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Bergman kernel function and canonical slit-mapping.

By

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1. Let D be a finitely connected domain in the z-plane which contains the points $z=0$ and $z=\infty$, and bounded by n proper continua. According to the well-known Grunsky's theorem[1] in the theory of conformal mapping of multiply-connected domains there exists one and only one function which, in the neighborhood of $z=\infty$, has a Laurent expansion of the form

$$w=s_0(z)=z+\frac{c}{z}+\cdots, \tag{1}$$

and at the origin $s_0(0)=0$ and $s'_0(0)=a_0$, and which maps D conformally onto a whole plane slit along n arcs on a finite number of logarithmic spirals having the same angle of inclination $\theta/2$ and the same asymptotic point $z=0$.

In the present paper we shall derive an inequality involving the coefficient a_0 appearing in (1) and the outer logarithmic area L of the complement (with respect to the whole plane) of the domain D, namely:

$$Re(-e^{-i\theta}\log a_0)-\frac{|\log a_0|^2}{\log(A/B)}\geqq\frac{L}{2\pi}, \tag{2}$$

where A and B are constants which will be explained in the section 3.

It suffices to prove the inequality (2) in the case when the boundary continua of D are closed analytic curves $C_1, C_2\cdots,C_n$, for it is known that D can be approximated by an increasing sequence of domains having such boundaries for which the mapping functions corresponding to (1) will converges to $s_0(z)$, so that (2) will continue to hold in the limit, when L is interpreted in the manner explained above.

In the proof of the above theorem we utilize the following lemma on Bergman kernel function[2]:

Lemma. *Let* $K(z,\bar{t})=\sum_{k=1}^{\infty}\varphi_k(z)\cdot\overline{\varphi_k(t)}$ *be the Bergman kernel function of the domain* D *where every function* $\varphi_k(z)$ $(k=1,2,\cdots)$ *is single-valued analytic and possesses a uniform indefinite integral* $\Psi_k(z)=\int_{\infty}^{z}\varphi_k(z)dz$.

Then there holds

$$\int_{\bar\infty}^{\bar z}\int_{\infty}^{z}K(z,\bar{t})\,dz d\bar{t}=\sum_{k=1}^{\infty}|\,\Psi_k(z)\,|^2.$$

Therefore the right-hand side is determined independently of the particular choice of the complete orthonormal system $\{\varphi_k(z)\}$.

Remembering that the series $\sum_{k=1}^{\infty}\varphi_k(z)\overline{\varphi_k(t)}$ may be termwise integrated with respect to both variables z and $\bar{t}$ $(z,t\in D)$ because of the uniform boundedness of the partial sum $\sum_{k=1}^{n}\varphi_k(z)\overline{\varphi_k(t)}$, the lemma is easily proved.

2. Since the boundary of D is for the present assumed to be consisted of analytic curves, it follows that $s_0(z)$ remains analytic there as well as in the interior of D. Taking the form of $s_0(z)$ in the neighborhoods of $z=0$ and $z=\infty$, and its behavior on each boundary curve $C_i(i=1,2,\cdots,n)$ into acccount, we see that the function $\log\frac{s_0(z)}{z}$ is single-valued analytic and has a finite Dirichlet integral

$$I=\iint_D\left|\frac{d}{dz}\log\frac{s_0(z)}{z}\right|^2 d\tau,\quad (d\tau=dxdy,\ z=x+iy).\tag{3}$$

I is real and non-negative, vanishing if and only if $s_0(z)\equiv z$, that is, if and only if D is identical with the domain onto which it is mapped. Now, by means of Green's theorem, the Dirichlet integral (3) can be transformed into an integral taken along the boundary curves of D, as follows;

$$\begin{aligned}I=&\frac{1}{2i}\sum_{k=1}^{n}\int_{C_k}\overline{\log\frac{s_0(z)}{z}}\cdot\frac{d}{dz}\left(\log\frac{s_0(z)}{z}\right)dz\\=&\frac{1}{2i}\left\{\sum_{k=1}^{n}\int_{C_k}\overline{\log s_0(z)}\cdot\frac{d}{dz}\log s_0(z)dz-\sum_{k=1}^{\infty}\int_{C_k}\overline{\log s_0(z)}\frac{dz}{z}\right.\\&\left.-\sum_{k=1}^{n}\int_{C_k}\overline{\log z}\frac{d}{dz}\log s_0(z)\cdot dz+\sum_{k=1}^{n}\int_{C_k}\overline{\log z}\frac{dz}{z}\right\},\end{aligned}\tag{4}$$

the sense of integration being positive with respect to the interior of D.

