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A Note on Bounded Functions

By

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Biernacki proved that a polynomial of degree n which admits $z=0$ as zero of order p is p-valent in the circle $|z|<(p/n)|z_1|$, z_1 being the nearest zero-point from the origin (except the origin).[1)]

Prof. A. Kobori proved briefly this theorem by using his theorem.[2)] Specialy, when $p=1$, the polynomial is univalent in the circle $|z|<(1/n)|z_1|$. In this paper we shall consider a regular and bounded function $f(z)$ in $|z|<1$ instead of a polynomial.

First, we shall prove a

Lemma. For $0<x_1\leqq x_2\leqq\cdots\leqq x_n\leqq\cdots<1$, *holds the inequality*

$$\sum_{i=1}^{\infty}(1-x_i^2)\leqq\frac{x_1^2(1-P^2)}{P^2},$$

where P means $\prod_{i=1}^{\infty}x_i$ *and* $P\neq 0$. *The equality holds only for the case of one* x_1.

In the general case we prove it by means of the mathematical induction. Let the lemma be true for the case of $x_1, x_2,\dots, x_m$, that is,

$$\sum_{i=1}^{m}(1-x_i^2)\leqq\frac{x_1^2(1-P_m^2)}{P_m^2}$$

be true where $P_m=\prod_{i=1}^{m}x_i$, then

$$\begin{aligned}\sum_{i=1}^{m+1}(1-x_i^2)&\leqq\frac{x_1^2(1-P_m^2)}{P_m^2}+(1-x_{m+1}^2)\\&<\frac{x_1^2-x_1^2x_{m+1}^2P_m^2}{P_m^2x_{m+1}^2}\\&=\frac{x_1^2(1-P_{m+1}^2)}{P_{m+1}^2}.\end{aligned}$$

Thus the lemma is proved, and the equality can not hold except in the case of only one x_1.

Let $w=f(z)$ be regular and bounded ($|f(z)|<M$) in $|z|<1$ and be normalized such that $f(0)=0$ and $f'(0)=1$. Let $z_1, z_2, \dots$ be zero-points of $f(z)$ in $0<|z|<1$, and let $z_1, z_2, \dots, z_n$ be zero-points of $f(z)$ in $0<|z|<R<1$, where $|z_i|=r_i$ and $r_1 \leqq r_2 \leqq \cdots \leqq r_n$, then we can write as follows:

$$f(z)=\frac{z}{R}\left(\prod_{i=1}^{n}\frac{R(z-z_i)}{R^2-\bar{z}_i z}\right)g(z). \qquad (1)$$

Here $g(z)$ is regular and bounded ($|g(z)|<M$) in $|z|<R$ and vanishes nowhere in this circle. By (1), we have

$$z\frac{f'}{f}=1+\sum_{i=1}^{n}\left(\frac{z}{z-z_i}+\frac{\bar{z}_i z}{R^2-\bar{z}_i z}\right)+z\frac{g'}{g}. \qquad (2)$$

As

$$Re\left\{\frac{z}{z-z_i}+\frac{\bar{z}_i z}{R^2-\bar{z}_i z}\right\}=Re\left\{\frac{1}{1-z/z_i}+\frac{1}{1-\bar{z}_i z/R^2}\right\}$$

$$=-Re\left\{\frac{(z/z_i)(1-r_i^2/R^2)}{(1-z/z_i)(1-\bar{z}_i z/R^2)}\right\}$$

$$\geqq-\frac{(r/r_i)(1-r_i^2/R^2)}{(1-r/r_i)(1-r_i r/R^2)}$$

$$\geqq-\frac{(r/r_1)(1-r_i^2/R^2)}{(1-r/r_1)(1-r_1 r/R^2)},$$

so we have

$$Re\left\{\sum_{i=1}^{n}\left(\frac{z}{z-z_i}+\frac{\bar{z}_i z}{R^2-\bar{z}_i z}\right)\right\}\geqq-\frac{R^2 r}{(r_1-r)(R^2-r_1 r)}\sum_{i=1}^{n}\left(1-\frac{r_i^2}{R^2}\right).$$

Further, by the lemma

$$\sum_{i=1}^{n}\left(1-\frac{r_i^2}{R^2}\right)<\sum_{i=1}^{\infty}(1-r_i^2)<\frac{r_1^2[1-(\prod_{i=1}^{\infty}r_i)^2]}{(\prod_{i=1}^{\infty}r_i)^2}.$$

Since $\prod_{i=1}^{\infty}r_i\geqq 1/M$, we have

$$\sum_{i=1}^{n}\left(1-\frac{r_i^2}{R^2}\right)<r_1^2(M^2-1)$$

We have, therefore,

$$Re\left\{\sum_{i=1}^{n}\left(\frac{z}{z-z_i}+\frac{\bar{z}_i z}{R^2-\bar{z}_i z}\right)\right\}>-\frac{R^2 r r_1^2(M^2-1)}{(r_1-r)(R^2-r_1 r)}. \qquad (3)$$

Since $|g(z)| < M$ in $|z| < R$, by the well-known theorem, we have

$$|g(z)| \geqq M\frac{R|g(0)|-Mr}{MR-r|g(0)|}.$$

Further, using Schwarz's lemma, we obtain

$$|g'(z)| \leqq \frac{R}{M}\,\frac{M^2-|g(z)|^2}{R^2-|z|^2} \leqq \frac{RM(M^2-|g(0)|^2)}{(MR-r|g(0)|)^2}.$$

