MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXVIII, Mathematics No. 2, 1953.

On the non-linear differential equation

By

Taro YOSHIZAWA

(Received June 22, 1953)

---

1. About special forms of non-linear differential equations of the second order, the boundedness of solutions and the existence of a periodic solution have been discussed by various authors; Cartwright, Littlewood, Reuter and others.

Now generalizing the problems, we consider a system of differential equations,

$$(1)\qquad \begin{cases} \dfrac{dx}{dt}=f(t, x, y) \\ \dfrac{dy}{dt}=g(t, x, y) \end{cases},$$

where $f(t, x, y)$ and $g(t, x, y)$ are *continuous* in the domain

$$\Delta_1:\quad 0\leqq t<+\infty,\quad -\infty<x<+\infty,\quad -\infty<y<+\infty.$$

The non-linear differential equation of the second order is a special case of (1).

At first, we shall prove two lemmas in order to discuss the boundedness theorem for the solutions of (1).

Lemma 1. *Let A_1 and B_1 be two positive constants (A_1 and B_1 may be arbitrarily great) and $\mathfrak{A}$ be the domain*

$$|x|<A_1,\quad |y|<B_1.$$

Suppose that there exists a continuous function $\Phi(x, y)$ satisfying the following conditions in the domain

$$\Delta_2:\quad 0\leqq t<+\infty,\ (x, y)\in\mathfrak{A}^c,$$

where $\mathfrak{A}^c$ is the complement of $\mathfrak{A}$ in $[-\infty<x<+\infty,\ -\infty<y<+\infty]$; namely the conditions are that

1° $\Phi(x, y)>0$,

2° *$\Phi(x, y)$ tends to zero uniformly for y and x respectively when $|x|$ or $|y|$ becomes infinity,* 3° *$\Phi(x, y)$ satisfies locally the Lipschitz condition with regard to (x, y) and in the interior of this domain Δ_2, we have*

$$(2) \qquad \varliminf_{h\to 0}\frac{1}{h}\{\Phi(x+hf(t,x,y),\, y+hg(t,x,y))-\Phi(x,y)\}\geqq \varepsilon>0,$$

where ε may be arbitrarily small, but it is a fixed positive number when x and y are bounded.

Then for any solution of (1) $x=x(t)$, $y=y(t)$, *being given two arbitrary positive numbers α and β, if we have $|x(t_0)|\leqq\alpha$, $|y(t_0)|\leqq\beta$ at an arbitrary $t=t_0$, then there exist two positive constants L_1 and M_1 depending only on α and β such that*

$$|x(t)|<L_1, \quad |y(t)|<M_1$$

for $t\geqq t_0$, where, of course, $L_1>\alpha$ and $M_1>\beta$.

Proof. Let us assume $\alpha>A_1$ and $\beta>B_1$, for this case alone is worth to consider. Let $\mathfrak{R}$ be the domain

$$|x|<\alpha, \quad |y|<\beta.$$

Then, by the conditions 1° and 2°, there are two positive numbers L_1 and M_1 such as

$$(3) \qquad \min_{\bar{\mathfrak{R}}-\mathfrak{R}} \Phi(x, y) > \max_{\bar{\mathfrak{R}}^*-\mathfrak{R}^*} \Phi(x, y),$$

where $\mathfrak{R}^*$ denotes the domain $[|x|<L_1 \ |y|<M_1]$ and $\bar{\mathfrak{R}}$ indicates the closure of $\mathfrak{R}$ and so on. Now suppose that some solution of (1) $x=x(t)$, $y=y(t)$ such as $(x(t_0), y(t_0))\in\bar{\mathfrak{R}}$ arrives at the boundary of $\mathfrak{R}^*$, i.e. $\bar{\mathfrak{R}}^*-\mathfrak{R}^*$ when t increases. Then, by the continuity of the solution, it is easy to see that there exist two values of t, say t_1 and t_2, such that $(x(t_1), y(t_1))\in\bar{\mathfrak{R}}-\mathfrak{R}$, $(x(t_2), y(t_2))\in\bar{\mathfrak{R}}^*-\mathfrak{R}^*$ and $(x(t), y(t))\in\mathfrak{R}^*-\bar{\mathfrak{R}}$ for $t_1<t<t_2$. Here consider the function $\Phi(x(t), y(t))$ and then this function is increasing along the solution of (1) by the condition 3°. Hence we have