On the other hand we observe that on each boundary curve C_k there holds an important relation for the mapping function $w=s_\theta(z)$

$$\overline{\log s_\theta(z)}=e^{-i\theta}\log s_\theta(z)+c_k \qquad \text{on } C_k(k=1,\cdots,n), \qquad (5)$$

c_k being constant. Now we shall calculate each term of (4) by means of (5) in the following manner. At first we obtain

$$\text{the 1st integral}=\sum_k\int_{C_k}\overline{\log s_\theta(z)}\frac{d}{dz}\log s_\theta(z)dz$$

$$=e^{-i\theta}\sum_k\int_{C_k}\log s_\theta(z)\frac{d}{dz}\log s_\theta(z)dz+\sum_k c_k\int_{C_k}\frac{d}{dz}\log s_\theta(z)dz \quad \text{(by (5))}$$

$$=e^{-i\theta}\sum_k\left[\frac{1}{2}(\log s_\theta(z))^2\right]_{C_k}+\sum_k c_k[\log s_\theta(z)]_{C_k} \qquad (6)$$

$$=0.$$

Next, being in the neighborhood of $z=0$

$$\log\frac{s_\theta(z)}{z}=\log a_\theta+O(z) \qquad \text{and} \qquad \frac{d}{dz}\log s_\theta(z)=\frac{1}{z}+O(1),$$

we have

$$\sum_k\int_{C_k}\overline{\log\frac{s_\theta(z)}{z}}\cdot\frac{d}{dz}\log s_\theta(z)\cdot dz$$

$$=e^{i\theta}\sum_k\overline{\int_{C_k}\log\frac{s_\theta(z)}{z}\cdot d\log s_\theta(z)}, \text{ (by (5))} \qquad (7)$$

$$=e^{i\theta}\overline{(2\pi i\log a_\theta)} \quad \text{(by residue theorem)}.$$

From (6) and (7) we obtain

$$\text{the 3rd integral}=\sum_k\int_{C_k}\overline{\log z}\frac{d}{dz}\log s_\theta(z)dz$$

$$=e^{i\theta}\cdot 2\pi i\,\overline{\log a_\theta}. \qquad (8)$$

Integrating the 2nd term of (4) by parts, we obtain

$$\text{the 2nd integral} = \sum_k \int_{C_k} \overline{\log s_0(z)} \frac{dz}{z}$$

$$= \sum_k \left\{ [\overline{\log s_0(z)} \cdot \log z]_{C_k} - \int_{C_k} \log z \cdot \overline{\frac{d}{dz} \log s_0(z) dz} \right\} \quad (9)$$

$$= e^{-i\theta} 2\pi i \cdot \log a_0, \text{ (-conjugate complex number of (8)).}$$

Finally we obtain

$$\text{the 4th integral} = \sum_k \int_{C_k} \overline{\log z} \frac{dz}{z}$$

$$= -2i \sum_k \frac{-1}{2i} \int_{C_k} \overline{\log z} \frac{dz}{z} = -2iL. \quad (10)$$

Putting (6), (7), (8), (9) and (10) in (4), we have

$$I = 2\pi Re(-e^{-i\theta} \log a_\theta) - L \quad (\geqq 0)^{3)}. \quad (11)$$