Hence we have

$$\left|z\frac{g'}{g}\right| \leqq rR\frac{M^2-|g(0)|^2}{(MR-r|g(0)|)(R|g(0)|-Mr)}.$$

And yet since $|g(0)| = R\prod_{i=1}^{n}\frac{R}{r_i} \geqq \frac{R^2}{r_1}$, we have

$$\left|z\frac{g'}{g}\right| \leqq r\frac{M^2r_1^2-R^4}{(Mr_1-Rr)(R^3-Mr_1r)}.$$

Thus we get

$$Re\left\{z\frac{g'}{g}\right\} \geqq -\left|z\frac{g'}{g}\right| \geqq -r\frac{M^2r_1^2-R^4}{(Mr_1-Rr)(R^3-Mr_1r)}. \qquad (4)$$

Therefore from (2), (3) and (4) we have

$$Re\left\{z\frac{f'(z)}{f(z)}\right\} > 1-\frac{R^2rr_1^2(M^2-1)}{(r_1-r)(R^2-r_1r)}-r\frac{M^2r_1^2-R^4}{(Mr_1-Rr)(R^3-Mr_1r)}.$$

For $R \to 1$, finally we have

$$Re\left\{z\frac{f'(z)}{f(z)}\right\} \geqq 1-\frac{rr_1^2(M^2-1)}{(r_1-r)(1-r_1r)}-\frac{r(M^2r_1^2-1)}{(Mr_1-r)(1-Mr_1r)}.$$

Thus, if R_1 is a minimum positive root of the equation

$$1-\frac{rr_1^2(M^2-1)}{(r_1-r)(1-r_1r)}-\frac{r(M^2r_1^2-1)}{(Mr_1-r)(1-Mr_1r)}=0, \qquad (5)$$

$f(z)$ is univalent and star-shaped in the circle $|z| < R_1$ since $Re\{z(f'(z)/f(z))\} > 0$ for $|z| < R_1$. As the equation (5) reduces to the following biquadratic equation

$$\begin{aligned} &Mr_1^2r^4-(M^3r_1^3+2M^2r_1^3+Mr_1)r^3 \\ &\quad +(M^4r_1^4+3M^2r_1^2+M^2r_1^4+2Mr_1-r_1^2)r^2 \qquad (5') \\ &\quad -(M^3r_1^3+2M^2r_1^3+Mr_1)r+Mr_1^2=0, \end{aligned}$$

we have the following.

Theorem 1. Let $w=f(z)$ *be a regular and bounded* ($|f(z)| < M$) *function in* $|z|<1$ *and be normalized such that* $f(0)=0$ *and* $f'(0)=1$. *Further suppose that* z_1 *be* $\neq 0$ *and the nearest zero-point from the origin, then* $f(z)$ *is univalent and star-shaped with respect to the origin in the circle* $|z|<R_1$, *where* R_1 *is a smallesl positive root of equation* (5').

Since $|f(z)/z|<M$ for $|z|<1$, $f(z)/z$ has no zero-point for $|z|<1/M$; so we have $r_1 \geqq 1/M$. Hence in the case $r_1=1/M$, from (5') we have, as a special case $R_1=M-\sqrt{M^2-1}$. This is Landau-Dieudonné's result.[3)]

If the function $f(z)$ is regular and admits the origin as a zero-point of order p we can write $f(z)=z^p g(z)$. So if $g(z)$ satisfies $Re\{z(g'(z)/g(z))\}>-p$ for $|z|<\rho$, then by Kobori's theorem $z^p g(z)$ i.e. $f(z)$ is p-valent for $|z|<\rho$.[4)] So by the method analogous to the precedent, we can obtain the following

Theorem 2. Let $f(z)$ *be regular and bounded* ($|f(z)|<M$) *in* $|z|<1$ *and be normalized such that*

$$f(z)=z^p+a_{p+1}z^{p+1}+\cdots\cdots.$$

Let z_1 ($\neq 0$) *be the nearest zero-point from the origin, so, if* R_p *is the smallest positive root of the equation*

$$p-\frac{rr_1^2(M^2-1)}{(r_1-r)(1-r_1r)}-\frac{r(M^2r_1^2-1)}{(Mr_1-r)(1-Mr_1r)}=0,$$

then $f(z)$ *is* p*-valent in the circle* $|z|<R_p$.

Now since $|f(z)/z^p|<M$ for $|z|<1$, $f(z)/z^p$ has no zero-point in $|z|<1/M$. Hence, as a special case, if we put $r_1=1/M$, we have $R_p=M_p-\sqrt{M_p^2-1}$, where $M_p=(1/2)[(M+1/M)+(1/p)(M-1/M)]$. This is nothing but the Loomis's result.[5)]

In conclusion I wish to express my hearty thanks to Prof. T. Matsumoto and Prof. A. Kobori for their guidances during my research.

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References.

1) Montel; Leçons sur les fonctions univalentes ou multivalentes. p. 27.

2) A. Kobori; Sur la multivalence d'une familles des fonctions analytiques. Proc. Imp. Acad. No. 5 (1938).

3) J. Dieudonné; Recherches sur quelques problèmes aux polynômes et aux fonctions bornées d'une variables complex. Ann. d. l'Ec. Sup., 48. (1931).

4) A. Kobori; loc. cit.

5) L. H. Loomis; The radius and modulus of n-valence for analytic functions whose first n-1 derivatives vanish at a point. Bul. Amer. Math. Soc. 46. (1940).