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Extensions of the Ground Field in the Theory of Algebraic Differential Equations[1)]

By

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1. Let $\mathfrak{F}$ be an ordinary or a partial differential field (d. f.) of characteristic zero, $\mathfrak{G}$ a differential extension field (d. e. f.) of $\mathfrak{F}$, and $y_1, \cdots, y_n$ n independent differential indeterminates over $\mathfrak{G}$. The ring of differential polynomials (d. p.) of $y_1, \cdots, y_n$ over $\mathfrak{F}$ will be denoted by $\mathfrak{R}=\mathfrak{F}\{y_1, \cdots, y_n\}$ and that over $\mathfrak{G}$ by $\mathfrak{S}=\mathfrak{G}\{y_1, \cdots, y_n\}$. The derivatives of $y_1, \cdots, y_n$ (y-derivatives) are supposed completely ordered.

Let $\mathfrak{p}$ be a nontrivial prime differential ideal (d. i.) of dimension r in $\mathfrak{R}$. If $y_{i_1}, \cdots, y_{i_r}$ form a parametric set for $\mathfrak{p}$ and if $y_{i_{r+1}}, \cdots, y_{i_n}$ are the others of the y's, then $\mathfrak{p}$ has a characteristic set $\langle A_1, \cdots, A_s\rangle$ such that the leaders of the A_i are derivatives of $y_{i_{r+1}}, \cdots, y_{i_n}$ and that some derivative of each $y_{i_{r+k}}$ $(1 \leq k \leq n-r)$ appears as the leader of one of the A_i. Such a characteristic set $\langle A_1, \cdots, A_s\rangle$ will be called a characteristic set of $\mathfrak{p}$ with respect to the parametric set $\langle y_{i_1}, \cdots, y_{i_r}\rangle$.

THEOREM. *The extension ideal $\mathfrak{p}\mathfrak{S}$ is a perfect d. i. The dimensions of the essential prime divisors $\mathfrak{P}_1, \cdots, \mathfrak{P}_t$ of $\mathfrak{p}\mathfrak{S}$ are all equal to the dimension of $\mathfrak{p}$. Every parametric set for $\mathfrak{p}$ is such a set for every $\mathfrak{P}_j$, and a characteristic set of every $\mathfrak{P}_j$ with respect to such a parametric set has the same leaders as that of $\mathfrak{p}$.*

For the "ordinary" case, the assertion about the leaders of characteristic sets amounts to that the order of every $\mathfrak{P}_j$ is equal to that of $\mathfrak{p}$, and our assertion is equivalent to Ritt's assertion (Ritt [4], p. 51). We can prove that the theorem holds true also for the "partial" case. We shall prove the theorem for the general

1) For the terminologies of this paper, see Ritt [4] and Kolchin [1], [2] and [3].

case (ordinary or partial) by means of minute observations on the relationship between prime d. i. of $\mathfrak{R}$ and prime ideals of certain rings of polynomials. The proof could not be made shorter than the one described below (§ 3), even if we made use of the resolvent for the general case as Ritt has done for the ordinary case. Our method enables us further to obtain interesting results (§ 4) concerning extensions of the ground field.

2. Let $\langle A\rangle=\langle A_1, \cdots, A_s\rangle$ be a chain in $\mathfrak{R}$. If we denote by $w_1, \cdots, w_s$ the leaders of $A_1, \cdots, A_s$ respectively, then w_j is higher than w_i provided $j>i$, and every A_i contains no y-derivative higher than w_s. Let us denote by $v_1, \cdots, v_p$ all the y-derivatives which are not higher than w_s and which are other than $w_1, \cdots, w_s$, and consider the ring $\mathfrak{J}=\mathfrak{F}[v_1, \cdots, v_p, w_1, \cdots, w_s]$ of polynomials of $v_1, \cdots, v_p, w_1, \cdots, w_s$ over $\mathfrak{F}$. If $v_1, \cdots, v_p, w_1, \cdots, w_s$ are taken in this order, then $\langle A\rangle$ is a chain in $\mathfrak{J}$. Ritt has proved[2] that *$\langle A\rangle$ is a characteristic set of a prime d. i. of $\mathfrak{R}$ if and only if it is a characteristic set of a prime ideal of $\mathfrak{J}$.*

Let $\langle A\rangle$ be a characteristic set of a prime d. i. $\mathfrak{p}$ of $\mathfrak{R}$ and such a set of a prime ideal $\mathfrak{q}$ of $\mathfrak{J}$. If $\mathfrak{R}$ is considered as a ring of polynomials of all y-derivatives over $\mathfrak{F}$, then $\mathfrak{J}$ is a subring of $\mathfrak{R}$, and $\mathfrak{p}$ is clearly a prime ideal of $\mathfrak{R}$. We can see that $\mathfrak{p}=\mathfrak{q}\cap\mathfrak{J}$. Because, $\langle A\rangle$ is a characteristic set either for $\mathfrak{q}$ or for $\mathfrak{p}\cap\mathfrak{J}$, and a prime ideal of $\mathfrak{J}$ is uniquely determined by any one of its characteristic sets.

