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Some remarks on local rings, II

By

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This paper contains some applications of the theory of Henselian rings to the theory of local rings. Main purpose of the present paper is to prove the following two assertions:

I) If an integrally closed local integrity domain $\mathfrak{o}$ is of finitely generated type over a valuation ring of characteristic 0 or over a field of arbitrary characteristic[1], then the completion of $\mathfrak{o}$ is an integrally closed integrity domain.

II) If an integrally closed local integrity domain $\mathfrak{o}$ is of finite type over a regular local ring $\mathfrak{r}$[1] then the completion of $\mathfrak{o}$ is an integrity domain, provided that $\mathfrak{r}$ contains a complete Noetherian local ring $\mathfrak{s}$ such that $\mathfrak{r}$ is a quotient ring of $\mathfrak{s}$ with respect to a prime ideal.

These results generalize and supplement a result (Theorem 5) in my previous paper[2]. By the way, we add a proof of the following:

Let C be the complex number field and let $\mathfrak{o}_n$ and $\bar{\mathfrak{o}}_n$ be the rings of convergent and formal power series respectively in n variables $z_1, \cdots, z_n$ over C. Then if $\mathfrak{p}$ is a prime ideal of $\mathfrak{o}_n$ then $\mathfrak{p}\bar{\mathfrak{o}}_n$ is also a prime ideal.

As for terminology, see my other papers on Henselian rings[3].

§ 1. Henselian regular local rings.

THEOREM 1. *Let* $\mathfrak{r}$ *be a Henselian regular local ring and let* $\mathfrak{m}$ *be*

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1) As for definition, see § 3 below.

2) Some remarks on local rings, to appear in Nagoya Math. J., 6, which will be referred as [L.R.] in the present paper.

3) On the theory of Henselian rings, Nagoya Math. J., 5, pp. 45–57: On the theory of Henselian rings II, to appear in Nagoya Math. J., They will be referred as [H.R. I] and [H.R. II] respectively in the present paper.

its maximal ideal. Assume that $\mathfrak{r}$ *satisfies the following two conditions*:

1) *If* $\mathfrak{p}$ *is a minimal prime ideal of* $\mathfrak{r}$, *then any finite integral extension of* $\mathfrak{r}/\mathfrak{p}$ *is algebraically closed in its completion.*

2) *Any almost finite integral extension of* $\mathfrak{r}$ *is a finite* $\mathfrak{r}$*-module.*

Then $\mathfrak{r}$ *is algebraically closed in its completion* $\bar{\mathfrak{r}}$.

LEMMA A. If an element b of $\bar{\mathfrak{r}}$ is purely inseparable over $\mathfrak{r}$, then $b \in \mathfrak{r}$.

Proof. Let p be the characteristic of $\mathfrak{r}$. We may assume that $b^p \in \mathfrak{m}$ and that b^p has no p-hold divisor in $\mathfrak{r}$. Let c be a prime divisor of b^p in $\mathfrak{r}$. Then the valuation determined by a prime divisor of $c\bar{\mathfrak{r}}$ ramifies in $R(b)$ with respect to R, where R is the quotient field of $\mathfrak{r}$. This contradicts to our assumption that $c\bar{\mathfrak{r}}$ is a prime ideal.

Now let $\mathfrak{o}$ be the totality of separably algebraic elements over $\mathfrak{o}$ among elements of $\bar{\mathfrak{r}}$. Then by Lemma A we have only to show that $\mathfrak{o}=\mathfrak{r}$. Let L be the quotient field of $\mathfrak{o}$. Then it is evident that $\mathfrak{o}=L \cap \bar{\mathfrak{r}}$.

LEMMA B. If y is an element of $\mathfrak{o}$, then $y\bar{\mathfrak{r}} \cap \mathfrak{o}=y\mathfrak{o}$.

Proof is easy since $\mathfrak{o}=L \cap \bar{\mathfrak{r}}$.

Corollary. If $y \in \mathfrak{m}$ and $y \notin \mathfrak{m}^2$ then $y\mathfrak{o}$ is a prime ideal.

We take an element z of $\mathfrak{m}$ which is not in $\mathfrak{m}^2$.

