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Some Remarks on Lüroth's Theorem

By

Pierre SAMUEL

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We give here a purely field theoretic proof of a generalization of Lüroth's theorem, recently pointed out by J. Igusa.[1] We need the following result:

Theorem 1. *Let K' and K'' be two finitely generated extensions of the same algebraic dimension of an infinite field k, and let (t) be a finite set of independent variables over K' and over K''. If $K'(t) = K''(t)$, K' and K'' are isomorphic extensions of k.*

Let us write $K'=k(x')$, $K''=k(x'')$, $(x')=(x_i')$ and $(x'')=(x_j'')$ being finite sets of quantities. There exist rational functions f_i, g_j with coefficients in k such that $x_i'=f_i(x_j'', t)$, $x_j''=g_j(x_i', t)$. Since k is infinite we may choose in k a set of quantities (a) such that the $f_i(x_j'', a)$ and $g_j(x_i', a)$ have non vanishing denominators. Since (a) is a specialization of (t) both over K' and K'', we have $x_i'=f_i(g_j(x', a), a)$. If we denote by $(\bar{x}'')$ the set of quantities $(g_j(x_i', a))$, the fields $k(x')$ and $k(\bar{x}'')$ are equal. Since $(\bar{x}'')$ is a specialization of (x'') over k, and since $k(x'')$ and $k(\bar{x}'')$ have the same algebraic dimension over k, $(\bar{x}'')$ is a generic specialization of (x'') over k,[2] and the fields $k(x'')$, $k(\bar{x}'')$ are isomorphic. QED.

> **Remark.** When the field k is algebraically closed, the proof applies to the following more general situation: if (t) is a finite set of quantities such that $k(t)$ is linearly disjoint from K' and from K'' over k, and if $K'(t)=K''(t)$, then K' and K'' are isomorphic extensions of k. This result is closely related to a question discussed by B. Segre.[3]

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1) "On a theorem of Lüroth", this JOURNAL, Vol. 26 (1951).

2) See A. Weil's *Foundations of Algebraic Geometry*, Amer. Math. Soc., Coll. Publ., 29 (1946), chap. II, th. 1 and 3.

3) "Sur un problème de M. Zariski", Colloque d'Algèbre et Théorie des Nombres, Paris (CNRS), 1950.

We now come to the generalization of Lüroth's theorem:

Theorem 2. *Let* $(t) = (t_1, \cdots, t_n)$ *be a finite set of independent variables over an infinite field* k. *Then every one-dimensional sub-extension* K *of* $k(t)$ *is a simple transcendental extension of* k.

We may suppose, for example, that $t_2, \cdots, t_n$ are independent variables over K. Then $K(t_2, \cdots, t_n)$, which is contained in $k(t_1, \cdots, t_n)$, is, by Lüroth's theorem, a simple transcendental extension $k(t_2, \cdots, t_n, x)$ of $k(t_2, \cdots, t_n)$. Since $K(t_2, \cdots, t_n) = k(x)(t_2, \cdots, t_n)$, K is isomorphic to $k(x)$ by th. 1. QED.

Remark. Theorem 2 may be extended to the case of finite basic field k: replacing k by an algebraic extension, we see that K has genus 0; we then notice (cf. Igusa's paper) that a rational curve over a finite field k has a rational point over k.

CORNELL UNIVERSITY