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On the Stiefel Characteristic Classes of a Riemannian Manifold

By

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Introduction. One of the chief problems in differential geometry in the large is the inquiry on relations between the *Stiefel characteristic classes* of a compact orientable manifold and a Riemannian metric defined on it. The determination of the formulas which express the characteristic classes in terms of differential forms has been discussed in the paper of Allendoerfer [1],* as an argument analogous to the proof of the Allendoerfer-Weil formula.

In the present paper we shall deal with this subject from a different point of view which is more geometrical. It will be shown that the consideration of a given frame function can be reduced to the simplest case in virtue of the homotopy theory of fibre bundles, and that the formulas can be found naturally from the well-known result due to Chern [4], making use of the induced metric on a submanifold. Thus, we make clear the intrinsic properties of the differential forms appearing in the formulas.

1. Preliminaries and notations. Let $\boldsymbol{R}^n$ be a compact connected orientable Riemannian manifold of dimension n and class $\geqq 4$, and let $\mathfrak{B}^{n-1}$ denote the tangent sphere bundle over it. Then we can get in a certain way the associated bundle $\mathfrak{B}^q$ $(0 \leqq q \leqq n-1)$ of $\mathfrak{B}^{n-1}$ having the Stiefel manifold $Y^q = V_{n,\,n-q}$ as fibre. Each element of $\mathfrak{B}^q$ may be an $(n-q)$-frame in $\boldsymbol{R}^n$,

$$\mathfrak{b}^q = P\mathfrak{e}_{q+1}\ \mathfrak{e}_{q+2}\cdots\mathfrak{e}_n,$$

where P is a point of $\boldsymbol{R}^n$ and $\mathfrak{e}_{q+1}, \mathfrak{e}_{q+2}, \cdots, \mathfrak{e}_n$ are mutually orthogonal unit vectors of $\boldsymbol{R}^n$ with origin P. If we define a map $p: \mathfrak{B}^q \rightarrow \mathfrak{B}^{q+1}$ by

$$p\mathfrak{b}^q = P\mathfrak{e}_{q+2}\cdots\mathfrak{e}_n,$$

*) Numbers in square brackets refer to the bibliography.

the elements of $\mathfrak{B}^q$ also constitute a sphere bundle over $\mathfrak{B}^{q+1}$, and the map p fills the rôle of the projection of this bundle structure. Furthermore through the map p we obtain stepwise projections:

$$\mathfrak{B}^0 \to \mathfrak{B}^1 \to \cdots \to \mathfrak{B}^{n-1} \to \boldsymbol{R}^n,$$

and any composition of them is the projection of a bundle structure.

Let $\pi_i(Y^q)$ denote the i-th homotopy group of the Stiefel manifold Y^q; and in particular we abbreviate $\pi_q(Y^q)$ by π_q. It is well-known that $\pi_i(Y^q)=0$ for $i<q$, and

$$\pi_q=\begin{cases}\text{infinite cyclic} & \text{if } q \text{ even or } q=n-1,\\ \text{cyclic of order 2} & \text{if } q \text{ is odd and } q<n-1.\end{cases}$$

For an integer c, let us define the element $\alpha_q\cdot c$ by

$$\alpha_q\cdot c=\begin{cases}c & \text{if } q \text{ is even or } q=n-1,\\ 0 \text{ or } 1\equiv c \pmod 2 & \text{if } q \text{ is odd and } q<n-1.\end{cases}$$

Then the operation α_q may be regarded as the natural homomorphism from the additive group of integers onto π_q.