Furthermore, we can prove that, *if S_i is the separant of A_i $(i=1, \cdots, s)$,*

(1) $\mathfrak{p}=\{A_1, \cdots, A_s\} : S_1\cdots S_s$,

(2) $\mathfrak{q}=\{A_1, \cdots, A_s\}_0 : S_1\cdots S_s$,

where $\{A_1, \cdots, A_s\}$ is the perfect d. i. generated in $\mathfrak{R}$ by $A_1, \cdots, A_s$ and $\{A_1, \cdots, A_s\}_0$ the perfect ideal generated in $\mathfrak{J}$ by $A_1, \cdots, A_s$. In fact, if G any d. p. of $\{A_1, \cdots, A_s\} : S_1\cdots S_s$, then $S_1\cdots S_s G\in\{A_1, \cdots, A_s\}\subset\mathfrak{p}$, and $G\in\mathfrak{p}$ as $S_1, \cdots, S_s$ are all not contained in $\mathfrak{p}$. Thus, we get $\{A_1, \cdots, A_s\} : S_1\cdots S_s\subset\mathfrak{p}$. Conversely, if $G\in\mathfrak{p}$, every zero of $\langle A\rangle$ is a zero of $S_1\cdots S_s G$ and $S_1\cdots S_s G\in\{A_1, \cdots, A_s\}$ by the analogue for d. p. of the Hilbert theorem of zeros.[3] Thus, $G\in\{A_1, \cdots, A_s\} : S_1$

2) See Ritt [4], pp. 107-108. His proof for the ordinary case can be easily carried over to the general case.

3) For the ordinary case, see Ritt [4], p. 108 and pp. 27-28. We can see easily that the assertions are also true for the general case.

$\cdots S_s$, and $\mathfrak{p} \subset \{A_1, \cdots, A_s\} : S_1 \cdots S_s$. This establishes the equality (1). The equality (2) can be proved similarly.[4]

3. We shall now prove the theorem of § 1, using the notations of that section.

If $\mathfrak{S}$ is considered as a ring of polynomials of all y-derivatives over $\mathfrak{G}$ and $\mathfrak{R}$ as that over $\mathfrak{F}$, we see that $\mathfrak{p}\mathfrak{S}$ is a perfect ideal of $\mathfrak{S}$, since any prime ideal of a ring of polynomials generates a perfect ideal for any extension of the ground field. Hence, $\mathfrak{p}\mathfrak{S}$ is a perfect d. i. of $\mathfrak{S}$; and this proves the first point of the theorem.

Let $\langle A \rangle = \langle A_1, \cdots, A_s \rangle$ be a characteristic set of $\mathfrak{p}$, and w_i the leader of A_i, S_i being its separant $(i=1, \cdots, s)$. We arrange, as we have done in § 2, all the y-derivatives which are not higher than w_s in a sequence $v_1, \cdots, v_p, w_1, \cdots, w_s$. Let $\mathfrak{J} = \mathfrak{F}[v_1, \cdots, v_p, w_1, \cdots, w_s]$ be the ring of polynomials of v, w over $\mathfrak{F}$ and $\overline{\mathfrak{J}} = \mathfrak{G}[v_1, \cdots, v_p, w_1, \cdots, w_s]$ that over $\mathfrak{G}$. Then, $\langle A \rangle$ is a characteristic set of the prime ideal $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{J}$. Let the decomposition of $\mathfrak{q}\overline{\mathfrak{J}}$ be $\mathfrak{q}\overline{\mathfrak{J}} = \mathfrak{Q}^{(1)} \cap \cdots \cap \mathfrak{Q}^{(t)}$, $\mathfrak{Q}^{(j)}$ $(j=1, \cdots, t)$ being the essential prime divisors of $\mathfrak{q}\overline{\mathfrak{J}}$ in $\overline{\mathfrak{J}}$, then $\mathfrak{Q}^{(j)} \cap \mathfrak{J} = \mathfrak{q}$ $(j=1, \cdots, t)$. If $\langle \nu, \omega \rangle = \langle \nu_1, \cdots, \nu_p, \omega_1, \cdots, \omega_s \rangle$ is a generic zero over $\mathfrak{G}$ of $\mathfrak{Q}^{(j)}$ for any j, then $\langle \nu, \omega \rangle$ is such a zero over $\mathfrak{F}$ of $\mathfrak{q}$. Hence, $\mathfrak{Q}^{(j)}$ has a characteristic set of the form $\langle \overline{A}^{(j)} \rangle = \langle \overline{A}_1^{(j)}, \cdots, \overline{A}_s^{(j)} \rangle$, where $\overline{A}_i^{(j)}$ has the same leader w_i as A_i $(i=1, \cdots, s)$. The separant of $\overline{A}_i^{(j)}$ will be denoted by $\overline{S}_i^{(j)}$ $(i=1, \cdots, s)$. By § 2, $\langle \overline{A}^{(j)} \rangle$ is a characteristic set of some prime d. i. $\mathfrak{P}^{(j)}$ of $\mathfrak{S}$, and $\mathfrak{P}^{(j)} \cap \overline{\mathfrak{J}} = \mathfrak{Q}^{(j)}$.

