MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A,
Vol. XXVII, Mathematics No. 2, 1952.

# On the Existence of Systems of Periodic Solutions for Several Nonlinear Circuits.

By

Sigeru MIZOHATA

(Received March 24, 1952)

The authour has obtained with M. Yamaguti a theorem on the existence of periodic solutions for nonlinear differential equations by a simple method.[1] Now we consider the natural extension of this method to nonlinear systems. Recently D. Graffi[2] has proved the existence of periodic systems for a type of nonlinear circuits. Here we will show a general principle which guarantees the existence of periodic systems for this type of nonlinear system, i. e.

$$\sum_{j=1}^{n} L_{ij}\ddot{x}_j + f_i(x_i)\dot{x}_i + \varphi_i(x_i) = p_i(t) \quad (i=1, 2, \dots, n), \tag{1}$$

where $L_{ij}=\text{const.}$, $L_{ij}=L_{ji}$, $\sum_{i,j} L_{ij}\xi_i\xi_j > 0$, when $|\xi_1| + \dots |\xi_n| \neq 0$, $p_i(t) \equiv p_i(t+\omega)$, $\int_0^\omega p_i(t)dt = 0$, $(i=1, 2, \dots, n)$, and the functions $f_i(x_i)$, $\varphi_i(x_i)$ are continuous and moreover the latter fulfil the condition of Lipshitz,[3] and $p_i(t)$ are continuous.

And, as examples, we will show Graffi's example (example I) and another of van der Pol's type (example II).

The principle is as follows:

*THEOREM. The system (1) possesses at least one system of periodic solution* $(x_1(t), \dots, x_n(t))$, $(x_i(t+\omega) \equiv x_i(t))$, *if the following conditions are fulfilled,*

*i)* $\operatorname{sgn} x_i \cdot \varphi_i(x_i) > 0$ for $|x_i| > q$, $\Phi_i(x_i) = \int_0^{x_i} \varphi_i(x_i)dx_i \to +\infty$, $(|x_i| \to \infty)$, $(i=1, 2, \dots, n)$.

*ii) there exist two constants* $r_0$ *and* $\varepsilon$ *such that*

$$A(x\,;\,t)^{4)} = \sum_{i,j} \tilde{L}_{ij}\varphi_i(x_i)\,[F_j(x_j) - P_j(t)] \geq \varepsilon(>0)$$

*for* $\sqrt{x_1^2 + \dots + x_n^2} \geq r_0$, *where* $F_i(x_i) = \int_0^{x_i} f_i(x_i)dx_i$, $P_i(t) = \int_0^t p_i(t)\,dt$

*and*

$$(\tilde{L}_{ij})\cdot(L_{ij})=(L_{ij})\cdot(\tilde{L}_{ij})=\begin{pmatrix}1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1\end{pmatrix}=(e).$$

PROOF. Put

$$y_i=\sum_{j=1}^{n}L_{ij}\dot{x}_j+F_i(x_i)-P_i(t).$$

Then the system of equations (1) is changed as follows:

$$\left.\begin{aligned}\dot{x}_i&=\sum_j \tilde{L}_{ij}\{y_j-F_j(x_j)+P_j(t)\}\\ \dot{y}_i&=-\varphi_i(x_i).\end{aligned}\right\}\qquad(2)$$

Now consider the quantity,

$$P(x_1,\dots,x_n;\ y_1,\dots,y_n)=P(x;\ y)=\sum_{i,j}\tilde{L}_{ij}y_iy_j/2+\sum_{i=1}^{n}\Phi_i(x_i),$$

then $\frac{d}{dt}P(x(t),\ y(t))=\sum_{i,j}\tilde{L}_{ij}y_i\{-\varphi_j(x_j)\}+\sum_i\varphi_i(x_i)\cdot[\sum_j\tilde{L}_{ij}\{y_j-F_j(x_j)$

$$+P_j(t)\}]=-\sum_{i,j}\tilde{L}_{ij}\varphi_i(x_i)\cdot[F_j(x_j)-P_j(t)]=-A(x;\ t).$$

The hypersurface $P(x;\ y)=C$, (we denote this by $S_c$), encloses in the 2n-dimensional space $(x_1,\dots x_n, y_1,\dots,y_n)$ a region $\mathfrak{D}$ homeomorphic to the interior of a sphere.