$$\Phi(x(t_1), y(t_1))<\Phi(x(t_2), y(t_2)).$$

This contradicts the inequality (3). Therefore any solution of (1) having the initial point which belongs to $\bar{\mathfrak{R}}$ cannot arrive at $\bar{\mathfrak{R}}^*-\mathfrak{R}^*$, that is to say, we have two positive constants L_1 and M_1 such as

$$|x(t)|<L_1, \quad |y(t)|<M_1,$$

and now, clearly by (3), these depend only on α, β and are independent of t_0 and the solutions.

Lemma 2. *Under the same assumptions as those in Lemma* 1, *let* $\mathfrak{B}$ *be the domain*

$$|x| < A_2, \ |y| < B_2$$

for two arbitrary positive constants $A_2(>A_1)$ *and* $B_2(>B_1)$.

Then for any solution $(x(t), y(t))$ *such as* $(x(t_0), y(t_0)) \in \overline{\mathfrak{B}} - \overline{\mathfrak{A}}$ *at* $t=t_0$ (t_0 *being arbitrary, but fixed*), *we have*

$$(x(t), \ y(t)) \in \overline{\mathfrak{A}}$$

for some $t(>t_0)$.

Proof. For the solution $(x(t), y(t))$ satisfying

$$(x(t_0), \ y(t_0)) \in \overline{\mathfrak{B}}$$

at $t=t_0$, we have, by Lemma 1, two positive constants L_2 and M_2 independent of particular solutions such that

$$|x(t)| < L_2, \ |y(t)| < M_2$$

when $t \geqq t_0$. Now let $\mathfrak{B}^*$ the domain such as $|x| < L_2$, $|y| < M_2$ and consider the function $\Phi(x, y)e^{-Nt}(N>0)$ in the domain

$$\Delta_3: \quad t_0 \leqq t < +\infty, \ (x, y) \in \overline{\mathfrak{B}}^* - \mathfrak{A}.$$

Then this function satisfies clearly following conditions; namely

1° this is a positive continuous function in Δ_3,

2° this tends to zero uniformly for $(x, y) \in \overline{\mathfrak{B}}^* - \mathfrak{A}$ as $t \to \infty$.

Now for two points (t, x, y) and (t, x', y') in Δ_3, we have

$$\begin{aligned} &|\Phi(x, y)e^{-Nt} - \Phi(x', y')e^{-Nt}| \\ &= e^{-Nt}|\Phi(x, y) - \Phi(x', y')| \leqq C(|x-x'| + |y-y'|), \end{aligned} \tag{4}$$

where C is a suitable positive constant. Therefore it *satisfies the Lipschitz condition* with regard to (x, y).

Moreover we have

$$\begin{aligned} &\varliminf_{h \to 0} \frac{1}{h}\{e^{-N(t+h)}\Phi(x+hf, y+hg) - e^{-Nt}\Phi(x, y)\} \\ &= \varliminf_{h \to 0} \frac{1}{h}\{e^{-N(t+h)}[\Phi(x+hf, y+hg) - \Phi(x, y) + \Phi(x, y) - e^{Nh}\Phi(x, y)]\} \\ &= \varliminf_{h \to 0} \left\{\frac{1}{h}e^{-N(t+h)}[\Phi(x+hf, y+hg) - \Phi(x, y)] - \Phi(x, y)\frac{e^{Nh}-1}{h}e^{-N(t+h)}\right\} \\ &\geqq e^{-Nt}\varliminf_{h \to 0} \frac{1}{h}\{\Phi(x+hf, y+hg) - \Phi(x, y)\} - e^{-Nt}N\Phi(x, y) \end{aligned}$$

$$=e^{-Nt}\left\{\lim_{h\to 0}\frac{\Phi(x+hf,y+hg)-\Phi(x,y)}{h}-N\Phi(x,y)\right\}.$$

$$\geqq e^{-Nt}\{\varepsilon-N\Phi(x,y)\} \qquad \text{(by the condition 3° in Lemma 1).}$$

On the other hand, we can have

$$\varepsilon-N\max_{(x,y)\in\overline{\mathfrak{B}}^*-\mathfrak{A}}\Phi(x,y)\geqq 0$$

by choosing N suitably small, since $\Phi(x,y)$ is positive and continuous in $\overline{\mathfrak{B}}^*-\mathfrak{A}$.