Let $\overline{G}$ be any d. p. of $\mathfrak{P}^{(1)} \cap \cdots \cap \mathfrak{P}^{(t)}$. If $\langle A \rangle$ is considered as a chain in $\mathfrak{S}$, then there exist non-negative integers $t_1, \cdots, t_s$ such that $S_1^{t_1} \cdots S_s^{t_s} \overline{G} \equiv \overline{H} \mod [A_1, \cdots, A_s]\mathfrak{S}$, where $[A_1, \cdots, A_s]$ is the d. i. which is generated in $\mathfrak{R}$ by $A_1, \cdots, A_s$, and hence, $[A_1, \cdots, A_s]$ $\mathfrak{S}$ the d. i. generated $\mathfrak{S}$ by $A_1, \cdots, A_s$, and where $\overline{H}$ has no proper derivative of $w_1, \cdots, w_s$. As $\overline{G} \in \mathfrak{P}^{(j)}$ for every j and $A_i \in \mathfrak{q} \subset \mathfrak{Q}^{(j)} \subset \mathfrak{P}^{(j)}$ $(i=1, \cdots, s;\ j=1, \cdots, t)$, we get $\overline{H} \in \mathfrak{P}^{(j)}$ $(j=1, \cdots, t)$. Hence, by (1) of § 2, there is a non-negative integer a such that $(\overline{S}_1^{(j)} \cdots \overline{S}_s^{(j)} \overline{H})^a \in [\overline{A}_1^{(j)}, \cdots, \overline{A}_s^{(j)}]$, the d. i. which is generated in $\mathfrak{S}$ by $\overline{A}_1^{(j)}, \cdots, \overline{A}_s^{(j)}$. By a procedure similar to the one which is used by Ritt [4] p. 30, we can find a non-negative integer b such that $(\overline{S}_1^{(j)} \cdots \overline{S}_s^{(j)})^{a+b} \overline{H}^a = \overline{A}_1^{(j)} \overline{P}_1^{(j)} + \cdots + \overline{A}_s^{(j)} \overline{P}_s^{(j)}$ $(\overline{P}_i^{(j)} \in \mathfrak{S})$. Some y-derivatives other than v_1,

4) See Ritt [4], p. 106 and p. 87.

$\cdots, v_p, w_1, \cdots, w_s$ may be contained in $\bar{H}$ and $\bar{P}_i^{(j)}$ $(i=1, \cdots, s)$; all such y-derivatives will be denoted by $v_{p+1}, \cdots, v_q$. Then, $\bar{H}^a$ and $\bar{P}_i^{(j)}$ $(i=1, \cdots, s)$ can be written as linear combinations of distinct power products of $v_{p+1}, \cdots, v_q$ over $\bar{\mathfrak{F}}$. Let $\bar{C}$ be any one of the coefficients of the expression for $\bar{H}^a$. Then, we see that $(\bar{S}_1^{(j)} \cdots \bar{S}_s^{(j)})^{a+b}\bar{C}$ is a linear combination of $\bar{A}_1^{(j)}, \cdots, \bar{A}_s^{(j)}$ over $\bar{\mathfrak{F}}$, and that $(\bar{S}_1^{(j)} \cdots \bar{S}_s^{(j)})^{a+b}\bar{C} \in \mathfrak{Q}^{(j)}$. Hence, $\bar{C} \in \mathfrak{Q}^{(j)}$ since no $\bar{S}_i^{(j)}$ is contained in $\mathfrak{Q}^{(j)}$. As $\bar{C}$ does not depend on j, we have $\bar{C} \in \mathfrak{Q}^{(1)} \cap \cdots \cap \mathfrak{Q}^{(t)} = \mathfrak{q}\mathfrak{F}$, and hence, $\bar{H}^a \in \mathfrak{q}\mathfrak{S} \subset \mathfrak{p}\mathfrak{S}$. Therefore, $(S_1^{t_1} \cdots S_s^{t_s}\bar{G})^a \in \mathfrak{p}\mathfrak{S}$, and $S_1 \cdots S_s \bar{G} \in \mathfrak{p}\mathfrak{S}$ since $\mathfrak{p}\mathfrak{S}$ is a perfect ideal. If we write $\bar{G} = \sum_k \gamma_k G_k$ $(\gamma_k \in \mathfrak{G}, G_k \in \mathfrak{R})$ as a linear combination over $\mathfrak{R}$ of $\gamma_1, \gamma_2, \cdots$, which are linearly independent over $\mathfrak{F}$, then $S_1 \cdots S_s G_k \in \mathfrak{p}$. Hence, $G_k \in \mathfrak{p}$, and $\bar{G} \in \mathfrak{p}\mathfrak{S}$. Thus, we have established the inclusion $\mathfrak{P}^{(1)} \cap \cdots \cap \mathfrak{P}^{(t)} \subset \mathfrak{p}\mathfrak{S}$.