3. We now introduce two canonical mapping functions $P(z)$ and $Q(z)$. $P(z)$ effects the conformal mapping of D onto the whole plane slit along circular arcs centered at the origin and satisfies the same normalized conditions with $s_\theta(z)$, *i. e.* $\lim_{z\to\infty} P(z)/z = 1$, $P(0)=0$, and $P'(0)=A$. $Q(z)$ effects the conformal mapping of D onto the whole plane slit along radial segments toward the origin and the same normalized conditions with $s_\theta(z)$, *i. e.* $\lim_{z\to\infty} Q(z)/z=1$, $Q(0)=0$ and $Q'(0)=B$. It is well-known that the function $\log \frac{P(z)}{Q(z)}$ is single-valued analytic and its Dirichlet integral $2\pi \log(A/B)$[4] (>0). Next we consider the following two functions

$$g(z) = \frac{\frac{d}{dz}\left(\log \frac{s_\theta(z)}{z}\right)}{[2\pi Re(-e^{-i\theta} \log a_\theta) - L]^{1/2}}, \quad (12)$$

and

$$h(z) = \frac{\frac{d}{dz}\left(\log \frac{P(z)}{Q(z)}\right)}{[2\pi \log(A/B)]^{1/2}}. \quad (13)$$

We can easily assert that $\iint_D |g(z)|^2 d\tau = 1$ (by (11)), $\iint_D |h(z)|^2 d\tau = 1$, and the uniform integrals of $g(z)$ and $h(z)$ are given by $\log \frac{s_0(z)}{z} \Big/ [2\pi Re(-e^{-i\theta} \log a_0) - L]^{1/2}$ and $\log \frac{P(z)}{Q(z)} \Big/ [2\pi \log (A/B)]^{1/2}$, respectively, i. e.

$$\Psi_1^{(1)}(z) = \int_\infty^z g(t)dt = \frac{\log \frac{s_0(z)}{z}}{[2\pi Re(-e^{-i\theta}\log a_0) - L]^{1/2}}, \quad \Psi_1^{(1)}(\infty) = 0, \tag{14}$$

$$\Psi_1^{(2)}(z) = \int_\infty^z h(t)\,dt = \frac{\log \frac{P(z)}{Q(z)}}{[2\pi \log(A/B)]^{1/2}}, \quad \Psi_1^{(2)}(\infty) = 0. \tag{15}$$

Now let two different complete systems be constructed, begining with the functions $g(z)$ and $h(z)$ respectively. At first we adopt the function $g(z)$ for $\varphi_1(z)$ belonging to the system $\{\varphi_k(z)\}$. Then we get the relation

$$\sum_{k=1}^{\infty} | \Psi_k^{(1)}(0)|^2 \geqq | \Psi_1^{(1)}(0)|^2 = \frac{|\log a_0|^2}{2\pi Re(-e^{-i\theta}\log a_0) - L}. \tag{16}$$

where $\Psi_k^{(1)}(z) = \int_\infty^z \varphi_k(t)dt$. Next we adopt the function $h(z)$ for $\varphi_1(z)$. Since $\varphi_k(z)$ $(k \geqq 2)$ belonging to the system $\{\varphi_k(z)\}$ is orthogonal to $h(z)$, we obtain

$$0 = \iint_D \varphi_k(z)\overline{h(z)}d\tau \qquad (k \geqq 2)$$

$$= \frac{2\pi\{\Psi_k^{(2)}(0) - \Psi_k^{(2)}(\infty)\}}{[2\pi \log(A/B)]^{1/2}}\text{[5)]},$$

where $\Psi_k^{(2)}(z) = \int_\infty^z \varphi_k(t)dt$. Therefore $\Psi_k^{(2)}(0) = 0$ $(k \geqq 2)$. Accordingly

$$\sum_{k=1}^{\infty} | \Psi_k^{(2)}(0) |^2 = | \Psi_1^{(2)}(0) |^2 = \frac{\log(A/B)}{2\pi}. \tag{17}$$

By (16), (17) and the lemma we obtain

$$\sum_{k=1}^{\infty} | \Psi_k^{(1)}(0)|^2 = \sum_{k=1}^{\infty} | \Psi_k^{(2)}(0)|^2,$$

and therefore

$$\frac{\log(A/B)}{2\pi} \geqq \frac{|\log a_0|^2}{2\pi Re(-e^{-i\theta}\log a_0)-L}. \tag{18}$$

Thus there holds the following

Theorem *Let D be a finitely connected domain in the z-plane which contains the points $z=0$ and $z=\infty$. Let $s_0(z)$, $P(z)$ and $Q(z)$ be the above mentioned mapping functions. Then there holds the following inequality*

$$Re(-e^{-i\theta}\log a_0) - \frac{|\log a_0|^2}{\log(A/B)} \geqq \frac{L}{2\pi},$$

where $a_0=s'_0(0)$, $A=P'(0)$ and $B=Q'(0)$.