Then the above mentioned inequality becomes always non-negative in the interior of Δ_3. Hence by this fact (4) we have

3° $\Phi(x,y)e^{-Nt}$ is non-decreasing along any solution of (1).

Now suppose that for some solution $(x(t),y(t))$ of (1) such as $(x(t_0),y(t_0))\in\overline{\mathfrak{B}}-\overline{\mathfrak{A}}$ at $t=t_0$, we should not have

$$(x(t),y(t))\in\overline{\mathfrak{A}}$$

for any t. For this solution, consider the function

$$e^{-Nt}\Phi(x(t),y(t)),$$

and then this is a non-decreasing function of t by 3°, while by 2° we have some T such that

$$\min_{(x,y)\in\overline{\mathfrak{B}}-\mathfrak{A}}\Phi(x,y)e^{-Nt_0}>\max_{(x,y)\in\overline{\mathfrak{B}}^*-\mathfrak{A}}\Phi(x,y)e^{-NT}.$$

There arises a contradiction and hence we have $(x(t),y(t))\in\overline{\mathfrak{A}}$ for some t.

2. Directly we have the following boundedness theorem by the above lemmas.

Theorem 1. *Suppose that the same assumptions as those in Lemma 1 hold good. Then all the solutions are ultimately bounded, i.e. there are positive constants A_3 and B_3 (independent of the particular solution considered) such that*

$$|x(t)|<A_3,\quad |y(t)|<B_3 \tag{5}$$

for any solution $(x(t),y(t))$ of (1) satisfying $(x(t_0),y(t_0))\in E^2$ (E^2 being the 2-dimensional Euclidian space) at $t=t_0$ (t_0 being arbitrary, but fixed) and for $t>T_0(\geqq t_0)$ (T_0 depending on the particular solution).

Proof. For the solutions which satisfy $|x| \leqq A_1$, $|y| \leqq B_1$ at $t=t_0$, i.e. start from $\bar{\mathfrak{A}}$, there exist by Lemma 1 two positive constants A_3 and B_3 such as $|x| < A_3$, $|y| < B_3$. Any solution starting from $\mathfrak{A}^c$ enters into $\bar{\mathfrak{A}}$ at some $t=T$ by Lemma 2 (since A_2 and B_2 in Lemma 2 being arbitrary). Then continuing this solution, we have by Lemma 1

$$|x| < A_3 \quad |y| < B_3,$$

that is, all the solutions are ultimately bounded.

Remark 1. Of course, A_3 and B_3 are independent of t_0, but T_0 depends on t_0.

Remark 2. The conditions in Lemma 1 and those in Lemma 2 may be considered *independently*; namely in Lemma 1 we have without using ε in (2)

$$\varlimsup_{h \to 0} \frac{1}{h} \{ \Phi(x+hf, y+hg) - \Phi(x, y) \} \geqq 0,$$

while in Lemma 2 we may assume again the existence of a similar function $\Psi(t, x, y)$.

Remark 3. We can generalize the problems for a general system of differential equations.

3. Next we can easily prove the following existence theorem of a periodic solution of (1) by aid of the above mentioned theorem and Lemma 3 below.

Theorem 2. *Suppose that the same conditions as those in Lemma 1 and the condition for the uniqueness of solutions in Cauchy-problem (Okamura's necessary and sufficient condition*[1]*) hold good. Moreover suppose that*

$$f(t+\omega, x, y) = f(t, x, y)$$

and

$$g(t+\omega, x, y) = g(t, x, y).$$

Then (1) *has at least a periodic solution of period* ω.

This theorem is proved, remarking the fact that by the uniqueness condition the transformation T of the point $P_0(x_0, y_0)$ in the plane $t=0$ into the point $P_1(x_1, y_1)$ on the same solution in the plane $t=\omega$ is *a (1,1) continuous transformation of the plane into*

(1) Okamura; Mem. Coll. Sci. Kyoto Univ. A 24 (1942), p. 22.

itself and the conditions assumed for $f(t, x, y)$, $g(t, x, y)$, by aid of the above boundedness theorem and the following lemma or Massera's theorem[2] about which we shall state some notes later.