Decompose $\boldsymbol{R}^n$ into a cell complex K^n of class $\geqq 4$, and let K^q be its q-dimensional skeleton. Then there exists an $(n-r+1)$-frame function F of class $\geqq 2$ on a neighborhood of K^{r-1}, since $\pi_i(Y^{r-1})=0$ for $i<r-1$. We will assume by a tacit understanding that chains and maps which will be introduced in our argument are of suitable class if necessary. The justification of these assumptions may be easily proved from the fact that $\boldsymbol{R}^n$ is of class $\geqq 4$. Since $\boldsymbol{R}^n$ is orientable, the associated bundle of coefficients $\mathfrak{B}^q(\pi_q)$ is a product bundle; thus, a cochain of K^n with coefficients in $\mathfrak{B}^q(\pi_q)$ can be regarded as an ordinary cochain with coefficients in π_q. Let $c(F)$ be the *obstruction cocycle* of F. The present purpose is to deduce an expression of the cohomology class to which $c(F)$ belongs and which is a topological invariant of $\boldsymbol{R}^n$. In order to achieve this, we first restrict attention only to an oriented r-cell $\varDelta^r$ of K^n $(2\leqq r\leqq n)$. The boundary of $\varDelta^r$, $\partial\varDelta^r=\Sigma^{r-1}$, is an oriented $(r-1)$-sphere. We will use, for instance, the notation $\mathfrak{B}^{r-1}_\Sigma$ which means the portion of bundle $\mathfrak{B}^{r-1}$ over Σ^{r-1}. The frame function F is then considered to be a cross-section: $\Sigma^{r-1}\to\mathfrak{B}^{r-1}_\Sigma$. The cross-section pF: $\Sigma^{r-1}\to\mathfrak{B}^r_\varDelta$ has a differentiable extension over $\varDelta^r$, because $\pi_{r-1}(Y^r)=0$. We denote it by G, and set $G\varDelta^r=E^r$ and $F\Sigma^{r-1}=\tilde{S}^{r-1}$. Then E^r is a cell contained in $\mathfrak{B}^r$ and $\tilde{S}^{r-1}$ is a sphere contained in $\mathfrak{B}^{r-1}$. Their ori-

entation may be induced naturally from that of Δ^r, and clearly $\partial E^r = p\tilde{S}^{r-1}$. The statement on an expression for the element $c(F)\cdot\Delta^r$ in π_{r-1} by integrals over the domains $\tilde{S}^{r-1}$ and E^r, will occupy the main part of this paper.

We shall denote by I the interval of real numbers $0 \leqq t \leqq 1$. In general, let $\mathfrak{B}$ be a bundle over Σ with the projection ρ, and let $F(x)$, $x \in \Sigma$, be a cross-section. A map $k: \Sigma \times I \rightarrow \mathfrak{B}$ is said to be a homotopy of the cross-section F, if $\rho k(x, t) = x$ for all t and $k(x, 0) = F(x)$. Define $k_t: \Sigma \rightarrow \mathfrak{B}$ by $k_t(x) = k(x, t)$, and then k_t is a cross-section. Two cross-sections k_0 and k_1 are said to be homotopic, in symbols $k_0 \simeq k_1$.

2. The auxiliary cross-section F_0. The first step of our consideration is to construct a special cross-section F_0 from the given cross-section F.

LEMMA 1. *There exists a cross-section F_0: $\Sigma^{r-1} \rightarrow \mathfrak{B}^{r-1}$ which satisfies the following conditions.*

(i) *$F \simeq F_0$, and so $c(F)\cdot\Delta^r = c(F_0)\cdot\Delta^r$.*

(ii) *For each frame $P\mathfrak{e}_r\ \mathfrak{e}_{r+1}\cdots\mathfrak{e}_n \in F_0\Sigma^{r-1}$, the vectors $\mathfrak{e}_{r+1},\cdots,\mathfrak{e}_n$ are normal to Δ^r, and so $\mathfrak{e}_r$ is tangent to Δ^r.*

(iii) *The cross-section pF_0: $\Sigma^{r-1} \rightarrow \mathfrak{B}^r$ has an extension over Δ^r with the following property: if we denote it by G_0, each frame of $G_0\Delta^r$ is normal to Δ^r.*

