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Stochastic Differential Equations in a Differentiable Manifold (2)

By

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§ **1.** Let $\pi(t)$ be any Markov process in an r-dimensional differentialble manifold M with the transition probability:

$$(1\cdot1) \qquad F(t, p; s, E) = P(\pi(s) \in E/\pi(t) = p).$$

As is well-known, the generating operator A_t of this process is defined as follows:

$$(1\cdot2) \qquad (A_t f)(p) = \lim_{\Delta\to+0} \frac{1}{\Delta}\int_M [f(q) - f(p)]\ F(t, p; t+\Delta, dq).$$

We shall consider here the process whose generating operator A_t is expressible in the form:

$$(1\cdot3) \qquad (A_t f)(x) = a^i(t, x)\frac{\partial f}{\partial x^i}(x) + \frac{1}{2}\ B^{ij}(t, x)\frac{\partial^2 f}{\partial x^i\,\partial x^j}(x),$$

where x is the local coordinate and f is a bounded function of class C_1. (1.3) is equiualent to the following (1.3′):

$$(1.3') \quad \begin{cases} \dfrac{1}{\Delta}\displaystyle\int_U (y^i - x^i) F(t, x; t+\Delta, dy) \longrightarrow a^i(t, x), \\ \dfrac{1}{\Delta}\displaystyle\int_U (y^i - x^i)(y^j - x^j) F(t, x; t+\Delta, dy) \longrightarrow B^{ij}(t, x),\ (\Delta \to +0) \\ \dfrac{1}{\Delta}\displaystyle\int_U F(t, x; t+\Delta, U^c) \to 0. \end{cases}$$

We can easily see that (B^{ij}) is symmetric and positive-definite and that a^i and B^{ij} are transformed in the following way:

$$(1.4') \qquad \begin{aligned} \bar{a}^i &= a^k \frac{\partial \bar{x}^i}{\partial x^k} + \frac{1}{2} B^{kl} \frac{\partial^2 \bar{x}^i}{\partial x^k \partial x^l} \\ \bar{B}^{ij} &= B^{kl} \frac{\partial \bar{x}^i}{\partial x^k}\ \frac{\partial \bar{x}^j}{\partial x^l} \end{aligned}$$

The purpose of the present paper is to find a continuous Markov process whose generating operator is given by (1.3) when a^i and B^{ij} satisfy some regularity conditions. This problem has been discussed by K. Yosida [4] and S. Itô [6] by means of parabolic differential equations. We shall here make use of stochastic differential equations established in our previous paper [1].

We have shown in [1] that if B^{ij} is written in the form

$$B^{ij}=\sum_{\nu=1}^{r}b_\nu^i\ b_\nu^j \tag{1·5}$$

by a system of r vectors $b_\nu=(b_\nu^1,\cdots,b_\nu^r)$, $\nu=1,2,\cdots,r$, having some appropriate regularity properties, then such a process is determined as the solution of the following stochastic differential equations:

$$d\xi^i=a^i(t,\xi)dt+b_\nu^i(t,\xi)d\beta^\nu \tag{1·6}$$

where $\beta=(\beta^1(t),\cdots,\beta^r(t))$ is an r-dimensional Wiener process.

(B^{ij}) being symmetric and positive-definite, it may be always expressible in the form (5) in many different ways, but we cannot always find the continuous vector system $\{b_\nu\}$ even if the tensor (B^{ij}) satisfies regularity conditions. Such possibility depends on the topological structure of the space M; for example, it is possible in the case of Brownian motions in Lie groups [2]; while it is impossible in the case of those on the surface of 3-sphere [3] [5]. At first sight one may consider that this fact raises an essential dfficulty in our method of stochastic differential equations, but as is shown in this paper, we can overcome it by considering the vector system satisfying regularity conditions *locally*, whose existence is easily verified.