If $A(x;\ t)\geqq 0$, the proposition is true. In fact, we have $\frac{d}{dt}P=-A(x;\ t)\leqq 0$ for every trajectory which issues from a point on $S_c$, therefore this trajectory never goes out from $\bar{\mathfrak{D}}$ at this point, therefore every trajectory issuing from a point of $\bar{\mathfrak{D}}$ remains in that domoin for increasing $t$.

If Min $A(x;\ t)=-m(m>0)$ for $|x|\leqq r_0$, it happens that $\frac{d}{dt}P=-A(x;\ t)$ is positive in $|x|\leqq r_0$, therefore a curve $(x(t), y(t))$ which issues at the time $t_0$ from a point of the hypersurface $S_c$ for $|x(t_0)|<r_0$ may pass through the exterior domain of $S_c$ for $t>t_0$.

But we shall see as follows that this curve will soon enter into $\mathfrak{D}$, and this curve remains bounded. The proof is as follows: we choose $R_0$ sufficiently large (for example $R_0>r_0\left(1+6\frac{m}{\varepsilon}\right)$), and hereafter consider in the domain $|x| < R_0$ in the $x$-space. Our assertion is to prove that every solution $(x(t), y(t))$ which issues from a point $(x(t_0), y(t_0)) \in S_c$, where $|x(t_0)| < r_0$, and passes through the exterior domain of $S_c$ will again intersects $S_c$ for $|x(t)| < R_0$ and enter into $\mathfrak{D}$, and $(x(t), y(t))$ is bounded for $t \geqq t_0$, if $C$ is chosen sufficiently large.

To prove this, we change the coordinates $y_i \rightarrow Y_i$ by

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} \tilde{L}_{ij} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Then the equations (2) are changed as follows:

$$\left.\begin{aligned} \dot{x}_i &= Y_i - \sum_j \tilde{L}_{ij}(F_j - P_j) = Y_i - \psi_i(x, t) \\ \dot{Y}_i &= -\sum_j \tilde{L}_{ij} \cdot \varphi_j(x_j). \end{aligned}\right\} \tag{3}$$

The hypersurface is defined as

$$\frac{1}{2} \sum_{i,j=1}^{n} L_{ij} Y_i Y_j + \sum_{i=1}^{n} \Phi_i(x_i) = C.$$

If $C$ is chosen sufficiently large, for $|x| < R_0$ max $(|Y_1|, \dots, |Y_n|)$ on $S_c$ becomes uniformly large. Taking into account the fact that $\psi_i(x, t)$ $(i=1, 2, \dots, n)$ are bounded for $|x| < R_0$, we see that, if $C$ is chosen sufficiently large, the trajectory $(x(t))$ in the $x$-space corresponding to the curve $(x(t), y(t))$ which issues from a point of $S_c$ for $|x(t_0)| < r_0$ will pass through the domain $|x| < R_0$ in sufficiently short time. In fact, for every point $(x, Y)$ on $S_c$, one of $Y_i$, $Y_{i_p}$ must be sufficiently large, while $\dfrac{dY_i}{dx_i} = \dfrac{\dot{Y}_i}{\dot{x}_i} = \dfrac{-\sum_j \tilde{L}_{ij}\varphi_j(x_j)}{Y_i - \psi_i(x, t)}$ shows that $\left|\dfrac{\delta Y_{i_p}}{Y_{i_p}}\right|$ is sufficiently small for such a trajectory, where $\delta Y_{i_p}$ is the variation of $Y_{i_p}$ corresponding to the part of trajectory such that $|x(t)| \leqq R_0$. Therefore, taking account of $\dot{x}_i = Y_i - \psi_i(x, t)$, we see that the point $(x(t))$ will pass through the domain $|x| \leqq R_0$ in sufficiently short time.

It follows that the variation $\delta Y$ is sufficiently small, therefore the direction cosines of the curve $(x_1(t), x_2(t), \dots, x_n(t))$ are almost constant, that is the curve $(x(t))$ behaves like a straight line, i. e. $\dfrac{dx_i}{ds} = \alpha_i + \varepsilon_i(s)$ $(i=1, 2, \dots, n)$ $\sum_{i=1}^{n} \alpha_i^2 = 1$, $\varepsilon_i$ are uniformly small, where $s$ is the arc length of the trajectory $(x(t))$ in the $x$-space.

