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On the differential equation of Carathéodory's type

By

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In this paper we shall study the differential equation of Carathéodory's type:

$$\frac{dy}{dx}=f(x, y)$$

where $f(x, y)$ is defined in the strip

$$S: \quad a\leqq x\leqq b, \quad -\infty<y<+\infty,$$

measurable with respect to x, continuous with respect to y and dominated in S by a summable function $k(x)$ of x alone. It is known[1] that the above equation has solutions in the interval

$$I: \quad a\leqq x\leqq b,$$

in the following sense: there exist in I absolutely continuous functions $\varphi(x)$ such that $\varphi(x)=\varphi(a)+\int_a^x f(t,\varphi(t))dt$ in I, and then $\varphi'(x)=f(x, \varphi(x))$ in a measurable subset of I, having the same measure as I and depending upon the particular solution $\varphi(x)$.

Recently Prof. G. Scorza Dragoni[2] has proved that we can determine a measurable subset E of I such that E is of the same measure as I and every absolutely continuous solution satisfies the differential equation on the set E.

The purpose of this paper is to give a simple proof of this theorem.

§ 1. Let $g(x)$ be a measurable function of x in I such that $|g(x)|\leqq h(x)$, $h(x)$ being summable in I. Moreover, suppose that $g(x)$ is continuous on each of sets e_1, e_2......, closed and satisfying $\lim_{n\to\infty} m(e_n)=b-a$. Let e'_n denote the set of the points of e_n of density 1 for e_n and e the union of all $e'_n (n=1, 2,\cdots\cdots)$. Then e is a measurable set of measure $b-a$, and $g(x)$ is approximately continuous[3] at all points of e.

Put for $x \in I$

$$g^+(x)=\begin{cases} g(x) & \text{if } g(x) \geqq 0, \\ 0 & \text{if } g(x) < 0, \end{cases}$$

$$g_n(x)=\begin{cases} g^+(x) & \text{if } g^+(x) \leqq n, \\ n & \text{if } g^+(x) > n, \end{cases}$$

$$(n=1,\ 2,\cdots\cdots).$$

Then we have in I $0 \leqq g_1(x) \leqq g_2(x) \leqq \cdots \leqq g_n(x) \leqq \cdots\cdots$ and $\lim_{n\to\infty} g_n(x) = g^+(x)$. Each of the functions $g^+(x)$ and $g_n(x)\,(n=1, 2,\cdots\cdots)$ is approximately continuous at all points of e, and moreover each $g_n(x)$ is bounded on e, so that we have[4]

$$\frac{d}{dx}\int_a^x g_n(t)dt=g_n(x) \quad \text{for } x \in e.$$

With regard to the function $h(x)$, also consider an analogous sequence of functions $h_n(x)\,(n=1\ 2,\cdots\cdots)$ such that for $x \in I$

$$h_n(x)=\begin{cases} h(x) & \text{if } h(x) \leqq n, \\ n & \text{if } h(x) > n, \end{cases}$$

$$(n=1,\ 2,\cdots\cdots).$$

Then, it follows that in I $0 \leqq h_1(x) \leqq h_2(x) \leqq \cdots \leqq h_n(x) \leqq \cdots\cdots\cdots$, $\lim_{n\to\infty} h_n(x)=h(x)$ and $0 \leqq g_n(x) \leqq h_n(x)$ $(n=1,\ 2,\cdots\cdots)$. If we write

$$H_n(x)=\int_a^x \{h(t)-h_n(t)\}\,dt,$$

the derivatives $H_n'(x)\,(n=1,\ 2,\cdots\cdots)$ exist in a measurable subset j of I which has the same measure as I. And evidently we have $H_1'(x) \geqq H_2'(x) \geqq \cdots \geqq H_n'(x) \geqq \cdots$ and $\lim_{n\to\infty} H_n'(x)=0$ at all points of j. Moreover, write

$$G_n(x)=\int_a^x \{g^+(t)-g_n(t)\}\,dt.$$

Since $h(x)-h_n(x) \geqq g^+(x)-g_n(x) \geqq 0$ for any point of I, we obtain

$$H_n'(x) \geqq \overline{D}\ G_n(x) \geqq 0^{5)} \qquad \text{for } x \in j.$$

Hence we have

$$\lim_{n\to\infty} \overline{D}\ G_n(x)=0 \qquad \text{for } x \in j.$$

If we denote the common part of the sets e and j by k, k is a measurable set and of the same measure as I. Since

$$\frac{d}{dx}\int_a^x g_n(t)dt=g_n(x) \qquad \text{for } x \in k,$$

we have

$$\overline{D}\ G_n(x)=\overline{D}\int_a^x g^+(t)\,dt-\frac{d}{dx}\int_a^x g_n(t)dt$$
$$=\overline{D}\int_a^x g^+\ (t)dt-g_n(x).$$

