisfying

$$d\Phi\mathfrak{A}_{12}^{-1}\in I_P \text{ and } d\Phi\mathfrak{A}_{12}^{-1}\in I_{\boldsymbol{P}} \text{ for all } P \text{ and } \boldsymbol{P}.$$

For $n(P)=1$, from $d\Phi=\Phi k$, $\Phi\mathfrak{A}_{12}^{-1}k\in I_P$ and $d\Phi\mathfrak{A}_{12}^{-1}\in I_P$ are equivalent. And for $n(P)>1$, $\Phi\theta_0^{-1}\Delta^{-1}k\in I_{\boldsymbol{P}}$ and $d\Phi=\Phi k_{\boldsymbol{P}}=\Phi k\boldsymbol{P}^{n-1}$ shows the equivalence of $d\Phi\mathfrak{A}_{12}^{-1}\in I_{\boldsymbol{P}}$ and $\Phi\mathfrak{A}_{12}^{-1}k\in I_{\boldsymbol{P}}$.

Theorem 1. (*Witt's theorem for given "Signatures."*)

$$\dim \mathfrak{A}_{12}=\deg \mathfrak{A}_{12}-G+1-m\sum_{P}\sum_{i=1}^{m}\langle\frac{d_i}{n(P)}\rangle+\dim \tilde{\mathfrak{A}}^{21},$$

where $\tilde{\mathfrak{A}}^{21}$ *is reguarded as the dimension of* $d\Phi$ *which satisfies*

$$d\Phi\mathfrak{A}_{12}^{-1}\in I_P \text{ and } I_{\boldsymbol{P}} \text{ for all } P \text{ and } \boldsymbol{P},$$

Remark: In our case, the genus G of S is easily computed, and we have

$$G=m^2(g-1)+1,$$

where g is the genus of the function-field K.

§ 5. Relation to the Riemann-Roch-Weil's theorem.

If we are given two divisors θ and θ' of degree r and r' respectively, the rank of the modul generated by the following r by r' matrix Φ of K which satisfies the condition

$$\theta_P\Phi\theta_P'^{-1}\in I_P^{(r,r')} \text{ and } \theta_{\boldsymbol{P}}\Phi\theta_{\boldsymbol{P}}'^{-1}\in I_{\boldsymbol{P}}^{(r,r')} \text{ for all } P \text{ and} \boldsymbol{P},$$

is denoted by $N(\theta, \theta')$, where $I_P^{(r,r')}$ and $I_{\boldsymbol{P}}^{(r,r')}$ denote the modul of all r by r' matrices of o_P and $o_{\boldsymbol{P}}$ respectively. (See [1] Chapitre I, Cf. [5]). Using theorem 1, this number $N(\theta, \theta')$ is easily computed.

The Kroneckerian product $\theta\times{}^t\theta'^{-1}$ *i.e.*

$$\theta\times{}^t\theta'^{-1}=\prod_{n(P)=1}\theta_P\times{}^t\theta'^{-1}_P\prod_{n(P)>1}\theta_{\boldsymbol{P}}\times{}^t\theta'^{-1}_{\boldsymbol{P}}$$

gives also a divisor of $K_{rr'}$ in our sense. If we denote by $\dim(\theta\times{}^t\theta'^{-1})$ the rank of the modul generated by the elements of $K_{rr'}$ which are determined by the conditions

$$\theta_P\times{}^t\theta'^{-1}_P.\ \Phi\in I_P \text{ and } \theta_{\boldsymbol{P}}\times{}^t\theta'^{-1}_{\boldsymbol{P}}\cdot\Phi\in I_{\boldsymbol{P}} \text{ for all } P \text{ and } \boldsymbol{P}.$$

So we can easily verify that

$$\dim\ (\theta\times{}^t\theta'^{-1})=rr'\ N(\theta,\theta'). \tag{6}$$

On the other hand, by theorem 1

$$\dim\ (\theta\times{}^t\theta'^{-1})=\deg\ (\theta\times{}^t\theta^{-1})-G+1$$
$$-rr'\sum_P\sum_{i=1}^{r}\sum_{i'=1}^{r'}\left\langle\frac{d_i-d'_{i'}}{n(P)}\right\rangle+\dim\ ({}^t\tilde{\theta}'\times\tilde{\theta}^{-1}\cdot\boldsymbol{k}),$$

where $\tilde{\theta}=\prod_{n(P)=1}\theta_P\prod_{n(P)>1}\theta_{0P}$ and $\tilde{\theta}'=\prod_{n(P)=1}\theta'_P\prod_{n(P)>1}\theta'_{0P}$.

But using (5) and the remark of theorem 1, we have

$$\dim\ (\theta\times{}^t\theta^{-1})=rr'\,[r'\,I(\theta)-rI(\theta')]-(rr')^2\,(g-1)$$
$$-rr'\sum_P\sum_{i=1}^{r}\sum_{i'=1}^{r'}\left\langle\frac{d_i-d'_{i'}}{n(P)}\right\rangle+\dim\ ({}^t\tilde{\theta}'\times\tilde{\theta}^{-1}\cdot\boldsymbol{k}), \tag{7}$$

and $\dim\ ({}^t\tilde{\theta}'\times\tilde{\theta}^{-1}\cdot\boldsymbol{k})$ represents the number of linearly independent differential matrices $d\Phi$, which satisfies

$$d\Phi.\,{}^t\theta'_P\times\theta_P^{-1}\in I_P \text{ and } d\Phi.\ {}^t\theta'_{\boldsymbol{P}}\times\theta_{\boldsymbol{P}}^{-1}\in I_{\boldsymbol{P}} \text{ for all } P \text{ and } \boldsymbol{P}.$$

It is clear that this is rr'-times of the number $\sigma(\theta,\theta')$ of linearly independent r by r' differential matrices $d\Phi$ of K, which satisfies

$$\theta'_Pd\Phi\theta_P^{-1}\in I_P^{(r,r')} \text{ and } \theta'_{\boldsymbol{P}}d\Phi\theta_{\boldsymbol{P}}^{-1}\in I_{\boldsymbol{P}}^{(r,r')} \text{ for all } P \text{ and } \boldsymbol{P}.$$

So we have proved, by dividing the both side of (7) by rr'.

Theorem 2. (*Weil's theorem.*)

$$N(\theta,\theta')=r'I(\theta)-rI(\theta')-rr'(g-1)+\sum_P\sum_{i=1}^{r}\sum_{i'=1}^{r'}\left\langle\frac{d_i-d'_{i'}}{n(P)}\right\rangle$$
$$+\sigma(\theta,\theta')$$

where $\sigma(\theta,\theta')$ denotes the number of linearly independent r by

r' *differential matrices* $d\Phi$ *of* K.

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