To prove the inverse inclusion, it is sufficient to show that any d. p. G of $\mathfrak{p}$ is contained in every $\mathfrak{P}^{(j)}$. For any $G \in \mathfrak{p}$, there exist non-negative integers $t_1, \cdots, t_s$ such that $S_1^{t_1} \cdots S_s^{t_s} G \equiv H$ mod $[A_1, \cdots, A_s]$, where H contains no proper derivative of $w_1, \cdots, w_s$. And, by the proceduae which was used above, there are non-negative integers a and b such that $(S_1 \cdots S_s)^{a+b} H^a = A_1 P_1 + \cdots + A_s P_s$ $(P_i \in \mathfrak{R})$. Let $\langle \eta \rangle = \langle \eta_1, \cdots, \eta_n \rangle$ be a generic zero over $\mathfrak{G}$ of $\mathfrak{P}^{(j)}$ for any j, and denote by $\langle \nu_1, \cdots, \nu_p, \omega_1, \cdots, \omega_s \rangle$ the values of $\langle v_1, \cdots, v_p, w_1, \cdots, w_s \rangle$ for $\langle \eta \rangle$, then $\langle \nu, \omega \rangle$ is a generic zero over $\mathfrak{G}$ of $\mathfrak{Q}^{(j)}$ and such a zero over $\mathfrak{F}$ of $\mathfrak{q}$. Now, H and P_i $(i=1, \cdots, s)$ may contain some y-derivatives other than $\langle v, w \rangle$; all such y-derivatives will be denoted again by $v_{p+1}, \cdots, v_q$. And, we write $H^a = \sum_k L_k V_k$ and $P_i = \sum_k Q_{ik} V_k$ for every i $(L_k, Q_{ik} \in \mathfrak{F})$, where $V_1, V_2, \cdots$ are distinct power products of $v_{p+1}, \cdots, v_q$. Then, we get in $\mathfrak{F}$ the equation $(S_1 \cdots S_s)^{a+b} L_k = A_1 Q_{1k} + \cdots + A_s Q_{sk}$ for each k. If we substitute $\langle \nu, \omega \rangle$, for $\langle v, w \rangle$, then $(S_1 \cdots S_s)^{a+b} L_k$ vanishes, since A_i are all contained in $\mathfrak{q}$. If any S_i vanished for $\langle \nu, \omega \rangle$, $\mathfrak{Q}^{(j)}$ would contain S_i, and so would do $\mathfrak{q} = \mathfrak{Q}^{(j)} \cap \mathfrak{F}$, and this would be a contradiction. Hence, L_k vanishes for $\langle \nu, \omega \rangle$, and $L_k \in \mathfrak{Q}^{(j)}$. Therefore, we can find non-negative integers $e_1, \cdots, e_s$ such that $\bar{I}_1^{(j)e_1} \cdots \bar{I}_s^{(j)e_s} L_k$ is a linear combination of $\bar{A}_1^{(j)}, \cdots, \bar{A}_s^{(j)}$ over $\bar{\mathfrak{F}}$, where $\bar{I}_i^{(j)}$ is the initial of $\bar{A}_i^{(j)}$ $(i=1, \cdots, s)$. Thus, we can get a power product $\bar{J}$ of $\bar{I}_1^{(j)}, \cdots, \bar{I}_s^{(j)}$ such that $\bar{J}H^a$ is a linear combination of $\bar{A}_1^{(j)}, \cdots, \bar{A}_s^{(j)}$ over

$\mathfrak{S}$. Hence, $\bar{J}H^{a}\in\mathfrak{P}^{(j)}$. As $\bar{I}_{r}^{(j)}$ is not contained in $\mathfrak{P}^{(j)}$, we get $H\in\mathfrak{P}^{(j)}$. Since $A_i\in\mathfrak{q}\subset\mathfrak{Q}^{(j)}\subset\mathfrak{P}^{(j)}$, we see that $S_1^{t_1}\cdots S_s^{t_s}G\in\mathfrak{P}^{(j)}$. As we know that S_i is not contained in $\mathfrak{Q}^{(j)}=\mathfrak{P}^{(j)}\cap\bar{\mathfrak{F}}$, S_i is not contained in $\mathfrak{P}^{(j)}$. Therefore, $G\in\mathfrak{P}^{(j)}$. Thus, we have established the inclusion $\mathfrak{p}\mathfrak{S}\subset\mathfrak{P}^{(1)}\cap\cdots\cap\mathfrak{P}^{(t)}$, and the proof of $\mathfrak{p}\mathfrak{S}=\mathfrak{P}^{(1)}\cap\cdots\cap\mathfrak{P}^{(t)}$ is completed. As $\mathfrak{Q}^{(1)},\cdots,\mathfrak{Q}^{(t)}$ are the essential prime divisors of $\mathfrak{q}\bar{\mathfrak{F}}$, it is clear that $\mathfrak{P}^{(1)},\cdots,\mathfrak{P}^{(t)}$ are such divisors of $\mathfrak{p}\mathfrak{S}$. The other assertions of the theorem are already shown to be true in the course of the above description.

