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On the Primary Difference of Two Frame Functions in a Riemannian Manifold.

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In a previous paper* we have expressed the Stiefel characteristic classes in terms of the forms Π^r and Ω^r. Now, we intend to express the *deformation cochain* of two frame functions by these forms. We make clear the geometrical meaning of the proof of theorem given in the said paper, and we show that Π^r may be regarded as a form which represents the *primary difference.*

We shall use, throughout this paper, the same notations as in the preceding paper.

1. The deformation cochain $d(f_0, h, f_1)$. Let f_0 and f_1 be two cross-sections: $K^r \to \mathfrak{B}^r$ $(1 \leqq r \leqq n-1)$, whose obstruction cocycles we denote by $c(f_0)$ and $c(f_1)$ respectively. Since $\pi_i(Y^r)=0$ for $i<r$, there exists a homotopy

$$h: f_0|K^{r-1} \simeq f_1|K^{r-1}.$$

The interval I is regarded as a cell complex consisting of one 1-cell I and the 0-cells 0 and 1. Let $\bar{0}$, $\bar{1}$ be the generators of the group of 0-cochains with integral coefficients; and let $\bar{I}$ denote a generator of the group of 1-cochains chosen so that $\delta\bar{0}=-\bar{I}$, $\delta\bar{1}=\bar{I}$. We may regard naturally $\mathfrak{B}^r \times I$ as a bundle over $K^n \times I$; and a cross-section φ of the part of $\mathfrak{B}^r \times I$ over the r-dimensional skeleton of $K^n \times I$, is constructed by

$$(1) \quad \begin{aligned} &\varphi(x, 0)=(f_0(x), 0), \ \varphi(x, 1)=(f_1(x), 1) && \text{for } x \in K^r, \\ &\varphi(x, t)=(h(x, t), t) && \text{for } x \in K^{r-1},\ t \in I. \end{aligned}$$

Then an obstruction cocycle $c(\varphi)$ is defined. If we set

$$(2) \qquad d(f_0, h, f_1) \times \bar{I} = (-1)^{r+1}\{c(\varphi) - c(f_0) \times \bar{0} - c(f_1) \times \bar{1}\},$$

the $(r+1)$-cochain $d(f_0, h, f_1) \times \bar{I}$ of $K^n \times I$ with coefficients in π_r is

*) *On the Stiefel characteristic classes of a Riemannian manifold,* these Memoirs, this number. We shall quote the paper as "[1]".

zero on $K^n \times 0 \cup K^n \times 1$; and there exists a unique r-cochain $d(f_0,h,f_1)$ of K^n with coefficient in π_r, which is called the *deformation cochain*. The coboundary formula

$$\delta d(f_0,h,f_1) = c(f_0) - c(f_1) \tag{3}$$

holds; and so, in case that f_0 and f_1 are extendable over K^{r+1}, $d(f_0, h, f_1)$ is a cocycle. Being $\varDelta^r$ an r-cell of K^n, $\varDelta^r \times I$ is an $(r+1)$-cell of $K^n \times I$, and

$$\partial(\varDelta^r \times I) = \Sigma^{r-1} \times I + (-1)^r(\varDelta^r \times 1 - \varDelta^r \times 0), \tag{4}$$

where $\Sigma^{r-1} = \partial\varDelta^r$. Applying both sides of (2) to $\varDelta^r \times I$, it follows that

$$d(f_0,h,f_1) \cdot \varDelta^r = (-1)^{r+1}\, c(\varphi) \cdot (\varDelta^r \times I). \tag{5}$$

Since $\pi_r(Y^{r+1}) = 0$, there exists a homotopy $\bar{h}$: $pf_0|K^r \simeq pf_1|K^r$ such that $ph = \bar{h}$ on K^{r-1}. Moreover pf_0 and pf_1 have the extensions g_0 and g_1 over K^{r+1}. Then, a cross-section ψ of the part of $\mathfrak{B}^{r+1} \times I$ over the $(r+1)$-dimensional skeleton of $K^n \times I$, is constructed by

$$\begin{aligned} &\psi(x, 0) = (g_0(x), 0), \quad \psi(x, 1) = (g_1(x), 1) \quad \text{for } x \in K^{r+1}, \\ &\psi(x, t) = (\bar{h}(x, t), t) \qquad\qquad\qquad\quad \text{for } x \in K^r,\ t \in I. \end{aligned} \tag{6}$$

Clearly $\psi(\varDelta^r \times I)$ is a cell which has the sphere $\psi\partial(\varDelta^r \times I)$ as boundary. We choose an interior point ξ of $\varDelta^r \times I$ and denote by Y_ξ^r the fibre of $\mathfrak{B}^r \times I$ over ξ. If a contruction of $\psi\partial(\varDelta^r \times I)$ over $\psi(\varDelta^r \times I)$ into $\psi\xi$ is chosen to sweep out each point of $\psi(\varDelta^r \times I) - \psi\xi$ once and only once, then a covering homotopy of this contructon may give an extension of φ over $\varDelta^r \times I - \xi$ and may carry φ into a map φ_ξ: $\partial(\varDelta^r \times I) \to p^{-1}(\psi\xi) \subset Y_\xi^r$. Moreover $p^{-1}(\psi\xi)$ is an r-sphere in which Π^{r+1} is reduced to the form

$$(-1)^{r+1} \frac{\Gamma(\frac{r+1}{2})}{2\pi^{\frac{1}{2}(r+1)}} \omega_{1,r+1} \cdots \omega_{r,\, r+1}\,.$$