Then we consider the variation of $P$ along the trajectory

$$\delta P=\int\frac{dP}{dt}dt=\int\dot{P}/\dot{s}\,ds=\int_{|x(t)|<r_0}\dot{P}/\dot{s}\,ds+\int_{r_0\leq|x(t)|\leq R_0}\dot{P}/\dot{s}\,ds.$$

In the second member, the second term is negative and the first term may be positive. But we easily see $\delta P$ negative.

In fact, consider the ratio of the two functions to integrate,

$$\frac{\dot{P}/\dot{s}\text{ in }|x(t)|\leq r_0}{|\dot{P}/\dot{s}\text{ in }r_0\leq|x(t)|\leq R_0|}=\frac{\dot{P}\text{in }|x|\leq r_0}{|\dot{P}\text{in }r_0\leq|x|\leq R_0|}\cdot\frac{\dot{s}\text{ in }r_0\leq|x|\leq R_0}{\dot{s}\text{ in }|x|\leq r_0}$$

taking into accout the fact that the second factor is nearly equal to 1, and $(\dot{P}\text{in }|x|\leq r_0)\leqq m$, $(\dot{P}\text{in }r_0\leq|x|\leq R_0)\leqq-\varepsilon$, we see that the ratio is less than $\frac{m}{\varepsilon}\times 2$. Secondly the arc length of the part $|x(t)|<r_0$ is less than $3r_0$, because the trajectory behaves like a straight line, and that of the second term is greater than $R_0-r_0$, therefore we see that

$$\int_{|x(t)|\leq r_0}\dot{P}/\dot{s}\,ds:\left|\int_{r_0\leq|x(t)|\leq R_0}\dot{P}/\dot{s}\,ds\right|\leqq\frac{m}{\varepsilon}\cdot 2\cdot\frac{3r_0}{R_0-r_0}=\frac{r_0}{R_0-r_0}\times 6\cdot\frac{m}{\varepsilon}.$$

As we have chosen $R_0$ so as to let this ratio be less than 1, we see that $\delta P<0$.

Now we shall consider the continuous mapping $T$: $(x(t_0),y(t_0))\to(x(t_0+\omega),\ y(t_0+\omega))$, ($\omega$ is the common period of $p_i(t)$), where $(x(t_0+\omega),y(t_0+\omega))$ is the point of the trajectory at the time $t_0+\omega$ corresponding to the initial value $(x(t_0),y(t_0))$. We shall prove that, if $(x(t_0),y(t_0))\in S_c$, the image $(x(t_0+\omega),y(t_0+\omega))$ belongs to $\mathfrak{D}$.

We shall prove this by contradiction, so we suppose that there exists a point $(x(t_0+\omega),\ y(t_0+\omega))$ not belonging to $\mathfrak{D}$, in this case the initial point $(x(t_0),y(t_0))$ fulfils either i) $|x(t_0)|>R_0$ or ii) $|x(t_0)|\leq R_0$

In the first case, if we take $\varepsilon(>0)$ sufficiently small, we have $|x(t_0+\varepsilon)|>R_0$ and that $(x(t_0+\varepsilon),y(t_0+\varepsilon))\in\mathfrak{D}$ (because at the point $x(t_0)$ such that $|x(t_0)|>R_0$, we have $\frac{d}{dt}P\leq-\varepsilon$), therefore there would exist a time $\tau$ between $t_0+\varepsilon$ and $t_0+\omega$ such that $(x(\tau),\ y(\tau))\in S_c$, $t_0+\varepsilon<\tau\leqq t_0+\omega$ and that $(x(t),\ y(t))\in\mathfrak{D}$, for $t_0+\varepsilon\leqq t<\tau$.

On the other hand, we should have $|x(\tau)|<r_0$ (as we have

$\frac{d}{dt}P \geq 0$ at this point), therefore by tracing the trajectory in the reverse sense for time, and taking account of the relation $\delta P < 0$ (which we have just obtained by the above analysis), there would exist a time $\tau'$ between $t_0+\varepsilon$ and $\tau$ such that

$(x(\tau'), y(\tau')) \notin \mathfrak{D}$, (that is, lies in the exterior domain of $S_c$). This contradicts with the choice of $\tau$. In the second case, i. e. if $|x(t_0)| \leq R_0$, by the above analysis, we can choose $\varepsilon(>0)$ such that $|x(t_0+\varepsilon)| > R_0$, where $0<\varepsilon<\omega$, and that $(x(t_0+\varepsilon), y(t_0+\varepsilon)) \in \mathfrak{D}$. This situation is entirely the same as i), thus in both cases, we meet with the contradiction.