4. In the preceding section we saw that $\mathfrak{p}\mathfrak{S}$ and $\mathfrak{q}\bar{\mathfrak{F}}$ have corresponding decompositions. From this fact, we can deduce some properties which, with the other results described below, are sufficient to treat various problems concerning extensions of the ground field. Henceforce, our development is deeply due to Weil [5] (Chap. I). Many of our propositions have analogous proofs as those of Weil [5]. Proofs will be given only for the propositions which depend essentially on the consideration of properties peculiar to d. p. We suppose that *various d. f. and elements (differential quantities), which will be treated together in the rest of this paper, are all contained in a common d. f.*

PROP. 1. *Let $\mathfrak{G}$ and $\mathfrak{H}$ be two d. f., and $\mathfrak{F}$ a common differential subfield of $\mathfrak{G}$ and $\mathfrak{H}$. If every (finite) set of differentially algebraically independent elements in $\mathfrak{G}$ over $\mathfrak{F}$ is still such over $\mathfrak{H}$, then every set of differentially algebraically independent elements in $\mathfrak{H}$ over $\mathfrak{F}$ is still such over $\mathfrak{G}$.*

When three d. f. $\mathfrak{F}$, $\mathfrak{G}$ and $\mathfrak{H}$ have the property described in prop. 1, we shall say, following Weil, that $\mathfrak{G}$ and $\mathfrak{H}$ are *differerentially independent over $\mathfrak{F}$*. It is easy to see that this notion is surely broader than the independence in the algebraic sense. Therefore, if $\mathfrak{F}$ is a common differential sûbfield of two d. f. $\mathfrak{G}$ and $\mathfrak{H}$, and if $\mathfrak{G}$ and $\mathfrak{H}$ are linearly disjoint over $\mathfrak{F}$, then they are differentially independent over $\mathfrak{F}$.

PROP. 2. *Let $\mathfrak{F}$ be a d. f., $\mathfrak{G}$ a d. e. f. of $\mathfrak{F}$, and $\langle\zeta_1,\cdots,\zeta_m\rangle$ a set of elements. Then, we have $dim_{\mathfrak{F}}\langle\zeta\rangle\geq dim_{\mathfrak{G}}\langle\zeta\rangle$* (we mean by the dimension, as in § 1, one in the differential sense), *and the equality $dim_{\mathfrak{F}}\langle\zeta\rangle$ holds if and only if $\mathfrak{G}$ and $\mathfrak{F}\langle\zeta\rangle$* (the d. f. which is obtained by adjoining to $\mathfrak{F}$ all the ζ-derivatives) *are differentially independent over $\mathfrak{F}$.*

PROP. 3. *Let $\mathfrak{F}$ be a d. f., and $\langle \eta_1, \cdots, \eta_m \rangle$ and $\langle \zeta_1, \cdots, \zeta_n \rangle$ two sets of elements; and let $\mathfrak{F}\langle \eta \rangle$ and $\mathfrak{F}\langle \zeta \rangle$ are linearly disjoint over $\mathfrak{F}$. Let $y_1, \cdots, y_m$, $z_1, \cdots, z_n$ are independent differential indeterminates over $\mathfrak{F}$, and put $\mathfrak{R}=\mathfrak{F}\{y_1, \cdots, y_m\}$, $\mathfrak{S}=\mathfrak{F}\{z_1, \cdots, z_n\}$ and $\mathfrak{T}=\mathfrak{F}\{y_1, \cdots, y_m, z_1, \cdots, z_n\}$; and let $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{P}$ be the prime d. i. defined by $\langle \eta \rangle$, $\langle \zeta \rangle$ and $\langle \eta, \zeta \rangle$ in $\mathfrak{R}$, $\mathfrak{S}$ and $\mathfrak{T}$ respectively. Then we have $\mathfrak{P}=(\mathfrak{p}, \mathfrak{q})\mathfrak{T}$.*

PROP. 4. *Let $\mathfrak{F}$ be a d. f., $\mathfrak{G}$ a d. e. f. of $\mathfrak{F}$ and $\langle \eta_1, \cdots, \eta_n \rangle$ a set of elements. Let $\mathfrak{F}\langle \eta \rangle$ and $\mathfrak{G}$ be differentially independent over $\mathfrak{F}$. And let $\mathfrak{F}'$ be the d. f. consisting of all those elements of $\mathfrak{G}$ which are differentially algebraic over $\mathfrak{F}$. Then $\mathfrak{F}'\langle \eta \rangle$ and $\mathfrak{G}$ are linearly disjoint over $\mathfrak{F}'$.*

