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On the Evaluation of the Derivatives of Solutions of $y''=f(x, y, y')$.

By

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1. Theorem. The principal part of the possibility of the prolongation of solutions of the ordinary differential equation of the second order,

$$\frac{d^2y}{dx^2}=f\left(x, y, \frac{dy}{dx}\right), \tag{1}$$

is on the evaluation of the derivatives of the solutions. On this subject Prof. Nagumo has given a sufficient condition in Proc. Physico-Math. Soc. of Japan, 3rd Series. Vol. 19 (1937)., and the late Prof. Okamura has obtained a more general and easier result in Functional Equations (in Japanese), Vol. 27, but it is also a sufficient condition.

Recently we have obtained a necessary and sufficient condition for the evaluation, by aid of the D-function having the properties like the distance has which Okamura had utilized in his research of necessary and sufficient conditions for the uniqueness of solutions in the Cauchy-problem.

Our theorem runs as follows.

Let $\mathfrak{L}$ be a bounded closed domain in xy-plane and $\mathfrak{L}^*$ be a three dimensional domain of (x, y, y'), where $(x, y)\in\mathfrak{L}$ and $-\infty<y'<+\infty$. Let $f(x, y, y')$ be defined and continuous in $\mathfrak{L}^*$. In order that, given a positive number α and for a suitable positive number $\beta(\alpha)$ $(>\alpha)$, if for any solution $y=y(x)$ of (1) *through a point (x_0, y_0) arbitrary in $\mathfrak{L}$, provided $|y'(x_0)|\leq\alpha$, we have*

$$|y'(x)|<\beta(\alpha),$$

so long as $y=y(x)$ lies in $\mathfrak{L}$ for $x_0\leq x$, it is necessary and sufficient that there exist two non-negative continuous functions $\Phi_i(x, y, y')$ $(i=1, 2)$ as follows; namely $\Phi_1(x, y, y')$ and $\Phi_2(x, y, y')$ are defined in

$$\Delta_1: \quad (x, y) \in \mathfrak{L}, \quad \alpha \leqq y' \leqq \beta$$

and

$$\Delta_2: \quad (x,y) \in \mathfrak{L}, \quad -\beta \leqq y' \leqq -\alpha$$

respectively and for $(x, y) \in \mathfrak{L}$,

$$\Phi_1(x, y, \alpha) > 0, \quad \Phi_1(x, y, \beta) = 0 \tag{2}$$

and

$$\Phi_2(x, y, -\alpha) > 0, \quad \Phi_2(x, y, -\beta) = 0\,, \tag{3}$$

and then they satisfy the Lipschitz condition,

$$| \Phi_i(x, y, y') - \Phi_i(x, \bar{y}, \bar{y}') | \leqq K_i(|y - \bar{y}| + |y' - \bar{y}'|) \quad (i=1, 2), \tag{4}$$

with regard to (y, y'), *where* K_i *are constants, and finally, for points of* Δ_1 *and* Δ_2,

$$\underline{D}^+_{[F]}\Phi_i(x, y, y') \geqq 0 \quad (i=1, 2) \tag{5}$$

respectively in the following sense.

(5) is Nagumo's notation, e. g. see "*Sur une sorte de distance relative à un système différentiel*" (Okamura, Proc. Ibid, Vol. 25 (1943) pp. 520—521). Now we consider the system of differential equations

$$\frac{dy_1}{dx} = y_2\,, \quad \frac{dy_2}{dx} = f(x, y_1, y_2), \tag{6}$$

which becomes by the vector notation

$$\frac{d\mathfrak{y}}{dx} = \boldsymbol{F}(x, \mathfrak{y}). \tag{7}$$

This is equivalent to (1). Then the above mentioned sense is that (5) holds for all points $(x_0, \mathfrak{y}_0)$ from which a regular curve i. e., continuous with continuous first derivatives, having the direction given by $\boldsymbol{F}(x_0, \mathfrak{y}_0)$ goes to the right in Δ_1 and Δ_2 respectively.

