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On a Property of the Domain of Regularity.

By

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In this short note we shall remark that the *analytic structure* of the domain of regularity is completely characterized by the *algebraic structure* of the ring of analytic functions on it. It would be hardly necessary to notice that the assumption of analytic completeness is indispensable. Although the proof is simple in the case of one variable, we can not treat the general case without a relatively recent result in function theory.

Let D be a (univalent) domain of regularity in the complex n-space C^n, where C means the field of complex numbers. We shall denote by

$$(z) = (z_1, z_2, \dots, z_n)$$

the complex coordinates in C^n. Let $\mathfrak{O}$ be the ring of all analytic functions in D, which we shall not topologize here. If $\mathfrak{P}$ is any maximal ideal in $\mathfrak{O}$, we can identify C as a subfield of the residue-class field $\mathfrak{O}/\mathfrak{P}$ in an obvious manner. The set of analytic functions in D, which vanish at a given point (a) of D, is a maximal ideal $\mathfrak{P}$ in $\mathfrak{O}$. We say that such a maximal ideal $\mathfrak{P}$ corresponds to the point (a) in D. Now not every maximal ideal is of this type as the following simple example shows.

Example. Let D be the whole C^n and put

$$F_j(z) = \frac{e^{2\pi i z_1}-1}{z_1 \cdot \Pi_{k=1}^{j}(z_1^2-k^2)} \quad (j=1, 2, \dots),$$

then these functions generate an ideal other than $\mathfrak{O}$ which has no common zero point. Therefore, by *Zorn's lemma*, this ideal is contained in some maximal ideal which has no common zero point.

Thus the problem arizes how to characterize *algebraically* the maximal ideal in $\mathfrak{O}$ which corresponds to the point of D. Here we notice that the residue-class field of the maximal ideal is C, when this corresponds to the point of D. We take the set

$$\hat{D} = \{ \mathfrak{P} \}$$

of all maximal ideals in $\mathfrak{O}$, which have this property. We shall then show that the maximal ideal $\mathfrak{P}$ in $\hat{D}$ corresponds to the point of D.

Let a_i be the residue-classes of $z_i (1 \leqq i \leqq n)$ modulo $\mathfrak{P}$, then the n linear polynomials $z_1 - a_1, z_2 - a_2, \ldots, z_n - a_n$ belong to $\mathfrak{P}$. We shall now ask whether the equation

$$F(z) - F(a) = \sum_{i=1}^{n} A_i(z)(z_i - a_i)$$

can be solved by n analytic functions $A_i(z) (1 \leqq i \leqq n)$ in D for any F in $\mathfrak{O}$. This question was known as *Weil's condition*[1] and is proved affirmatively by *Cartan-Oka's ideal theory*.[2] Therefore (a) must be a point of D and the n linear polynomials $z_i - a_i$ $(1 \leqq i \leqq n)$ form an ideal base of $\mathfrak{P}$.

On the other hand let $F_i (1 \leqq i \leqq n)$ be another ideal base of $\mathfrak{P}$ composed of just n functions, then we can find n^2 analytic functions $A_{ij}(z) (1 \leqq i, j \leqq n)$ in D such that

$$z_i - a_i = \sum_{j=1}^{n} A_{ij}(z) F_j(z) \quad (1 \leqq i \leqq n)$$

hold identically in D. If we differentiate them by z_k and then if we substitute (a) for (z), we get

$$\delta_{ik} = \sum_{j=1}^{n} A_{ij}(a) \left(\frac{\partial F_j}{\partial z_k} \right)_{(z)=(a)} \quad (1 \leqq i, k \leqq n).$$

Therefore the n functions $F_i(z)$ form the "local analytic coordinates" at (a). We have thus proved the following theorem, which is largely analogous to *Tannaka's duality*[3] in the theory of compact groups.

Theorem. *Let $\mathfrak{O}$ be the ring of analytic functions on a domain of regularity D in C^n and let $\hat{D}$ be the set of maximal ideals $\mathfrak{P}$ in $\mathfrak{O}$ such that $\mathfrak{O}/\mathfrak{P} = C$, then each $\mathfrak{P}$ has an ideal base composed of n elements. If we attach the analytic structure to $\hat{D}$ by taking them as the local analytic coordinates in an obvious manner, the analytic manifold so obtained has the same analytic structure as D.*

A direct consequence of this theorem is the following

Corollary. *Let D and D' be two domains of regularity in C^n*

and let $\mathfrak{O}$ and $\mathfrak{O}'$ be the rings of analytic functions in D and D' respectively, then every algebraic isomorphism φ from $\mathfrak{O}$ onto $\mathfrak{O}'$ induces an analytic homeomorphism Φ from D' onto D such that

$$\varphi \circ F = F \circ \Phi$$

holds for every F in $\mathfrak{O}$. In particular an algebraic automorphism of $\mathfrak{O}$ induces an analytic automorphism of D.

It must be remarked that a similar characterization of the compact complex manifold is impossible except the special case of one variable, even for the algebraic manifold.

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References

We shall use the same terminology as Bochner-Martin's book: Several Complex Variables, Princeton (1948).

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3) T. Tannaka, Dualität der nicht-kommutativen Gruppen, Tôhoku Math. J., vol. 53 (1935).