PROP. 5. *Let $\mathfrak{F}$ be a d. f., and $\mathfrak{G}$ a d. e. f. of $\mathfrak{F}$, and $\langle \eta_1, \cdots, \eta_n \rangle$ a set of elements. Let $y_1, \cdots, y_n$ be independent differential indeterminates over $\mathfrak{G}$, and put $\mathfrak{R}=\mathfrak{F}\{y_1, \cdots, y_n\}$ and $\mathfrak{S}=\mathfrak{G}\{y_1, \cdots, y_n\}$. Let $\mathfrak{p}$ and $\mathfrak{P}$ be the prime d. i. defined by $\langle \eta \rangle$ in $\mathfrak{R}$ and in $\mathfrak{S}$ respectively. Then $\mathfrak{p}\mathfrak{S}=\mathfrak{P}$ if and only if $\mathfrak{F}\langle \eta \rangle$ and $\mathfrak{G}$ are linearly disjoint over $\mathfrak{F}$.*

Proof. We determine $v_1, \cdots, v_p$, $w_1, \cdots, w_s$, $\mathfrak{F}$, and $\bar{\bar{\mathfrak{F}}}$ by means of a characteristic set $\langle A_1, \cdots, A_s \rangle$ of $\mathfrak{p}$ as we have done in § 3. $\langle \eta \rangle$ is a generic zero over $\mathfrak{F}$ of $\mathfrak{p}$; we denote by $\nu_1, \cdots, \nu_p$, $\omega_1, \cdots, \omega_s$ the values of $v_1, \cdots, v_p$, $w_1, \cdots, w_s$ for $\langle \eta \rangle$. Then, $\langle \nu, \omega \rangle$ is a generic zero over $\mathfrak{F}$ of $\mathfrak{q}=\mathfrak{p} \cap \mathfrak{F}$. Since $\mathfrak{q}\bar{\bar{\mathfrak{F}}}$ is a prime ideal of $\bar{\bar{\mathfrak{F}}}$ if and only if the two fields $\mathfrak{F}(\nu_1, \cdots, \nu_p, \omega_1, \cdots, \omega_s)$ and $\mathfrak{G}$ are linearly disjoint over $\mathfrak{F}$ (Weil [5], p. 15), this is the necessary and sufficient condition for the equality $\mathfrak{p}\mathfrak{S}=\mathfrak{P}$. We shall prove that the condition is equivalent to that $\mathfrak{F}\langle \eta \rangle$ and $\mathfrak{G}$ are linearly disjoint over $\mathfrak{F}$.

If $\mathfrak{F}\langle \eta \rangle$ and $\mathfrak{G}$ are linearly disjoint over $\mathfrak{F}$, so are clearly $\mathfrak{F}(\nu, \omega)$ and $\mathfrak{G}$. Conversely, suppose that $\mathfrak{F}(\nu, \omega)$ and $\mathfrak{G}$ are linearly disjoint over $\mathfrak{F}$. Now, assume that $\mathfrak{F}\langle \eta \rangle$ and $\mathfrak{G}$ are not linearly disjoint over $\mathfrak{F}$. Then, there must exist polynomials $B_1, \cdots, B_t$ of ν, ω and some finite set $\eta', \eta'', \cdots$ of η-derivatives over $\mathfrak{F}$ which are linearly independent over $\mathfrak{F}$, and which are linearly dependent over $\mathfrak{G}$; let $\sum_{k=1}^{t} \gamma_k B_k = 0$ ($\gamma_k \in \mathfrak{G}$, not all zere). If we replace $\langle \nu, \omega, \eta', \eta'', \cdots \rangle$ by the corresponding y-derivatives $\langle v, w, y', y'', \cdots \rangle$, and if we get from $B_1, \cdots, B_t$ polynomials $C_1, \cdots, C_t$ of $\langle v, w, y', y'', \cdots \rangle$ over $\mathfrak{F}$, then $\sum \gamma_k C_k$ must be contained in $\mathfrak{p}\mathfrak{S}$, since $\mathfrak{p}\mathfrak{S}$ is, by our hypothesis, equal to $\mathfrak{P}$ with $\langle \eta \rangle$ as a generic zero. Let all γ_k be written as linear combinations $\gamma_k=\sum_h \alpha_{kh}\varepsilon_h(\alpha_{kh}\in\mathfrak{F})$ of a set $\langle\varepsilon\rangle$ of linearly independent elements of $\mathfrak{G}$ over $\mathfrak{F}$. Then $\sum_k\sum_h \alpha_{kh}\varepsilon_h C_k\in\mathfrak{p}\mathfrak{S}$, and we must have $\sum_k \alpha_{kh}C_k\in\mathfrak{p}$ for every h, and $\sum_k \alpha_{kh}B_k$ must be zero. Therefore, all α_{kh} must be zero, and so must be all γ_k. Thus, we must be led to a contradiction. This proves that the linear disjointness of $\mathfrak{F}(\nu,\omega)$ and $\mathfrak{G}$ over $\mathfrak{F}$ implies that of $\mathfrak{F}\langle\eta\rangle$ and $\mathfrak{G}$ over $\mathfrak{F}$.