The left hand of (5), $\underline{D}^+_{[F]}\Phi_i(x_0, \mathfrak{y}_0)$, depends by the Lipschitz condition only on the direction $\mathfrak{y}'(x_0)$ at $x=x_0$ of the curve $\mathfrak{y} = \mathfrak{y}(x)$. Therefore, if (x, y, y') is an interior point, (5) may be replaced by

$$\varliminf_{t \to +0} \frac{1}{t} \{\Phi_i[x+t,\ y+ty',\ y'+tf(x, y, y')] - \Phi_i(x, y, y')\} \geqq 0$$

and moreover if $\Phi_i(x, y, y')$ be totally differentiable, then it reduces to

$$\frac{\partial \Phi_i}{\partial x} + \frac{\partial \Phi_i}{\partial y} y' + \frac{\partial \Phi_i}{\partial y'} f(x,y,y') \geqq 0.$$

Remark 1. In the above theorem, if "*given a positive number α and so on*" be replaced by "*given an arbitrary positive number α and so on*", we shall have $\Phi_i^\alpha(x, y, y')$ $(i=1, 2)$ for every α, instead of $\Phi_i(x, y, y')$.

Remark 2. Generalizing the problem, we can obtain the analogous condition for the system $\frac{dy_i}{dx}=f_i(x, y_1 \cdots, y_n)$ $(i=1,\cdots, n)$.

2. Proof of Theorem. At first, we prove that the condition is sufficient. When there exist the above mentioned functions $\Phi_i(x, y, y')$ $(i=1, 2)$ for a suitable positive number $\beta(\alpha)$ $(>\alpha)$, we suppose that, for a solution $y=y(x)$ which passes through a point (x_0, y_0) and satisfies $|y'(x_0)| \leqq \alpha$, we have $y'(x)=\beta$ at some point $x(>x_0)$. Then by the continuity of $y'(x)$, there exist two values of x, say x_1 and x_2, such that $y'(x_1)=\alpha$, $y'(x_2)=\beta$ and $\alpha<y'(x)<\beta$ for $x_1<x<x_2$. In this interval, we consider the function $\Phi_1(x, y(x), y'(x))$. By (2),

$$\Phi_1(x_1, y(x_1), y'(x_1)) = \Phi_1(x_1, y(x_1), \alpha) > 0$$

and

$$\Phi_1(x_2, y(x_2), y'(x_2)) = \Phi_1(x_2, y(x_2), \beta) = 0$$

and yet, by (5), $\Phi_1(x, y(x), y'(x))$ is non-decreasing with respect to x and therefore

$$\Phi_1(x_1, y(x_1), y'(x_1)) \leqq \Phi_1(x_2, y(x_2), y'(x_2)).$$

This contradicts with the inequality standing above. Hence $y'(x)=\beta$ can not hold, whence we have

$$y'(x) < \beta.$$

Similarly by aid of $\Phi_2(x, y, y')$, we can see that $-\beta<y'(x)$. Therefore we have in any case

$$|y'(x)| < \beta.$$

Next we show the condition to be necessary. For clearness, we will consider the equation (6) equivalent to (1). Let Q be a point (x^Q, y_1^Q, y_2^Q) in Δ_1 and P be a point such as $(x^P, y_1^P) \in \mathcal{L}$, $y_2^P=\beta$

and $x^Q \leqq x^P$. Now we suppose that, if $|y'(x_0)| \leqq \alpha$, then we have $y'(x) < \beta$. Now we construct the Okamura's D-function $D(Q, P)$[1] and we indicate by $\delta(Q)$ the minimum of $D(Q, P)$, when P displaces, yet satisfying $x^Q \leqq x^P$; namely we have

$$\delta(Q) = \min_{x^Q \leq x^P} D(Q, P). \tag{8}$$

This minimum exists and there exists a point $\bar{P}$ such as just $\delta(Q) = D(Q, \bar{P})$, since $D(Q, P)$ is a continuous function of P and the range of the displacement of P is a bounded closed domain. Moreover the continuity of $\delta(Q)$ concerning to Q results from that of $D(Q, P)$ with regard to (Q, P).