We are going to define a cross-section F_0 which satisfies the above conditions. Since any bundle over a cell is a product bundle, it is possible to define a cross-section G_0: $\Delta^r \rightarrow \mathfrak{B}^r$ so that each element of $G_0\Delta^r$ may be an $(n-r)$-frame *normal to* Δ^r. Moreover, since $\pi_i(Y^r) = 0$ for $i < r$, any two cross-sections: $\Sigma^{r-1} \rightarrow \mathfrak{B}^r_\Sigma$ are homotopic; of course $pF \simeq G_0$ on Σ^{r-1}. If k is a homotopy of the cross-section pF into G_0 on Σ^{r-1}, the second covering homotopy theorem concerning the bundle $\mathfrak{B}^{r-1}_\Sigma$ over $\mathfrak{B}^r_\Sigma$, the space Σ^{r-1}, the map F, and the homotopy k, asserts the existence of a homotopy $\tilde{k}$: $\Sigma^{r-1} \times I \rightarrow \mathfrak{B}^{r-1}_\Sigma$ covering k (i.e. $p\tilde{k} = k$) and being stationary with k (see [8], § 11·7). It is easily seen from the constructions of k and $\tilde{k}$ that $\tilde{k}$ is a homotopy of the cross-section F. Hence $\tilde{k}_1$ is a cross-section: $\Sigma^{r-1} \rightarrow \mathfrak{B}^{r-1}_\Sigma$. We define F_0 by $F_0 = \tilde{k}_1$. Obviously F_0 satisfies the conditions of lemma 1. The maps G_0, k and $\tilde{k}$, which have been introduced to define F_0, do not determine in a unique way. For convenience, however, we take them fixed throughout this paper, and set $G_0\Delta^r=E_0^r$ and $F_0\Sigma^{r-1}=\tilde{S}_0^{r-1}$. Clearly $\partial E_0^r=p\tilde{S}_0^{r-1}$.

3. The tangent vector field φ_0. Let $\mathfrak{T}^{r-1}$ be the tangent sphere bundle over Δ^r and let $\mathfrak{T}^0$ be its associated principal bundle. A map τ : $\mathfrak{B}^{r-1}\to\mathfrak{B}^{n-1}$ is defined by $\tau\mathfrak{b}^{r-1}=P\mathfrak{e}_r \in \mathfrak{B}^{n-1}$ with $\mathfrak{b}^{r-1}=P\mathfrak{e}_r\, \mathfrak{e}_{r+1}\cdots\mathfrak{e}_n \in \mathfrak{B}^{r-1}$. According to lemma 1, $\varphi_0=\tau F_0$ is a tangent vector field, that is, a cross-section: $\Sigma^{r-1}\to\mathfrak{T}^{r-1}$.

We consider Δ^r to be a Riemannian space induced by the Riemannian metric of R^n. Let A be an interior point of Δ^r. We may assume that Δ^r is included in a sufficiently small neighborhood of A, and that the points of Δ^r are determined by the normal coordinate with respect to the induced metric. Let $P \in \Delta^r$ be a point on a geodesic line $\widehat{AP_0}$ joining A to $P_0 \in \Sigma^{r-1}$. By parallel displacement from P_0, transport the vector $\varphi_0 P_0$ along $\widehat{AP_0}$ to P and to A. Denoting the resulting vectors by $\bar{\varphi}_0 P$ and by $\varphi_A P_0$ respectively, we obtain an extension $\bar{\varphi}_0$ of φ_0 over Δ^r-A and a map φ_A : $\Sigma^{r-1}\to S_A^{r-1}$, where S_A^{r-1} is the tangent unit sphere of Δ^r at A. The point A is possibly a singular point of the vector field $\bar{\varphi}_0$: $\Delta^r\to\mathfrak{T}^{r-1}$, and the index of $\bar{\varphi}_0$ at A equals to the degree of the map φ_A.

LEMMA 2. *If $D(\varphi_0)$ denotes the index of $\bar{\varphi}_0$ at A, then*

$$c(F_0)\cdot\Delta^r=\alpha_{r-1}\cdot D(\varphi_0).$$

PROOF. We can obtain an extension $\bar{F}_0$ of F_0 over Δ^r-A by $\tau\bar{F}_0=\bar{\varphi}_0$ and $p\bar{F}=G_0$. A map F_A: $\Sigma^{r-1}\to Y_A^{r-1}$ is given by $\tau F_A=\varphi_A$ and $pF_A=$ *constant map*: $\Sigma^{r-1}\to G_0 A$, where Y_A^{r-1} is the fibre of $\mathfrak{B}^{r-1}$ ovor A. From the constructions of $\bar{F}_0$ and F_A, it is clear that $c(F_0)\cdot\Delta^r$ is equal to the homotopy class of the map F_A in $\pi_{r-1}(Y_A^{r-1})$. Further, by the well-known relation between the generator of π_{r-1} and the degree of $(r-1)$-sphere map, we can set $c(F_0)\cdot\Delta^r=\alpha_{r-1}\cdot D(\varphi_0)$; for, $\tau F_A=\varphi_A$, and pF_A is a constant map. This proves our assertion.