4. We shall consider the special case where the domain D is given by

$$|z-1| > \sqrt{1-q}, \quad (0<q<1),$$

and confirm that in this case the equality holds in (2). In this case we have, in the neighborhood of $z=0$,[6]

$$P(z)=z\frac{z-1}{z-q}=\frac{1}{q}z+\cdots, \quad A=\frac{1}{q},$$

and

$$Q(z)=z\frac{z-q}{z-1}=qz+\cdots, \quad B=q.$$

Now we use the general relation obtained by Grunsky

$$s_0(z)=p(z)\{q(z)\}^t, \quad (t=e^{i\theta}),$$

where $p(z)=\sqrt{P(z)Q(z)}$ and $q(z)=\sqrt{Q(z)/P(z)}$. Then we get

$$\log s_0(z)=\frac{1-t}{2}\log P(z)+\frac{1+t}{2}\log Q(z),$$

therefore

$$\log a_0=\frac{1-t}{2}\log A+\frac{1+t}{2}\log B$$

and in the special case

$$\log\ a_0 = -e^{i\theta} \log \frac{1}{q}, \tag{19}$$

therefore

$$Re(-e^{-i\theta} \log a_0) = \log \frac{1}{q}. \tag{20}$$

By (19)

$$|\log a_0|^2 = \left(\log \frac{1}{q}\right)^2. \tag{21}$$

Hence the left-hand side of (2) assumes the value $\frac{1}{2}\log(1/q)$. On the other hand we shall calculate the logarithmic area of the complement of D;

$$L = \frac{1}{2i}\int_C \overline{\log z}\,\frac{dz}{z} \qquad (C;\ |z-1| = \sqrt{1-q}\ )$$

$$= \iint_{\tilde{D}} \left|\frac{1}{z}\right|^2 d\tau_z \qquad \left(\begin{array}{l} d\tau = dxdy, z = x+iy. \\ \tilde{D};\ \text{complement of } D. \end{array}\right)$$

$$= \iint_{|\zeta|<\sqrt{1-q}} \frac{d\tau_\zeta}{|\zeta+1|^2} \qquad (z=\zeta+1,\ \zeta=\xi+i\eta,\ d\tau_\zeta = d\xi d\eta)$$

$$= \iint_{|\zeta|<\sqrt{1-q}} \frac{r dr d\varphi}{|1+re^{i\varphi}|^2} \qquad (\zeta = re^{i\varphi})$$

$$\therefore\ L = \int_0^{2\pi}\int_0^{\sqrt{1-q}} \frac{r dr d\varphi}{1+2r\cos\varphi + r^2} \qquad (0<q<1)$$

$$= \int_0^{\sqrt{1-q}} \left\{\int_0^{2\pi} (1+re^{i\varphi})^{-1}(1+re^{-i\varphi})^{-1} d\varphi\right\} r dr$$

$$= \pi\left[\log\frac{1}{1-r^2}\right]_0^{\sqrt{1-q}} = \pi\log\frac{1}{q}.$$

Therefore the right-hand side of (2) also takes the same value $\frac{1}{2}\log(1/q)$. Thus the exactness of the inequality (2) is shown.

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2) S. Bergman: The kernel function and conformal mapping, Amer. Math. Soc. (1950).

3) By (11) we obtain an inequality simpler than (2), namely:

$$Re(-e^{-i\theta} \log a_0) \geqq L/2\pi.$$

We must remark that the analogous results have been recently obtained by Y. Komatu and M. Ozawa from more general point of view. See Y. Komatu and M. Ozawa; Conformal mapping of multiply connected domains, I. Kōdai Math. Sem. Reports Nos. 5 and 6 (1951) pp. 81-95.

4) P. R. Garabedian and M. Schiffer: Identities in the theory of conformal mapping, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 187-238.

5) loc. cit. 4)

6) loc. cit. 1)