Lemma 3.[3] *Let T be a (1,1) continuous transformation of the plane into itself, and let D_0 be a fixed domain and D a domain containing D_0 bounded by a closed Jordan curve J. Suppose that if P is a point of $\bar{D}$, every $T^n(P)$ lies in D_0 for all $n > n_0(P)$.*

Then there is a domain Δ depending on D having the following properties:

1° *Δ is bounded by a closed Jordan curve,*

2° *Δ contains D,*

3° *$T(\bar{\Delta})$ is contained in $\bar{\Delta}$.*

By this lemma *the Brouwer fixed point theorem* can now be applied to $\bar{\Delta}$, and so $\bar{\Delta}$ contains *a fixed point.*

If we *do not consider* the ultimate boundedness and discuss only the existence of a periodic solution, the conditions may be as follows: Namely for three pairs of positive constants (A_4, B_4), (A_5, B_5) and (A_6, B_6) $(A_4 < A_5 < A_6,\ B_4 < B_5 < B_6)$ we indicate the domains

$$|x| < A_i, \quad |y| < B_i \quad (i = 4, 5, 6)$$

by $\mathfrak{D}_1$, $\mathfrak{D}_2$ and $\mathfrak{D}_3$ respectively. And assume *the existence of two non-negative continuous functions $\varphi_1(x, y)$ and $\varphi_2(x, y)$* in the domains

$$0 \leqq t < +\infty, \quad (x, y) \in \bar{\mathfrak{D}}_2 - \mathfrak{D}_1$$

and

$$0 \leqq t < +\infty, \quad (x, y) \in \bar{\mathfrak{D}}_3 - \mathfrak{D}_2,$$

such that

$$\min_{(x,y) \in \bar{\mathfrak{D}}_1 - \mathfrak{D}_1} \varphi_1(x, y) > \max_{(x,y) \in \bar{\mathfrak{D}}_2 - \mathfrak{D}_2} \varphi_1(x, y)$$

and

$$\min_{(x,y) \in \bar{\mathfrak{D}}_2 - \mathfrak{D}_2} \varphi_2(x, y) > \max_{(x,y) \in \bar{\mathfrak{D}}_3 - \mathfrak{D}_3} \varphi_2(x, y)$$

respectively, and they *satisfy the Lipschitz condition* with regard to (x, y) and finally we have

---

(2) Wendel; Ann. Math. Stud. no 20. (Princeton, 1950), p. 226 or Massera; Bull. Amer. Math. Soc. Vol. 54 (1948), p. 636.

(3) Cartwright; Ann. Math. Stud. no. 20 (Princeton, 1950), p. 174.

$$\underline{\lim}_{h\to 0}\frac{1}{h}\{\varphi_i(x+hf, y+hg)-\varphi_i(x,y)\}\geqq 0 \qquad (i=1,2)$$

in the interior of each domain.

Moreover assume *the existence of a positive continuous function* $\psi(t,x,y)$ in the domain

$$\Delta^*: 0\leqq t<+\infty,\ (x,y)\in\bar{\mathfrak{D}}_3-\mathfrak{D}_1,$$

where $\psi(t,x,y)$ *converges to zero uniformly as* $t\to\infty$ and *satisfies the Lipschitz condition* with regard to (x,y) and finally in the interior of Δ^* *satisfies the inequality*

$$\underline{\lim}_{h\to 0}\frac{1}{h}\{\psi(t+h, x+hf, y+hg)-\psi(t,x,y)\}\geqq 0.$$

Of course, we *require the other conditions* in Theorem 2. In this case the period is $n\omega$, where n is a certain positive integer.