4. The integral formula for F_0. When a Riemannian metric of R^n is given, we can define in $\mathfrak{B}^0$ in a unique way a set of $n(n+1)/2$ linearly independent linear differential forms ω_λ, $\omega_{\lambda\mu}=-\omega_{\mu\lambda}$ $(\lambda,\mu=1,\cdots,n)$ which satisfy the equations (cf. [6], § 5)

$$dP=\sum_\lambda\omega_\lambda\,\mathfrak{e}_\lambda,\quad d\mathfrak{e}_\lambda=\sum_\mu\omega_{\lambda\mu}\,\mathfrak{e}_\mu, \tag{1}$$

$$d\omega_\lambda=\sum_\mu\omega_\mu\,\omega_{\mu\lambda}; \tag{2}$$

and the form $\Omega_{\lambda\mu}$, so-called the curvature form, is given by the equation

$$d\omega_{\lambda\mu}=\sum_{\nu=1}^{n}\omega_{\lambda\nu}\,\omega_{\nu\mu}+\Omega_{\lambda\mu}. \tag{3}$$

From what has been proved already, we see that an expression of $c(F)\cdot\Delta^r$ may be obtained by calculating $D(\varphi_0)$. In order to express $D(\varphi_0)$ in terms of differential forms, we may apply, making use of the induced metric, a well-known result for tangent vector field (cf. [4] and [5]). Then we get the following.

LEMMA 3. *The index $D(\varphi_0)$ is given by the formula*

$$(-1)^r D(\varphi_0)=\int_{\tilde{S}_0^{r-1}}\Pi^r+\int_{E_0^r}\Omega^r, \tag{4}$$

where Π^r and Ω^r are forms defined by

$$\Omega^r=\begin{cases}\dfrac{(-1)^{\frac{r}{2}}}{2^r\pi^{\frac{r}{2}}\left(\dfrac{r}{2}\right)!}\displaystyle\sum_{(i)}\epsilon_{i_1\ldots i_r}\Omega^r_{i_1i_2}\cdots\Omega^r_{i_{r-1}i_r} & \text{if } r \text{ is even},\\ 0 & \text{if } r \text{ is odd},\end{cases} \tag{5}$$

$$\Pi^r=\frac{(-1)^r}{2^r\pi^{\frac{1}{2}(r-1)}}\sum_{k=0}^{[\frac{1}{2}(r-1)]}(-1)^k\frac{1}{k!\,\Gamma\left(\frac{1}{2}(r-2k+1)\right)}\Phi^r_k, \tag{6}$$

with

$$\Omega^r_{ij}=\Omega_{ij}+\sum_{\sigma=r+1}^{n}\omega_{i\sigma}\,\omega_{\sigma j}, \tag{7}$$

$$\Phi^r_k=\sum_{(\alpha)}\epsilon_{\alpha_1\ldots\alpha_{r-1}}\,\Omega^r{}_{\alpha_1\alpha_2}\cdots\Omega^r{}_{\alpha_{2k-1}\alpha_{2k}}\omega_{\alpha_{2k+1}r}\cdots\omega_{\alpha_{r-1}r}, \tag{8}$$

$$(i,j=1,\cdots,r;\quad \alpha=1,\cdots,r-1).$$

It seems that we need a few explanations on the formula (4) in which integrals of forms in $\mathfrak{B}^0$ are defined over domains in $\mathfrak{B}^{r-1}$ and $\mathfrak{B}^r$. Though it will be shown generally in the next section that Π^r and Ω^r are, in fact, forms in $\mathfrak{B}^{r-1}$ and $\mathfrak{B}^r$ respectively, we can see here these integrals have intrinsic meanings, through the following statement on the relation between the formula (4) written in terms of $\mathfrak{B}^q$ and the formula for $\mathfrak{T}^{r-1}$.