By the hypothesis, there exist no solutions passing through a point Q lying on $y_2 = \alpha$ and a point P lying on $y_2 = \beta$ in Δ_1 and hence we have always

$$D(Q, P) > 0$$

and

$$\delta(Q) = \min D(Q, P) > 0,$$

because there exists a point $\bar{P}$ such as $\delta(Q) = D(Q, \bar{P})$. Clearly, if Q lies on $y_2 = \beta$, we have

$$\delta(Q) = 0,$$

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1) Since the domain is general, the construction of $D(Q, P)$ is based on the method, for example, stating in *Functional Equations*, Vol. 39 pp. 33–40 by Okamura. Namely we consider a system of ordinary differential equations $\frac{d\boldsymbol{y}}{dx} = \boldsymbol{F}(x, \boldsymbol{y})$, where $\boldsymbol{y} = (y_1, \cdots\cdots, y_n)$ is a n-dimensional vector and $\boldsymbol{F}(x, \boldsymbol{y})$ is a continuous function (n-dimensional vector) in a bounded closed set $\boldsymbol{B}$ in the space of $n+1$ dimensions. By using an independent variable τ, this system reduces to a system of $n+1$ differential equations $\frac{dx}{d\tau} = 1, \frac{d\boldsymbol{y}}{d\tau} = \boldsymbol{F}(x, \boldsymbol{y})$, and the latter is represented by

$$\frac{d\boldsymbol{x}}{d\tau} = \boldsymbol{U}(\boldsymbol{x}), \tag{A}$$

where $\boldsymbol{x} = (x, \boldsymbol{y})$ and $\boldsymbol{U}(\boldsymbol{x}) = (1, \boldsymbol{F}(x, \boldsymbol{y}))$. Then $\boldsymbol{U}(\boldsymbol{x})$ is continuous in $\boldsymbol{B}$. Now let P and Q be two points in $\boldsymbol{B}$ and a function $\boldsymbol{x}(\tau)$ be a function of bounded vaviation in $\tau_0 \leqq \tau \leqq \tau_1$ which satisfies $\boldsymbol{x}(\tau_0) = P$ and $\boldsymbol{x}(\tau_1) = Q$ and $\mathfrak{X}_{PQ}$ be the family of all such functions, where τ_0 and τ_1 are arbitrary and $-\infty < \tau_0 \leqq \tau_1 < \infty$. And put

$$D(P, Q) = \min_{\boldsymbol{x}(\tau) \in \mathfrak{X}_{PQ}} \underset{\tau_0 \leqq \tau \leqq \tau_1}{V} [\boldsymbol{x}(\tau) - \int \boldsymbol{U}(\boldsymbol{x}(\tau)) d\tau],$$

where V expresses the total variation.

and in the other cases where $\alpha < y_2 < \beta$, we have

$$\delta(Q) \geqq 0,$$

since $D(Q, P) \geqq 0$ by the definition of the D-function.

Let Q and R be two points, where $x^Q \leqq x^R$, and then there exist P_1 and P_2 such that $\delta(Q) = D(Q, P_1)$ and $\delta(R) = D(R, P_2)$ and $x^Q \leqq x^{P_1}$, $x^R \leqq x^{P_2}$. Since $x^Q \leqq x^R$, we have $x^Q \leqq x^{P_2}$, and hence, by the definition of $\delta(Q)$, we have evidently

$$\delta(Q) \leqq D(Q, R) + \delta(R).^{2)} \tag{9}$$

Moreover if $x^Q = x^R$, we have

$$|\,\delta(Q) - \delta(R)\,| \leqq \overline{QR}, \tag{10}$$

since $\delta(Q) \leqq D(Q, R) + \delta(R)$ and $\delta(R) \leqq D(Q, R) + \delta(Q)$, that is to say, it satisfies the Lipschitz condition with regard to (Q, R).