Now, since $\lim_{n\to\infty} \overline{D}\ G_n(x)=0$, and $\lim_{n\to\infty} g_n(x)=g^+(x)$ for all $x \in k$, we have

$$\overline{D}\int_a^x g^+(t)dt=g^+(x) \qquad \text{for all } x \in k.$$

Since $0\leqq \underline{D}\ G_n(x)\leqq \overline{D}\ G_n(x)$ for any n, we have $\lim_{n\to\infty} \underline{D}\ G_n(x)=0$ for all $x \in k$, that is $\underline{D}\int_a^x g^+(t)dt=g^+(x)$ for all $x \in k$. Therefore we can conclude that

$$\frac{d}{dx}\int_a^x g^+(t)dt=g^+(x) \qquad \text{for all } x \in k.$$

Finally, put

$$g^-(x)=\begin{cases} -g(x) & \text{if } g(x)\leqq 0, \\ 0 & \text{if } g(x)>0. \end{cases}$$

Then we have

$$\frac{d}{dx}\int_a^x g^-(t)dt=g^-(x) \qquad \text{for all } x \in k.$$

Since $g(x)=g^+(x)-g^-(x)$, we have

$$\frac{d}{dx}\int_a^x g(t)dt=g(x) \qquad \text{for all } x \in k.$$

§ 2. For the second member $f(x, y)$ of the differential equation considered in this paper, we can choose such a sequence[6] of closed subsets $e_1, e_2, \cdots\cdots$ of I, that $\lim_{n\to\infty} m(e_n)=b-a$ and $f(x, \varphi(x))$ is continuous on e_n, if $\varphi(x)$ is a continuous function x in I. Therefore, if we regard $f(x, \varphi(x))$ and $k(x)$ as respectively $g(x)$ and $h(x)$ in § 1, we obtain the following conclusion: *for the given* $f(x, y)$, *we can choose a measurable subset* E $(m(E)=m(I)=b-a)$ *of* I *such that, given a continuous function* $\varphi(x)$, *we have*

$$\frac{d}{dx}\int_a^x f(t, \varphi(t))dt=f(x, \varphi(x)) \qquad \text{for } x \in E.$$

Moreover, if $\varphi(x)$ *is a solution of the differential equation* $\frac{dy}{dx}=f(x, y)$, *then we have*

$$\varphi(x)=\varphi(a)+\int_a^x f(t, \varphi(t))dt,$$

and therefore for all $x \in E$

$$\varphi'(x)=f(x, \varphi(x)).$$

It is clear that the same proposition can be concluded for the system of differential equations

$$\frac{dy_i}{dx}=f_i(x, y_1, y_2, \cdots\cdots, y_n) \quad (i=1, 2, \cdots\cdots, n).$$

Remark: This consideration completes our previous paper.[7]

References

1) Carathéodory; *Vorlesungen über reelle Funktionen*, 2te Anf., pp. 665-672.

2) G. Scorza Dragoni; *Una applicazione della quasicontinuità semiregolare della funzioni misurabili rispetto ad una e continue rispetto ad un'altra variabile*, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12 (1952), pp. 55-61.

3) S. Saks; *Theory of Integral*, 2nd ed. p. 132, (10.6).

4) S. Saks; *loc. cit.*, p. 132, (10.7).

5) $\bar{D}\, G_n(x)$ means the upper derivative of $G_n(x)$.

6) G. Scorza Dragoni; *Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile*, Rend. Padova. Vol. 17 (1948), pp. 102-106; *loc. cit.* 2), p. 57.

7) Hayashi-Yoshizawa; *New treatise of solutions of a system of ordinary differential equations and its application to the uniqueness theorems*, these Memoirs **26** (1951), pp.225-233.

Correction

In our previous paper (*loc. cit.* 7)), after "*hyperplane*" (p. 228, line 10) insert "*defined*" and for "*for almost every point* (x, y) *in* G," (p. 233, lines 3-4) read "*for almost every* x *in* $0 \leqq x \leqq a$,".