Take a *repère mobile* $P\mathfrak{e}_1\cdots\mathfrak{e}_n$ such that, for $P\in\Delta^r$, the vectors $\mathfrak{e}_1,\cdots,\mathfrak{e}_r$ are tangent to Δ^r. Then, from (1), (2) and (3), the equations of induced Riemannian connection are written

$$\bar{d}P=\sum_i\omega_i\,\mathfrak{e}_i,\qquad \bar{d}\mathfrak{e}_i=\sum_j\omega_{ij}\,\mathfrak{e}_j\qquad (i,j=1,\cdots,r), \tag{9}$$

and its equations of structure are

(10) $$d\omega_i=\sum_j \omega_j\, \omega_{ji},$$

(11) $$d\omega_{ij}=\sum_{k=1}^{r} \omega_{ik}\, \omega_{kj}+\Omega^r_{ij},$$

where

$$\Omega^r_{ij}=\Omega_{ij}+\sum_{\sigma=r+1}^{n} \omega_{i\sigma}\, \omega_{\sigma j}.$$

It should be observed that forms ω_i, ω_{ij} and Ω^r_{ij} remain invariant under a change of repère mobile, if its tangent vectors $\mathfrak{e}_1,\cdots, \mathfrak{e}_r$ are left unchanged. We choose the normal vectors of the repère mobile such that $P\mathfrak{e}_{r+1}\cdots\mathfrak{e}_n \in E_0^r$. As to the term *repère mobile* due to Cartan, see [2] and [3].

Let $\bar{\mathfrak{B}}^{r-1}$ and $\bar{\mathfrak{B}}^0$ denote respectively the portion of $\mathfrak{B}^{r-1}$ and $\mathfrak{B}^0$ over E_0^r when we suppose them to be bundles over $\mathfrak{B}^r$. Then the map τ defined in the preceding section is a bundle homeomorphism of $\bar{\mathfrak{B}}^{r-1}$ onto $\mathfrak{T}^{r-1}$. A map $\bar{\tau}$ given by $p\tau=\bar{\tau}p$ is a homeomorphism of E_0^r onto Δ^r. Further, if we define a map μ: $\mathfrak{B}^0\rightarrow\mathfrak{B}^{n-r}$ by $\mu\mathfrak{b}^0=P\mathfrak{e}_1\cdots\mathfrak{e}_r \in \mathfrak{B}^{n-r}$ with $\mathfrak{b}^0=P\mathfrak{e}_1\cdots\mathfrak{e}_n \in \mathfrak{B}^0$, μ is a bundle homeomorphism of $\bar{\mathfrak{B}}^0$ onto $\mathfrak{T}^0$.

We regard ω_i, ω_{ij} and Ω^r_{ij} as forms in $\bar{\mathfrak{B}}^0$ induced by the inclusion map $\bar{\mathfrak{B}}^0\rightarrow\mathfrak{B}^0$. From these forms the homeomophism μ^{-1}: $\mathfrak{T}^0\rightarrow\bar{\mathfrak{B}}^0$ induces the forms in $\mathfrak{T}^0$, which we denote by $\bar{\omega}_i$, $\bar{\omega}_{ij}$ and $\bar{\Omega}^r_{ij}$ respectively to avoid any confusion. Clearly $\bar{\omega}_i$ and $\bar{\omega}_{ij}$ satisfy the equations (9) and (10) which are fundamental for induced Riemannian geometry; and $\bar{\Omega}^r_{ij}$ is its curvature form. Let $\bar{\Pi}^r$ and $\bar{\Omega}^r$ denote the forms given by putting $\bar{\omega}_i$, $\bar{\omega}_{ij}$ and $\bar{\Omega}^r_{ij}$ into (5) and (6). Then we may apply the result due to Chern.