Then if we put

$$\Phi_1(x, y_1, y_2) = \delta(Q) \qquad \text{with } Q = (x, y_1, y_2) \in \Delta_1, \tag{11}$$

it is easy to see that $\Phi_1(x, y_1, y_2)$ satisfies the conditions in our theorem.

Secondly we suppose that, if $|\,y'(x_0)| \leqq \alpha$, we have $-\beta < y'(x)$. In this case, similarly put

$$\bar{\delta}(Q) = \min_{x^Q \leqq x^P} D(Q, P) \tag{12}$$

in Δ_2, where $Q \in \Delta_2$ and $(x^P, y_1^P) \in \mathcal{L}$, $y_2^P = -\beta$ and $x^Q \leqq x^P$. Then $\bar{\delta}(Q)$ has the same properties as the above. Therefore, if we put

$$\Phi_2(x, y_1, y_2) = \bar{\delta}(Q) \quad \text{with } Q = (x, y_1, y_2) \in \Delta_2, \tag{13}$$

this $\Phi_2(x, y_1, y_2)$ is the desired function.

3. Remarks. In the preceding paragraph, we stated our theorem in the case where x *increases.* Yet for *decreasing* x we can obtain in the same way, a similar theorem for the solutions going *to the left.* In this case we may consider

$$\delta(Q) = \min_{x^Q \geqq x^P} D(P, Q)$$

in Δ_1. And if we put $\delta(Q) = \Psi_1(x, y_1, y_2)$ for $Q = (x, y_1, y_2) \in \Delta_1$,

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2) If Q and R lie on the same solution, $D(Q, R) = 0$. Therefore $\delta(Q) \leqq \delta(R)$, that is, it is non-decreasing along any solution.

$\Psi_1(x, y_1, y_2)$ has same properties as those of $\Phi_1(x, y_1, y_2)$, but we must have, instead of (5)

$$\bar{D}^{-}_{[F]}\Psi_1(x, y, y') \leqq 0. \tag{14}$$

Also let $\Psi_2(x, y, y')$ be defined in Δ_2 in the similar way, then it satisfies

$$\bar{D}^{-}_{[F]}\Psi_2(x, y, y') \leqq 0. \tag{15}$$

Hence by using $\Psi_1(x, y, y')$ and $\Psi_2(x, y, y')$ in place of $\Phi_1(x, y, y')$ and $\Phi_2(x, y, y')$ respectively in our theorem then we can obtain the necessary and sufficient condition for the evaluation of derivatives of solutions going to the left. Therefore the existence of such four functions $\Phi_i(x, y, y')$ and $\Psi_i(x, y, y')$ $(i=1, 2)$ gives the condition for the general case.

Finally we remark that $\Phi_i(x, y, y')$ and $\Psi_i(x, y, y')$ can be modified as follows; namely, for example, we can replace " $\Phi_1(x, y, \alpha) > 0$ *and* $\Phi_1(x, y, \beta) = 0$ " by the condition "*for* $(x, y) \in \mathfrak{L}$,

$$\min_{(x, y) \in \mathfrak{L}} \Phi_1(x, y, \alpha) > \max_{(x, y) \in \mathfrak{L}} \Phi_1(x, y, \beta)\text{''}. \tag{16}$$

As to the necessary condition, (16) holds good, because $\Phi_1(x,y,y')$ is continuous in the bounded closed domain Δ_1 and hence $\Phi_1(x,y,\alpha)$ is continuous just for $(x, y) \in \mathfrak{L}$ aud therefore we have

$$\min_{(x, y) \in \mathfrak{L}} \Phi_1(x, y, \alpha) > 0,$$

while clearly we have $\max_{(x, y) \in \mathfrak{L}} \Phi(x, y, \beta) = 0$. As to the sufficient condition, if (16) only holds good, our theorem is proved as we see evidently in the above proof. If our theorem be modified in this way, we can directly deduce Nagumo's condition and others.

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