As we have proved $T(S_c) \subset \mathfrak{D}$, ($T(S_c)$ is the image of $S_c$ by the transformation $T$), and as the mapping $T$ is topological, we have finally

$$T(\bar{\mathfrak{D}}) \subset \mathfrak{D}.$$

As the closed domain $\bar{\mathfrak{D}}$ is homeomorphic to $2n$-dimensional sphere (i. e. $\sum_{i=1}^{2n} X_i^2 \leq 1$), by applying Brouwer's fixed point theorem, we conclude that there exists at least one fixed point, that is

$$(x(t_0), y(t_0)) = (x(t_0+\omega),\ y(t_0+\omega)).$$

Therefore the trajectory corresponding to this initial value $(x(t_0),\ y(t_0))$ must be periodic.

Thus we have finished the proof of the theorem.

Example I. *Graffi's example.*

The system of differential equations is as follows:

$$\begin{cases} L_1\ddot{x}_1 + M\ddot{x}_2 + f_1(x_1)\dot{x}_1 + \varphi_1(x_1) = p_1(t), \\ M\ddot{x}_1 + L_2\ddot{x}_2 + f_2(x_2)\dot{x}_2 + \varphi_2(x_2) = p_2(t). \end{cases} \left(\begin{matrix} L_i,\ M>0,\ L_1L_2 - M^2 > 0 \\ p_i(t+\omega) \equiv p_i(t) \end{matrix}\right),$$

where $\frac{F_i(x_i)}{x_i} \to R_i$, $(|x_i| \to +\infty)$, and $\frac{\varphi_i(x_i)}{x_i} \to \frac{1}{C_i}$, $(|x_i| \to \infty)$, ($R_i$, $C_i$ are positive).

If the condition

$$4L_1L_2\frac{R_1}{C_1}\frac{R_2}{C_2} > M^2\left(\frac{R_1}{C_2} + \frac{R_2}{C_1}\right)^2 \tag{4}$$

be satisfied, the existence of at least one system of periodic solutions may be concluded.

PROOF. We have only to show that the condition ii) of the theorem is fulfilled.

$$\text{Here}\quad (L_{ij})=\begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix},\ (\tilde{L}_{ij})=\begin{pmatrix} \frac{L_2}{\delta} & -\frac{M}{\delta} \\ -\frac{M}{\delta} & \frac{L_1}{\delta} \end{pmatrix},\ (\delta=L_1L_2-M^2),$$

$$A(x_1, x_2;\ t)=\frac{1}{\delta}[L_2\varphi_1(x_1)\{F_1(x_1)-P_1(t)\}-M\varphi_1(x_1)\{F_2(x_2)-P_2(t)\}$$
$$-M\varphi_2(x_2)\{F_1(x_1)-P_1(t)\}+L_1\varphi_2(x_2)\{F_2(x_2)-P_2(t)\}]$$
$$=\frac{1}{\delta}\left\{L_2\frac{R_1}{C_1}x_1^2-M\left(\frac{R_2}{C_1}+\frac{R_1}{C_2}\right)x_1x_2+L_1\frac{R_2}{C_2}x_2^2\right\}+\lambda_1(x_1, x_2;\ t)x_1^2$$
$$+\lambda_2(x;\ t)x_1x_2+\lambda_3(x;\ t)x_2^2,$$

where $\lambda_i(x;t)\to 0$, when $\begin{cases}|x_1|\to\infty \\ |x_2|\to\infty\end{cases}$.

Therefore by the relation (5), we can find $\varepsilon$, and $L$ such that

$$A(x;\ t)\geq\varepsilon,\ \text{for}\ |x_i|\geq L\ \ (i=1,2).$$

On the other hand, from the hypotheses that $F_i(x_i)\to\pm\infty$, $\varphi_i(x_i)\to\pm\infty$ $(x_i\to\pm\infty)$, and $P_i(t)$ are bounded, we can find $L'(\geqq L)$ such that

$$A(x;\ t)\geqq\varepsilon\quad \text{for}\ |x_1|\leqq L,\ |x_2|\geqq L'.$$
$$\text{for}\ |x_2|\leqq L,\ |x_1|\geqq L'.$$

Therefore for $\sqrt{x_1^2+x_2^2}\geqq 2L'$, we have $A(x;\ t)\geqq\varepsilon$. Q.E.D.