PROP. 6. *Let $\mathfrak{F}\langle\eta_1,\cdots,\eta_n\rangle$ be a differentially algebraic extension of a d. f. $\mathfrak{F}$, and $\mathfrak{G}$ a d. e. f. of $\mathfrak{F}$. If $\mathfrak{F}$ is differentially algebraically closed in $\mathfrak{G}$, then $\mathfrak{F}\langle\eta\rangle$ and $\mathfrak{G}$ are linearly disjoint over $\mathfrak{F}$.*

Proof. Let $y_1,\cdots,y_n$ be independent differential indeterminates over $\mathfrak{G}$, and put $\mathfrak{R}=\mathfrak{F}\{y_1,\cdots,y_n\}$ and $\mathfrak{S}=\mathfrak{G}\{y_1,\cdots,y_n\}$. Let $\mathfrak{p}$ and $\mathfrak{P}$ be the prime d. i. defined by $\langle\eta\rangle$ in $\mathfrak{R}$ and $\mathfrak{S}$ respectively. By prop. 5, it is necessary only to prove the equality $\mathfrak{p}\mathfrak{S}=\mathfrak{P}$.

Let $\langle A_1,\cdots,A_s\rangle$ be a characteristic set of $\mathfrak{p}$, and let us use the same notations as in the proof of prop. 5. Since $\mathfrak{P}$ is one of essential prime divisors of $\mathfrak{p}\mathfrak{S}$, $\mathfrak{P}$ has, by the theorem of § 1, a characteristic set of the form $\langle\bar{A}_1,\cdots,\bar{A}_s\rangle$, where $\bar{A}_i$ has the leader w_i as $A_i(i=1,\cdots,s)$. If $v_1,\cdots,v_p$ are replaced by $\nu_1,\cdots,\nu_p$, we get from A_1 and $\bar{A}_1$ polynomials B_1 and $\bar{B}_1$ of w_1 over $\mathfrak{F}(\nu)$ and over $\mathfrak{G}(\nu)$ respectively. Ritt has proved[5] that B_1 is irreducible over $\mathfrak{F}(\nu)$ and that so is $\bar{B}_1$ over $\mathfrak{G}(\nu)$. While, B_1 and $\bar{B}_1$ have a common root ω_1, hence B_1 is divisible by $\bar{B}_1$, and the coefficients of $\bar{B}_1$ are algebraic over $\mathfrak{F}(\nu)$. Since $\nu_1,\cdots,\nu_p$ are algebraically independent over $\mathfrak{G}$, and since $\mathfrak{F}$ is algebraically closed in $\mathfrak{G}$, $\mathfrak{F}(\nu)$ is also algebraically closed in $\mathfrak{G}(\nu)$. Therefore, the coefficients of $\bar{B}_1$ must be contained in $\mathfrak{F}(\nu)$, and consequently $B_1=\bar{B}_1$ except for a factor in $\mathfrak{F}(\nu)$. Hence, we may conclude that $A_1=\bar{A}_1$ for they may be supposed irreducible polynomials of $v_1,\cdots,v_p$, w_1 over $\mathfrak{F}$ and over $\mathfrak{G}$ respectively. In the next place, replacing $v_1,\cdots,v_p$, w_1 by $\nu_1,\cdots,\nu_p$, ω_1, we get from A_2 and $\bar{A}_2$ polynomials B_2 and $\bar{B}_2$ of w_2 respectively. And, B_2 is irreducible over $\mathfrak{F}(\nu,\omega_1)$, and so is $\bar{B}_2$ over $\mathfrak{G}(\nu,\omega_1)$, as Ritt has proved.[5] Since they have a common root ω_2, B_2 is divisible by $\bar{B}_2$; and the coefficients of $\bar{B}_2$ are algebraic over $\mathfrak{F}(\nu,\omega_1)$. While, $\mathfrak{F}(\nu,\omega_1)$ and $\mathfrak{G}$ can be easily seen to be independent over $\mathfrak{F}$ in the algebraic sense, and consequently

5) See Ritt [4], pp. 88-90 and § 2 of this paper.

$\mathfrak{F}(\nu, \omega_1)$ is algebraically closed in $\mathfrak{G}(\nu, \omega_1)$. Hence, the coefficients of $\bar{B}_2$ are contained in $\mathfrak{F}(\nu, \omega_1)$, and consequently $B_2=\bar{B}_2$ except for a factor in $\mathfrak{F}(\nu, \omega_1)$. As it can be easily proved that there are non-negative integer a such that $\bar{I}_2^a A_2=\bar{P}\bar{A}_2$ ($\bar{P}$: a polynomial of v's, w_1 and w_2 over $\mathfrak{G}$), where $\bar{I}_2$ is the initial of $\bar{A}_2$, we may conclude that $A_2=\bar{A}_2$, because they may be supposed irreducible polynomials of y-derivatives. Continuing similar considerations, we can conclude that $A_i=\bar{A}_i (i=1, \cdots, s)$. This establishes the equality $\mathfrak{p}\mathfrak{S}=\mathfrak{P}$.