§ **2.** THEOREM. *Consider a differential operator*:

$$(A_tf)(x)=a^i(t,x)\frac{\partial f}{\partial x^i}(x)+\frac{1}{2}B^{ij}(t,x)\frac{\partial^2 f}{\partial x^i\,\partial x^j}(x) \tag{2·1}$$

and put

(2.2) $B_{ij}(t,x)=$ *the* (i,j)*-component of the inverse matrix of* $(B^{ij}(t,x))$. If $a^i(t,x)$, $B^{ij}(t,x)$ *and* $B_{ij}(t,x)$ *are all bounded*[1] *and*

1) The boundedness of $a^i(t,x)$ etc. is defined as follows.

By a canonical coordinate around p we understand a local coordinate which maps a neighbourhood of p onto the interior of the unit sphere in the r-dimensional space R^r and especially transforms p to the centre of the sphere.

$a^i(t,x)$ is called to be bounded in $0\leq t\leq 1$, $x\in M$, if and only if there exist a constant K and a canonical coordinate (x) around any point of M satisfying

$$|a^i(t,x)|<K,\ 0\leq t\leq 1,\ \|x\|<1.$$

continuous in t for each x and of class C_1 in x for each t, then there exists a continuous Markov procass $\pi(t)$, $0 \leq t \leq 1$, whose generating operator is given by (2.1). *The initial distribution i.e. the distribution of $\pi(o)$ is arbitrarily assigned.*

PROOF. 1°. By the assumption of the boundedness of a^i, B^{ij} and B_{ij} we shall define a canonical coordinate (x) around any point p of M with the canonical neighbourhood $U(p)$ such that

$$(2.3) \qquad |a^i(t, x)|, |B^{ij}(t, x)|, |B_{ij}(t, x)|, < K, \ 0 \leq t \leq 1, \ \|x\| < 1, \quad i, j = 1, 2, \cdots, r,$$

K being independent of p. Firstly we shall show that there exist a constant K_1 $(0 < K_1 < 1)$ independent of p and a vector system $(b^i_\nu(t, x))$ for $0 \leq t \leq 1$ and for $\| x \| < K_1$ which is continuous in t for each x and of class C_1 in x for each t and satisfies

$$(2.4) \qquad \sum_{\nu=1}^{n} b^i_\nu(t, x) \ b^j_\nu(t, x) = B^{ij}(t, x).$$

For brevity we shall write as

$$(2.5) \qquad B = B(t, x) = (B^{ij}(t, x)), \ B_0 = B(0, 0).$$

Since B_0 is symmetric and positive-definite we can express it as follows:

$$(2.6) \qquad B_0 = UD^2U', \ D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{pmatrix}, \ U = \text{orthogonal matrix.}$$

Put

$$(2.7) \qquad b_0 = UDU'.$$

Then we have

$$b_0 = b_0', \ B_0 = b_0^2, \text{ and } \|b_0^{-1}\| \leq \sqrt{\|B_0^{-1}\|}.^{2)}$$

Next we put

$$(2.8) \qquad C = b_0^{-1} \, B \, b_0^{-1} - E = b_0^{-1}(B - B_0)b_0^{-1}, \ E = \text{unit matrix,}$$

which satisfies

$$\|C\| \leq \|B_0\| \cdot \|B - B_0\| \leq K_2 \|x\|,$$

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2) We define $\| A \|$ by

$$\| A \| = \sup \{ \| Ax \| ; \| x \| < 1 \},$$

for which the following properties are easily verified:

$$\max_{ij} | a^{ij} | \leq \| A \| \leqq \sum_{ij} | a^{ij} | \text{ for } A = (a^{ij}),$$

$$\| U \| = 1 \text{ for any orthogonal } U,$$

$$\| A \ B \| \leq \| A \| \| B \|,$$

$$\| D_1 \ D_2 \| = \| D_1 \| \| D_2 \| \text{ for diagonal } D_1, D_2.$$

K_2 being a constant determined only by K, and so we obtain

(2.9) $$\|C\| < 1 \text{ for } \|x\| < K_1 = \max(\frac{1}{K_2}, 1).$$

Therefore

(2.10) $$\begin{cases} c = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} C^n \\ c_{(k)} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} n C \cdot \frac{\partial C}{\partial x^k} \end{cases}$$

is convergent in the above norm for $\|x\| < 1$, and so we easily see that

(2.11) $$\frac{\partial c^{ij}}{\partial x^k} = c^{ij}_{(k)}$$

in considering that $|a^{ij}| \leq \|A\|, i, j=1, 2, \cdots, r$ for $A=(a^{ij})$. Thus we see that c^{ij} are of class C_1 in x and continuous in t. We have clearly

$$c=c', \; c^2=E+C=b_0^{-1} B b_0^{-1} \; i.e. \; B=b_0 c^2 b_0=(b_0 c)(b_0 c)'.$$

We put

(2.12) $$b=(b^i_\nu)=b_0 c,$$

which satisfies the above-mentioned conditions.