Example II. *The system of van der Pol's type.*
The system is as follows:

$$\begin{cases} L_1\ddot{x}_1+M\ddot{x}_2+\mu_1(x_1^2-a_1)\dot{x}_1+k_1x_1=p_1(t), \\ M\ddot{x}_1+L_2\ddot{x}_2+\mu_2(x_2^2-a_2)\dot{x}_2+k_2x_2=p_2(t), \end{cases}\left(\begin{matrix} L_i,\ M>0,\ L_1L_2-M^2>0 \\ \mu_i>0,\ k_i>0,\ p_i(t+\omega)\equiv p_i(t)\end{matrix}\right).$$

If $M$ is sufficiently small with respect to $L_1, L_2$, or if $L_1, \mu_1, k_1$ differ slightly from $L_2, \mu_2, k_2$ respectively, that is, if

$$M<\sqrt[4]{L_1L_2k_1k_2\mu_1\mu_2}\cdot\text{Min}\left(\frac{\sqrt{L_2\mu_1k_1}}{\mu_1k_2},\ \frac{\sqrt{L_1k_2\mu_2}}{\mu_2k_1}\right),\tag{5}$$

we can conclude the existence of at least one periodic system.

PROOF.

$$A(x\,;t)=\frac{1}{\delta}\Big[L_2k_1x_1\Big\{\frac{\mu_1}{3}x_1^3-\frac{\mu_1a_1}{2}x_1^2-P_1(t)\Big\}+\dots\ \Big]$$
$$=\frac{1}{3\delta}\Big[L_2k_1\mu_1x_1^4-M(k_1\mu_2x_1x_2^3+k_2\mu_1x_1^3x_2)+L_1k_2\mu_2x_2^4\Big]+R(x\,;\,t).$$

To asssert that $A(x;\,t)\geqq\varepsilon(>0)$ for both sufficiently large $|x_1|$, $|x_2|$, we have only to show the first term is positive for $|x_1|+|x_2|\geq L(>0)$. For this purpose we consider the following ratio for $|x_1|+|x_2|>0$, $x_i\geq 0$, $\dfrac{L_2k_1\mu_1x_1^4+L_1k_2\mu_2x_2^4}{k_1\mu_2x_1x_2^3+k_2\mu_1x_1^3x_2}$, by easy computation we have,

$$\geqq\frac{1}{2}\cdot\frac{(\sqrt{L_2k_1\mu_1}x_1^2+\sqrt{L_1k_2\mu_2}x_2^2)^2}{k_1\mu_2x_1x_2^3+k_2\mu_1x_1^3x_2}\geqq\frac{\sqrt{L_2k_1\mu_2}x_1^2+\sqrt{L_1k_2\mu_2}\,x_2^2}{\mu_1k_2x_1^2+\mu_2k_1x_2^2}$$
$$\cdot\sqrt[4]{L_1L_2\mu_1\mu_2k_1k_2}\,.$$

As the first factor of this last member is greater than Min $\left(\dfrac{\sqrt{L_2\mu_1k_1}}{\mu_1k_2},\ \dfrac{\sqrt{L_1k_2\mu_2}}{\mu_2k_1}\right)$, and by the hypothesis (5), the ratio $\dfrac{L_2k_1\mu_1x_1^4+L_1k_2\mu_2x_2^4}{M(k_1\mu_2x_1x_2^3+k_2\mu_1x_1^3x_2)}\geqq(1+\delta)$, $(\delta>0)$. By the same reasoning as example I, we can conclude $A(x;\,t)\geqq\varepsilon(>0)$ for $|x|>r_0$, properly chosen. Q. E. D.

## Footnote.

1) S. Mizohata and M. Yamaguti: On the existence of periodic solutions of the non-linear differential equation, $\ddot{x}+a(x)\cdot\dot{x}+\varphi(x)=p(t)$. (This Mem.).

2) D. Graffi: Forced oscillations for several nonlinear circuits. p. 262–271. Ann. of Math. 54 (1951).

3) These conditions are only sufficient conditions to guarantee the unicity of trajectories in the phase space in which we will consider.

4) Hereafter we denote $(x_1,\cdots,x_n)$ by $x$, and $\sqrt{x_1^2+\cdots+x_n^2}$ by $|x|$. We should remark that when we write for example $\psi(x)$, it means $\psi(x_1,\cdots,x_n)$ defined in the $(x_1,\cdots,x_n)$-space (we denote this briefly by $x$-space), on the other hand when we write $\psi(x_i)$ it means a real function defined in $-\infty<x_i<+\infty$.