It has been known that $\bar{\Pi}^r$ and $\bar{\Omega}^r$ are indeed forms in $\mathfrak{T}^{r-1}$ such that $-d\bar{\Pi}^r=\bar{\Omega}^r$, and especially $\bar{\Omega}^r$ is a form in Δ^r. Let Σ^{r-1}_ε be a geodesic hypersphere in Δ^r of radius ε with center A, and let Γ^r_ε be the closed ring-shaped domain in Δ^r bounded by $\Sigma^{r-1}$and Σ^{r-1}_ε. If we set $\varphi_0\Sigma^{r-1}=\tilde{\Sigma}^{r-1}$, $\bar{\varphi}_0\Gamma^r_\varepsilon=\tilde{\Gamma}^r_\varepsilon$, and $\bar{\varphi}_0\Sigma^{r-1}_\varepsilon=\tilde{\Sigma}^{r-1}_\varepsilon$, then clearly $\partial\tilde{\Gamma}^r_\varepsilon=\tilde{\Sigma}^{r-1}-\tilde{\Sigma}^{r-1}_\varepsilon$; and hence

$$\int_{\tilde{\Sigma}^{r-1}} \bar{\Pi}^r - \int_{\partial\tilde{\Gamma}^r_\varepsilon} \bar{\Pi}^r = \int_{\tilde{\Sigma}^{r-1}_\varepsilon} \bar{\Pi}^r.$$

By applying Stokes' theorem and by considering that $\bar{\Omega}^r$ is a form in $\varDelta^r$, we have

$$-\int_{\partial\tilde{\Gamma}_\varepsilon^r}\bar{\Pi}^r=\int_{\tilde{\Gamma}_\varepsilon^r}\bar{\Omega}^r=\int_{\Gamma_\varepsilon^r}\bar{\Omega}^r,\quad \lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^r}\bar{\Omega}^r=\int_{\varDelta^r}\bar{\Omega}^r.$$

On the other hand, we know by [5] that

$$\lim_{\varepsilon\to 0}\int_{\tilde{\Sigma}_\varepsilon^{r-1}}\bar{\Pi}^r=(-1)^r D(\varphi_0).$$

Consequently it follows that

$$\int_{\tilde{\Sigma}^{r-1}}\bar{\Pi}^r+\int_{\varDelta^r}\bar{\Omega}^r=(-1)^r D(\varphi_0).$$

The formula (4) is nothing but an interpretation of this equation in terms of $\bar{\mathfrak{B}}^{r-1}$ and E_0^r through mediums of the homeomorphisms τ and $\bar{\tau}$.

As regards transformations of forms and chains by a map, see, for instance, [7], §§ 5–6.

5. The forms Π^r and Ω^r. In the above section the forms Π^r and Ω^r were considered only in $\bar{\mathfrak{B}}^0$. Now we consider them in whole $\mathfrak{B}^0$. They have the following formal properties.

LEMMA 4.

(i) $\Omega^r=-2\Pi^{r+1}$, *if r is even and $r<n$.*

(ii) *Π^r is a form in $\mathfrak{B}^{r-1}$.*

(iii) $-d\Pi^r=\Omega^r$.

(iv) *Ω^r is a form in $\mathfrak{B}^r$.*

The proof of this lemma is quite mechanical, so we only sketch its outline.

(i) From (7), obviously

$$\Omega_{ij}^r=\Omega_{ij}^{r+1}+\omega_{i,\,r+1}\ \omega_{r+1,\,j}.$$

By substituting this into the expression for Ω^r, we have $\Omega^r=-2\Pi^{r+1}$.

(ii) It is sufficient to establish the result for a coordinate neighborhood V on $\mathfrak{B}^{r-1}$. Let f and f^* be arbitrary local cross-sections: $\mathfrak{B}^{r-1}\to\mathfrak{B}^0$ defined on V. For a frame $\mathfrak{b}^{r-1}=P\mathfrak{e}_r\cdots\mathfrak{e}_n\in V$, if we write $f\mathfrak{b}^{r-1}=P\mathfrak{e}_1\cdots\mathfrak{e}_n$ and $f^*\mathfrak{b}^{r-1}=P\mathfrak{e}_1^*\cdots\mathfrak{e}_n^*$, we have the transformation of frames

$$(12)\quad \begin{cases}\mathfrak{e}_A^*=\mathfrak{e}_A & (A,\,B=r,\,r+1,\cdots,\,n),\\ \mathfrak{e}_\alpha^*=\sum\limits_\beta \theta_{\alpha\beta}\,\mathfrak{e}_\beta & (\alpha,\,\beta=1,\cdots,\,r-1),\end{cases}$$

where $\theta_{\alpha\beta}$ are functions of the local parameters of V and $(\theta_{\alpha\beta})$ is a proper orthogonal matrix of order $r-1$. From a form ω in $\mathfrak{B}^0$ the maps f and f^* induce forms in V which we denote by ω and ω^* respectively. Then we have easily