PROP. 7. *Let* $\mathfrak{F}$, $\mathfrak{G}$, $\langle \eta_1, \cdots, \eta_n \rangle$, $\langle y_1, \cdots, y_n \rangle$, $\mathfrak{R}$, $\mathfrak{S}$, $\mathfrak{p}$ *and* $\mathfrak{P}$ *be as in prop. 5. Then,* $\mathfrak{p}\mathfrak{S}=\mathfrak{P}$ *for every differentially algebraic extension field* $\mathfrak{G}$ *of* $\mathfrak{F}$ *if and only if* $\mathfrak{F}$ *is differentially algebraically closed in* $\mathfrak{F}\langle \eta \rangle$.

Proof. By pryp. 5, $\mathfrak{p}\mathfrak{S}$ is a prime d. i. for every differentially algebraic extension field $\mathfrak{G}$ of $\mathfrak{F}$ if and only if $\mathfrak{F}\langle \eta \rangle$ is linearly disjoint over $\mathfrak{F}$ with every such $\mathfrak{G}$. If $\mathfrak{F}$ is not differentially algebraically closed in $\mathfrak{F}\langle \eta \rangle$, it is easily proved that $\mathfrak{F}\langle \eta \rangle$ is not linearly disjoint over $\mathfrak{F}$ with at least one differentially algebraic extension field $\mathfrak{G}$ of $\mathfrak{F}$. Conversely, assume that $\mathfrak{F}$ is differentially algebraically closed in $\mathfrak{F}\langle \eta \rangle$, and that $\mathfrak{G}$ is any differentially algebraic extension field of $\mathfrak{F}$. Let $\langle \zeta_1, \cdots, \zeta_m \rangle$ be any set of linearly independent elements of $\mathfrak{G}$ over $\mathfrak{F}$. Applying prop. 6 to the fields $\mathfrak{F}\langle \zeta \rangle$, $\mathfrak{F}$ and $\mathfrak{F}\langle \eta \rangle$, we can see that $\mathfrak{F}\langle \zeta \rangle$ and $\mathfrak{F}\langle \eta \rangle$ are linearly disjoint over $\mathfrak{F}$. Consequently, $\langle \zeta_1, \cdots, \zeta_m \rangle$ is a set of linearly independent elements over $\mathfrak{F}\langle \eta \rangle$, and $\mathfrak{G}$ and $\mathfrak{F}\langle \eta \rangle$ are linearly disjoint over $\mathfrak{F}$.

PROP. 8. *Let* $\mathfrak{F}$ *be a d. f.,* $\mathfrak{G}$ *a d. e. f. of* $\mathfrak{F}$ *and* $\langle \eta_1, \cdots, \eta_n \rangle$ *a set of elements. If* $\mathfrak{G}$ *is differentially algebraically closed in* $\mathfrak{G}\langle \eta \rangle$ *and if* $\mathfrak{G}$ *and* $\mathfrak{F}\langle \eta \rangle$ *are linearly disjoint over* $\mathfrak{F}$, *then* $\mathfrak{F}$ *is differentially algebraically closed in* $\mathfrak{F}\langle \eta \rangle$.

PROP. 9. *Let* $\mathfrak{F}$ *be a d. f.,* $\mathfrak{G}$ *a d. e. f. of* $\mathfrak{F}$ *and* $\langle \eta_1, \cdots, \eta_n \rangle$ *a set of elements. If* $\mathfrak{G}$ *and* $\mathfrak{F}\langle \eta \rangle$ *are differentially independent over* $\mathfrak{F}$, *and if* $\mathfrak{F}$ *is differentially algebraically closed in* $\mathfrak{F}\langle \eta \rangle$, *then* $\mathfrak{G}$ *and* $\mathfrak{F}\langle \eta \rangle$ *are linearly disjoint over* $\mathfrak{F}$, *and* $\mathfrak{G}$ *is differentially algebraically closed in* $\mathfrak{G}\langle \eta \rangle$.

PROP. 10. *Let* $\mathfrak{F}$ *be a d. f., and* $\langle \eta_1, \cdots, \eta_m \rangle$ *and* $\langle \zeta_1, \cdots, \zeta_n \rangle$ *two sets of elements. If* $\mathfrak{F}$ *is differentially algebraically closed in* $\mathfrak{F}\langle \zeta \rangle$, *and in* $\mathfrak{F}\langle \zeta \rangle$, *and if* $\mathfrak{F}\langle \eta \rangle$ *and* $\mathfrak{F}\langle \zeta \rangle$ *are differentially independent* *over* $\mathfrak{F}$, *then* $\mathfrak{F}\langle\eta\rangle$ *and* $\mathfrak{F}\langle\zeta\rangle$ *are linearly disjoint over* $\mathfrak{F}$, *and* $\mathfrak{F}$ *is differentially algebraically closed in* $\mathfrak{F}\langle\eta, \zeta\rangle$.

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