4. In Massera's theorem, $f(t,x,y)$ and $g(t,x,y)$ *do not appear explicitly* in the conditions and *properties of solutions themselves* are woven into conditions. It is the part: " *if no solution of* (1) *tends to infinity in a finite time and if* (1) *has a solution* $(x(t), y(t))$ *which is bounded for* $t\geqq t_0$ *and so on* " The first half is the condition for the possibility of the continuation of solutions. About it the late Prof. Okamura has *already* obtained the necessary and sufficient condition in *Functional Equations* (in Japanese) Vol. 32 (1942). Namely *consider a system of differential equations*

$$(6)\qquad \begin{cases}\dfrac{dx}{dt}=f(t,x,y)\\[2mm] \dfrac{dy}{dt}=g(t,x,y),\end{cases}$$

where $f(t,x,y)$ *and* $g(t,x,y)$ *are continuous in the domain*

$$a\leqq t\leqq b,\ -\infty<x<+\infty,\ -\infty<y<+\infty.$$

Then in order that all solutions of (6) *are continuable to the right until they arrive at* $t=b$, *it is necessary aud sufficient that there exists a function* $\varphi(t,x,y)$ *satisfying the following conditions*; *namely*

1° $\varphi(t,x,y)$ *is a continuous function with the continuous first partial derivatives in*

$$a\leqq t\leqq b,\ r=\sqrt{x^2+y^2}\geqq r_0,$$

2° $\varphi(t_,, x, y)>0$ *and* $\varphi(t,x,y)$ *tends to zero uniformly for*

$a \leqq t \leqq b$ *as* $r \to +\infty$,

3° *we have always*

$$\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial x}f(t,x,y)+\frac{\partial\varphi}{\partial y}g(t,x,y)\geqq 0. \tag{7}$$

Therefore concerning with the equation (1), we may consider in $0 \leqq t \leqq T$ and there may be *such a function* $\varphi(t,x,y)$ *as the above mentioned for every* T. But when we will apply to an individual equation as the sufficient condition, it is convenient to assume instead of (7) as follows; namelv $\varphi(t,x,y)$ *satisfies locally the Lipschitz condition* with regard to (x,y) add *we have in the interior of the domain*

$$\varlimsup_{h\to 0}\frac{1}{h}\{\varphi(t+h,x+hf,y+hg)-\varphi(t,x,y)\}\geqq 0. \tag{8}$$

Also for the second half, as a sufficient condition, we assume the existence of a function such as in the lemmas, and then we shall be able to see that solutions starting from a suitable domain are bounded.

5. Example. Reuter's boundedness theorem (*The Journal of the London Mathematical Society, Vol. 27 (1952)*).

Consider the equation

$$\ddot{x}+F(\dot{x})+g(x)=p(t), \tag{9}$$

where $g(x)$ is continuous and

$$g(x)\ \mathrm{sgn}\ x\to\infty \quad \text{as } |x|\to\infty.$$

If $F(y)$ is continuous and $F(y)$ *sgn* $y\to\infty$ as $|y|\to\infty$ and $p(t)$ is continuous and bounded, the solutions of (9) *satisfy ultimately*

$$|x(t)|<A_7,\ |\dot{x}(t)|<B_7,$$

where A_7 and B_7 are independent of the particular solution considered.

In this case, we have instead of (9) only to consider the system

$$\dot{x}=y,\ \dot{y}=-F(y)-g(x)+p(t). \tag{10}$$

Then choosing sufficiently great positive numbers a and b suitably, we may define the function $\Phi(x,y)$ as follows; namely

$$\Phi(x, y)=\begin{cases} e^{u(x,y)} & (-\infty<x<+\infty,\ y\geqq b) \\ e^{u(x,y)-y+b} & (x\geqq a,\ |y|\leqq b) \\ e^{u(x,y)+2b} & (x\geqq a,\ y\leqq -b) \\ e^{u(x,y)+\frac{2b}{a}(x+a)-2b} & (|x|\leqq a,\ y\leqq -b) \\ e^{u(x,y)-2b} & (x\leqq -a,\ y\leqq -b) \\ e^{u(x,y)+y-b} & (x\leqq -a,\ |y|\leqq b), \end{cases}$$

where $u(x, y)=-\dfrac{y^2}{2}-G(x)$ and $G(x)=\int_0^x g(x)dx.$

Finally, I wish again to express my sincere thanks to Prof. Toshizo Matsumoto to whom I owe a great debt for his guidance in my researches, at the time of his retirement from the professor under the age-limit system.