$$\omega^*_A=\omega_A, \quad \omega^*_{AB}=\omega_{AB}, \quad \omega^*_{\alpha A}=\sum_\beta \theta_{\alpha\beta}\,\omega_{\beta A},$$

$$\Omega^*_{\alpha\beta}=\sum_{\gamma,\delta=1}^{r-1}\theta_{\alpha\gamma}\,\theta_{\beta\delta}\,\Omega_{\gamma\delta}, \quad \Omega^*_{\alpha r}=\sum_\beta \theta_{\alpha\beta}\,\Omega_{\beta r}, \quad \Omega^*_{rr}=\Omega_{rr}=0.$$

From these, we can easily prove that Ω^r_{ij} has the same transformation law as that of Ω_{ij} under the transformation (12); and so, noting that $|\theta_{\alpha\beta}|=1$, we see Φ^r_k remain invariant under (12). Hence Φ^r_k is a form in $\mathfrak{B}^{r-1}$, and also is Π^r.

(iii) For $r=n$ the formula $-d\Pi^n=\Omega^n$ is known: see [5]. Suppose that $r<n$. Then the equations

$$\text{(13)} \quad \begin{cases} d\Omega^r_{\alpha\beta}=-\sum_i \Omega^r_{\alpha i}\,\omega_{i\beta}+\sum_i \omega_{\alpha i}\,\Omega^r_{i\beta}, \\ d\omega_{\alpha r}=\sum_\beta \omega_{\alpha\beta}\,\omega_{\beta r}+\Omega^r_{\alpha r}, \end{cases}$$

also hold. Moreover $d\Phi^r_k$ is a form in $\mathfrak{B}^{r-1}$. Hence, if we substitute (13) into the expression for $d\Phi^r_k$, the terms involving $\omega_{\alpha\beta}$ will cancel each other. The conditions are entirely same as the case $r=n$, and they suffice to deduce $-d\Pi^r=\Omega^r$.

(iv) It is well-known that Ω^n is a scaler in $\boldsymbol{R}^n$; and the assertion is trivial for the case that r is odd. If r is even and $r<n$, Ω^r is equal to $-2\Pi^{r+1}$ which is a form in $\mathfrak{B}^r$.

Our lemma has been thus established. If $\tilde{C}^r$ is an r-chain in $\mathfrak{B}^{r-1}$, as an immediate consequence of the lemma we have by applying Stokes' theorem

$$\text{(14)} \qquad \int_{\partial\tilde{C}^r}\Pi^r=-\int_{\tilde{C}^r}\Omega^r=-\int_{p\tilde{C}^r}\Omega^r.$$

6. The integral formula for F. We prove now the following result.

THEOREM. *If the cross-section $G: K^r\to\mathfrak{B}^r$ is an extension of pF over K^r, the following formula holds:*

$$\text{(15)} \qquad (-1)^r\,c(F)\cdot\Delta^r=\alpha_{r-1}\cdot\left\{\int_{\tilde{S}^{r-1}}\Pi^r+\int_{E^r}\Omega^r\right\},$$

where $\tilde{S}^{r-1}=F\partial\Delta^r$ and $E^r=G\Delta^r$.

PROOF. From lemmas 1–3, we have

$$(16) \qquad (-1)^r c(F)\cdot\Delta^r = \alpha_{r-1}\cdot\left\{\int_{\tilde{S}_0^{r-1}} \Pi^r + \int_{E_0^r} \Omega^r\right\}.$$

Let us recall k and $\tilde{k}$ introduced in §2. Putting

$$k(\Sigma^{r-1}\times I) = (-1)^{r-1} D^r \subset \mathfrak{B}^r_\Sigma,$$
$$\tilde{k}(\Sigma^{r-1}\times I) = (-1)^{r-1}\tilde{D}^r \subset \mathfrak{B}^{r-1}_\Sigma,$$

we have

$$p\tilde{D}^r = D^r, \quad \tilde{S}_0^{r-1} - \tilde{S}^{r-1} = \partial\tilde{D}^r.$$