2^0. We shall define the subsets V, W and Q of R^r by the following conditions:

$$V: \|x\| < K_1, \quad W: \|x\| < \frac{2}{3}K_1, \quad Q: \|x\| < \frac{1}{3}K_1,$$

and denote by $V(p)$, $W(p)$ and $Q(p)$ the parts of the above assigned canonical coordinate neighbourhood $U(p)$ corresponding to V, W and Q respectively.

Let $\varphi(x)$ denote the function of $x \in R^r$,

$$\varphi(x)=1 \;(x \in W), =0 \;(x \in V^c), = \frac{2K_1-3\|x\|}{K_1} (x \in V-W).$$

Then $\varphi(x)$ satisfies the following conditions:

$$0 \leq \varphi(x) \leq 1, \; |\varphi(x)-\varphi(y)| \leq \frac{3}{K_1}\|x-y\|.$$

Since $a^i(t, x)$ and $b^i_\nu(t, x)$ depend on the neighbourhood $U(p)$, we shall denote them with $a^i(t, x; p)$ and $b^i_\nu(t, x; p)$ respectively. We put

$$\tilde{a}^i(t, x: p) = a^i(t, x; p) \ \ \varphi(x) \ \ (x \in U(p)), \ 0(x \in U(p))$$

$$\tilde{b}^i(t, x; p) = d^i_\nu(i, x; p) \ \ \varphi(x) \ \ (x \in U(p)), \ 0(x \in U(p))$$

and consider a stochastic differential equation

$$d\xi^i(t) = \tilde{a}^i(t, \xi; p) \ dt + \tilde{b}^i_\nu(t, \xi; p) \ d\beta^\nu(t), \ i=1, 2,\cdots, r, \ t_0 \leq t \leq 1,$$

where $\xi(t_0)$ is any assigned point q in $Q(p)$. As is shown in [1], this equation has a unique solution $\pi(t; U(p); t, q)$ which lies in $V(p)$ for $t_0 \leq t \leq 1$ with probability 1 and in $W(p)$ for $t_0 \leq t \leq t+\Delta$ with probabilty $1-o(\Delta)$, $o(\Delta)$ being independent of p and q.

Let Φ denote the distribution assigned as the probability law of $\pi(0)$. We shall consider an M-valued random variable π which is subject to Φ and an r-dimensional Wiener process $\beta(t) = (\beta^1(t), \cdots, \beta^r(t))$, $0 \leq t \leq 1$, independent of π. Since M satisfies the second countability axiom, M is covered by a countable system of $Q(p)$, say $Q(p_1)$, $Q(p_2)$,$\cdots$

Let $Q(p_f)$ be the first of $\{Q(p_n)\}$ that contains π. We shall define

$$\pi_1(t) = \pi(t; U(p_f); 0, \pi).$$

Next we shall define $\pi_2(t)$ to be equal to $\pi_1(t)$ as far as $\pi_1(t)$ remains in $W(p_f)$ and if $\pi_1(t)$ attains the boundary of $W(p_f)$ at $t=t_1$, then we shall define $\pi_2(t)$ $(t \geq t_1)$ by

$$\pi_2(t) = \pi_2(t; U(p_s); t_1, \pi_1(t_1)),$$

where $Q(p_s)$ is the first of $\{Q(p_n)\}$ that contains $\pi_1(t_1)$. In the same way we shall $\pi_n(t)$, $n=1, 2,\cdots$ recursively. We put

$$\pi(t) = \lim \ \pi_n(t).$$

This limit process exists and satisfies the conditions of our theorem, as is shown by the same idea as our previous paper [1].

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