Therefor, by Kronecker's formula we have

$$(-1)^{r+1} D(\varphi_\xi) = \int_{\varphi_\xi \partial(\varDelta^r \times I)} \Pi^{r+1},$$

where $D(\varphi_\xi)$ is the degree of r-sphere map φ_ξ. On the other hand we have directly

$$c(\varphi) \cdot (\varDelta^r \times I) = \alpha_r \cdot D(\varphi_\xi)$$

and

$$\int_{\varphi_\xi \partial(\Delta^r\times I)} \Pi^{r+1} = \int_{\varphi\partial(\Delta^r\times I)} \Pi^{r+1} + \int_{\psi(\Delta^r\times I)} \Omega^{r+1}.$$

It follows that

$$(-1)^{r+1}c(\varphi)\cdot(\Delta^r\times I) = \alpha_r\cdot\left\{\int_{\varphi\partial(\Delta^r\times I)} \Pi^{r+1} + \int_{\psi(\Delta^r\times I)} \Omega^{r+1}\right\}.$$

In view of (4) and (5), we obtain

$$(7)\quad d(f_0, h, f_1)\cdot\Delta^r = \alpha_r\cdot\left\{(-1)^r\int_{f_1\Delta^r - f_0\Delta^r} \Pi^{r+1} + \int_{\varphi(\Sigma^{r-1}\times I)} \Pi^{r+1} + \int_{\psi(\Delta^r\times I)} \Omega^{r+1}\right\}.$$

It is to be noted that this result does not depend on ψ. If z^r is an r-cycle of K^n with integral coefficients, then

$$(8)\quad (-1)^r d(f_0, h, f_1)\cdot z^r = \alpha_r\cdot\left\{\int_{f_1 z^r} \Pi^{r+1} - \int_{f_0 z^r} \Pi^{r+1} + (-1)^r\int_{\psi(z^r\times I)} \Omega^{r+1}\right\}.$$

In particular, if $pf_0 = pf_1$, taking $\bar{h}(x, t) = pf_0(x)$ for all $t \in I$, we have

$$(9)\quad (-1)^r d(f_0, h, f_1)\cdot z^r = \alpha_r\cdot\left\{\int_{f_1 z^r} \Pi^{r+1} - \int_{f_0 z^r} \Pi^{r+1}\right\},$$

which shows that in this case $d(f_0, h, f_1)$ depends only on f_0 and f_1. When $r = n-1$, (9) always holds.

The formula (17) in [1], § 6 means $d(F, \tilde{k}, F_0)\cdot\Sigma^{r-1} = 0$. And it is easy to see that (9) implies the theorem of [1]: since $c(*F)\cdot\Delta^r = 0$, we have $c(F)\cdot\Delta^r = \delta d(F, *F)\cdot\Delta^r = d(F, *F)\cdot\partial\Delta^r = d(F, *F)\cdot\Sigma^{r-1}$ by (3), and so the formula (15) in [1] follows from (9) taking account of (14) in [1], § 5.

2. The primary difference $\bar{d}(f_0, f_1)$. If f_0 and f_1 are cross-sections: $\mathbf{R}^n \to \mathfrak{B}^r$, then $d(f_0, h, f_1)$ is an r-cocycle whose cohomology class does not depend upon the choice of homotopy $h: f_0|K^{r-1} \simeq f_1|K^{r-1}$. This class denoted by $\bar{d}(f_0, f_1)$ is called the ***primary difference*** of f_0 and f_1. Let Z^r be an arbitrary homology class of $\mathbf{R}^n$ with integral coefficients, and choose a cycle z^r to represent Z^r. For f_0 and f_1 we take a cross-section ψ as (6). Then, from (8) and (9) we have the following formulas.

THEOREM. *If r is even or $r = n-1$,*

$$(10)\qquad (-1)^r\bar{d}(f_0, f_1)\cdot Z^r = \int_{f_1 z^r} \Pi^{r+1} - \int_{f_0 z^r} \Pi^{r+1};$$

and if r is odd and $r<n-1$,

$$(11)\qquad \bar{d}(f_0, f_1)\cdot Z^r \equiv \int_{f_1 z^r} \Pi^{r+1} - \int_{f_0 z^r} \Pi^{r+1} - \int_{\psi(z^r\times I)} \Omega^{r+1} \pmod 2.$$

So far as we consider the Stiefel characteristic classes, the form Π^r has been used only as a supplementary one, and we can regard Ω^r as an essential form which represents the class; for, the formula

$$(-1)^r \bar{c}_r(\boldsymbol{R}^n)\cdot Z^r = a_{r-1}\cdot \int_{G\cdot z^r} \Omega^r$$

holds when coefficients of a cycle z^r are integers. As against this, the formulas (8)—(11) show that $\bar{\Pi}^r$ is the very form which represents the primary difference or at least the deformation cochain, and that Ω^r merely assists the form Π^r in case that it is not closed.