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Note on Nonlinear Differential Equation of Catalysis[1)]

By

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1. Mr. Yoshiyuki Suehiro, a chemical engineer, has consulted the author about the solutions of his differential equation of catalysis by spherical tablets,

$$(1)\qquad \begin{cases} (m-1)\left(Da\dfrac{dy}{dx}-wa\dfrac{y}{t}\right)=wa\,, \\[2ex] \dfrac{dwa}{dx}dx=cy\dfrac{adx}{\gamma}\,; \end{cases}$$

or for the case $m=2$, eliminating w in (1), we have

$$(2)\qquad \frac{d}{dx}\left(\frac{Dat}{y+t}\frac{dy}{dx}\right)=\frac{ca}{\gamma}y\,;$$

or fully written,

$$(3)\qquad \frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}-\frac{1}{y+t}\left(\frac{dy}{dx}\right)^2-n^2\frac{y(y+t)}{t}=0\,,$$

where $n=\sqrt{C/D\gamma}$. In these equations, x means the distance of any point of the tablet from its centre; y is the concentration of the reacting substance at x; w is the quantity of mass flow of the reacted substance through the spherical surface of radius x in the tablet; a is the total area occupied by the pores on the spherical surface of radius x, and hence proportional to x^2, while the remainings are chemical constants, positive. Solutions for $x>0$, satisfying the conditions $\dfrac{dy}{dx}=0$ at $x=0$, are required.

2. We may suppose $t=1$ by writing y instead of ty (also $n=1$ by writing x instead of nx).

1) Read before the last autumn meeting of Japanese Mathematical Society held in Kyoto.

Putting $t=1$, and

(4) $\quad e^z=1+y$, we have $\dfrac{d\left(x^2\dfrac{dz}{dx}\right)}{dx}=n^2x^2y$.

Further putting

(5) $\quad \dfrac{\chi}{x}=z$, we have $\dfrac{d^2\chi}{dx^2}=n^2xy$.

The differential equation is very near to Emden's one[2)]

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=\theta^n.$$

By its knowledge we may prove the following proposition:

The solution $y(x)$ of (4) which is continuous in an interval in which $0<x$ and $y(+0)=a_0$ finite and determinate and greater than -1, is normal to y-axis, i.e., $y'(+0)=0$. We may define $y(0)\equiv y(+0)$, then $y(x)$ is continuous for $0\leq x$ and $y(0)=a_0$.

Since $z(x)=\log(1+y(x))$ is also continuous and $y(0)>-1$, $z(0)\,(\equiv z(+0))$ is also finite and determinate. Hence we have

$$\lim_{x\to+0}\chi(x)=\lim_{x\to+0}xz(x)=0,\quad \text{i.e.,}\quad \chi(0)\,(\equiv\chi(+0))=0.$$

Nextly we have

$$\left.\frac{dz}{dx}\right|_{x=+0}=\lim_{x\to+0}\frac{(\chi/x)-z(0)}{x}=\lim_{x\to+0}\frac{\chi-xz(0)}{x^2}$$

$$=\lim_{x\to+0}\frac{\dfrac{d\chi}{dx}-z(0)}{2x},$$

provided the last limit is determinate. On the other hand

$$\left.\frac{d\chi}{dx}\right|_{x=+0}=\lim_{x\to+0}\frac{\chi(x)-\chi(0)}{x}=\lim_{x\to+0}\frac{\chi}{x}=z(0).$$

From (5) we have

$$\frac{d\chi}{dx}=z(0)+n^2\int_0^x xy\,dx.$$

2) Emden, *Gaskugeln* (1907); Chandrasekhar, *Steller Structure* (1938); by the author, *Emden's differential Equation* (in japanese), Sankaidō & Co., (1945).

Hence
$$\lim_{x\to+0}\frac{\dfrac{d\chi}{dx}-z(0)}{2x}=\lim_{x\to+0}\frac{n^2xy(x)}{2}=0,$$

since by hypothesis $y(0)=a_0$ is finite. Hence we may conclude that

$$\left.\frac{dz}{dx}\right|_{x=+0}=0;$$

hence by (4), we have

$$\left.\frac{dy}{dx}\right|_{x=+0}=0. \qquad \text{Q. E. D.}$$

In the following for simplicity we take $a_0\geq 0$, since y is the concentration, not negative.

3. *Our equation has an integral near* $0\leq x$ *of the form:*

(6) $$y(x)=a_0+a_2x^2+a_4x^4+\cdots\cdots,$$

where $a_0\geq 0$ *is given, while the other coefficients can be found successively from the equation:*

(7) $$2a_2x+4a_4x^3+\cdots\cdots$$
$$=n^2\left(\frac{a_0}{3}x+\frac{a_2}{5}x^3+\cdots\cdots\right)(1+a_0+a_2x^2+a_4x^4+\cdots\cdots).$$

This power-series is dominated by

$$\psi(x)=A_0+A_2x^2+A_4x^4+\cdots\cdots,$$

where $A_0=a_0$ *and*

(8) $$\frac{d\psi}{dx}=\frac{n^2}{3}x\psi(1+\psi),$$

which gives

$$\psi(x)=\frac{a_0}{1+a_0}e^{n^2x^2/6}\bigg/\left(1-\frac{a_0}{1+a_0}e^{n^2x^2/6}\right).$$

Hence the convergence abscissa ρ *of* $y(x)$ *is*

$$\rho\geq\frac{\sqrt{6}}{n}\sqrt{\log\left(1+\frac{1}{a_0}\right)}.$$

Our solution $y(x)$ *with the initial condition* $y(0)=a_0$ *is unique on the right of y-axis and it is a power-series of* x *and* a_0, *with positive* *coefficients, so that our solution, written $y(x, a_0)$ increases with a_0 for fixed x.*

To obtain (7), integrating (4) we have

$$x^2 \frac{dz}{dx} = n^2 \int_0^x x^2 y\,dx\,,$$

or

$$x^2 \frac{dy}{dx} = n^2(1+y) \int_0^x x^2 y\,dx\,.$$

Putting (6), we have (7). From (7) we may find all the coefficients except a_0; they are positive for $a_0 > 0$.

To find $\psi(x)$, we consider instead of (7), the relation:

$$2A_2 x + 4A_4 x^3 + \cdots\cdots$$
$$= \frac{n^2}{3} (A_0 x + A_2 x^3 + \cdots)(1 + A_0 + A_2 x^2 + A_4 x^4 + \cdots).$$

We see easily that for $a_0 = A_0 (>0)$, we have $a_2 = A_2$, $a_4 < A_4, \cdots$ and that $\psi(x)$ satisfies (8).

It is evident that the solution $y(x)$ with the initial condition $y(0) = a_0 > 0$ is unique and it is continuous with respect to a_0; the first quadrant of the coordinate plane is swept by our integrals with $0 \leq a_0 < \infty$.

We remark that above considerations may easily be extended for the cases $a_0 > -1$ and $x < 0$.