Thus, application of (14) gives

$$(17) \qquad \int_{\tilde{S}_0^{r-1}} \Pi^r - \int_{\tilde{S}^{r-1}} \Pi^r = -\int_{D^r} \Omega^r.$$

It is easy to define a cross-section $*F$: $\Sigma^{r-1}\to\mathfrak{B}^{r-1}_\Sigma$ such that $p*F = pF$ and $c(*F)\cdot\Delta^r = 0$. That is, since E^r is a cell, there exists a cross-section g: $\mathfrak{B}^r \supset E^r \to \mathfrak{B}^{r-1}_{\Delta}$ and we may define $*F$ by $*F = gpF$. In fact $*F$ is extendable over Δ^r: one of its extension is provided by gG; and so $c(*F)\cdot\Delta^r = 0$. Put $gE^r = \tilde{E}^r$ and $*F\Sigma^{r-1} = *\tilde{S}^{r-1}$. Then clearly $*\tilde{S}^{r-1} = \partial\tilde{E}^r$ and $p\tilde{E}^r = E^r$. Let us construct a covering homotopy $*\tilde{k}$ from $*F$ and k in the same way as we have done $\tilde{k}$ from F and k; namely $p*\tilde{k} = p\tilde{k} = k$. If we set $*\tilde{k}_1\Sigma^{r-1} = *\tilde{S}_0^r$, the formula (17) for $*F$ is written

$$\int_{*\tilde{S}_0^{r-1}} \Pi^r - \int_{*\tilde{S}^{r-1}} \Pi^r = -\int_{D^r} \Omega^r;$$

and moreover, according to (14), it is modified as follows:

$$(18) \qquad \int_{*\tilde{S}_0^{r-1}} \Pi^r + \int_{E^r} \Omega^r = -\int_{D^r} \Omega^r.$$

From (17) and (18) it follows that

$$(19) \quad \int_{\tilde{S}_0^{r-1}} \Pi^r + \int_{E_0^r} \Omega^r = \int_{\tilde{S}^{r-1}} \Pi^r + \int_{E^r} \Omega^r + \int_{*\tilde{S}_0^{r-1}} \Pi^r + \int_{E_0^r} \Omega^r.$$

If we apply the operation α_{r-1} to both sides of (19) and use the formula (16) for F and for $*F$, we obtain finally

$$(-1)^r c(F)\cdot\Delta^r = \alpha_{r-1}\cdot\left\{\int_{\tilde{S}^{r-1}} \Pi^r + \int_{E^r} \Omega^r\right\} + (-1)^r c(*F)\cdot\Delta^r.$$

Since $c(*F)\cdot\Delta^r = 0$, the theorem follows. It is noteworthy that this result does not depend upon the choice of G.

It has been known already that the cochain defined by (15) is a cocycle whose cohomology class $\bar{c}_r(R^n)$, called the *Stielfel characteristic class*, does not depend upon the choice of a cross-section F defined on K^{r-1}. Let Z^r be an arbitrary homology class of R^n whose coefficients may be integers or integers mod m, where m is a prime number if r is odd or $r=n$, and $m=2$ if r is even and $r<n$. Then we can easily find the formula which gives $\bar{c}_r(R^n)\cdot Z^r$ in terms of differential forms, by employing an arbitrary cycle chosen to represent Z^r. We abridge it.

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Bibliography

1) Allendoerfer, C. B., *Characteristic cohomology classes in a Riemann manifold.* Ann. of Math., **51** (1950), 551–570.

2) Cartan, É., *La théorie des groupes finis et continus et la géométrie differentielle.* Paris, 1937.

3) Cartan, É., *Leçons sur la géométrie des espaces de Riemann.* Paris, 1928.

4) Chern, S., *The Gauss-Bonnet formula.* Ann. of Math. **45** (1944), 747–752.

5) Chern, S., *On the curvatura integra.* Ann. of Math., **46** (1945), 674–684.

6) Chern, S., *Some new viewpoints in differential geometry in the large.* Bull. Amer. Math. Soc., **52** (1946), 1–30.

7) de Rham, G., and Kodaira, K., *Harmonic integrals.* (mimeographed note) 1950.

8) Steenrod, N., *The topology of fibre bundles.* Princeton Univ. Press, 1951.