LEMMA C. Every element b of $\mathfrak{o}$ can be expressed in the following form:

$$b=b_0+b_1z+\cdots+b_{k-1}z^{k-1}+b'_kz^k, \quad b_0, \cdots, b_{k-1} \in \mathfrak{r}, \quad b'_k \in \mathfrak{o}.$$

Proof. Let $f(x)$ be the irreducible primitive polynomial over $\mathfrak{r}$ satisfied by b. Then $b \bmod z\bar{\mathfrak{r}}$ satisfies $f(x) \bmod z\mathfrak{r}[x]$. Since $f(x)$ is primitive, $f(x) \notin z\mathfrak{r}[x]$. Therefore $b \bmod z\bar{\mathfrak{r}}$ is algebraic over $\mathfrak{r}/z\mathfrak{r}$, whence $b \equiv b_0 \pmod{z\bar{\mathfrak{r}}}$ $(b_0 \in \mathfrak{r})$. Therefore we can write $b=b_0+b'_1z$ $(b'_1 \in \bar{\mathfrak{r}})$. Since $b, b_0, z \in \mathfrak{o}$, we have $b'_1 \in L$, hence $b'_1 \in \mathfrak{o}$. When we apply this for b'_1, we have our assertion when $k=2$ and repeating the similar method we have our assertion.

Corollary. $\mathfrak{o}/z^k\mathfrak{o}=\mathfrak{r}/z^k\mathfrak{r}$ for every $k=1, 2, \cdots$.

LEMMA D. $\mathfrak{o}$ is a dense subspace of $\bar{\mathfrak{r}}$.

Proof. We denote by $\mathfrak{q}$ the maxial ideal of $\mathfrak{o}$. We have only to prove that $\mathfrak{m}^k\bar{\mathfrak{r}} \cap \mathfrak{o} \subseteq \mathfrak{q}^k$ and it follows easily from Lemma C.

LEMMA E. If $\mathfrak{q}$ is a proper prime ideal of $\mathfrak{o}$, then $\mathfrak{q} \cap \mathfrak{r} \neq (0)$.

Proof. $\mathfrak{q}$ contains a non-zero element a. Let $a_0a^r+a_1a^{r-1}+\cdots+a_r=0$ $(a_i \in \mathfrak{r},\ a_r \neq 0)$. Then $a_r \in \mathfrak{q} \cap \mathfrak{r}$.

Now we assume that $\mathfrak{o} \neq \mathfrak{r}$ and let a_0 be an element of $\mathfrak{o}$ such that $a_0 \notin \mathfrak{r}$ and a_0 is integral over $\mathfrak{r}$. Let $\tilde{\mathfrak{o}}$ be the integral closure of $\mathfrak{r}[a_0]$ in $R(a_0)$. Then $\tilde{\mathfrak{o}}$ is a finite $\mathfrak{r}$-module: $\tilde{\mathfrak{o}} = \mathfrak{r}[a_0, a_1, \cdots, a_s]$. Let $g(x)$ be the irreducible monic polynomial over $\mathfrak{r}$ satisfied by a_0 and let d_0 be its discriminant. We set $r = [R(a_0) : R]$.

For the convenience, we denote by $a'_0, \cdots, a'_s$ elements $a_0, \cdots, a_s$ if they are considered as elements of the local ring $\tilde{\mathfrak{o}}$.

LEMMA F. $\mathfrak{o}'' = \mathfrak{o}[a'_0, a'_1, \cdots, a'_s]$ contains $\tilde{\mathfrak{o}}$ as a dense subspace.

Proof. It is evident that $d_0 \mathfrak{o}'' \subseteq \mathfrak{o}[a'_0]$ and that $d_0 \tilde{\mathfrak{o}} \subseteq \mathfrak{r}[a'_0]$. Since d_0 is not a zero divisor in the completion $\bar{\tilde{\mathfrak{o}}}$ of $\tilde{\mathfrak{o}}$, $\{\mathfrak{m}^k \tilde{\mathfrak{o}} : d_0 \tilde{\mathfrak{o}}\ (k = 1, 2, \cdots)\}$ forms a system of neighbourhoods of zero in $\tilde{\mathfrak{o}}$[4]. That is, there exists a natural number $n(k)$ for each k such that $\mathfrak{m}^{n(k)} \tilde{\mathfrak{o}} : d_0 \tilde{\mathfrak{o}} \subseteq \mathfrak{m}^k \tilde{\mathfrak{o}}$. If b is in $\tilde{\mathfrak{o}} \cap \mathfrak{p}^{n(k)} \mathfrak{o}''$, bd_0 is in $\mathfrak{p}^{n(k)} \mathfrak{o}[a'_0] \cap \tilde{\mathfrak{o}} = \mathfrak{m}^{n(k)} \tilde{\mathfrak{o}}$. Therefore $b \in \mathfrak{m}^k \tilde{\mathfrak{o}}$. This show that $\tilde{\mathfrak{o}}$ is a subspace of $\mathfrak{o}''$. That $\tilde{\mathfrak{o}}$ is dense in $\mathfrak{o}''$ follows easily.

LEMMA G. If $\mathfrak{q}''$ is a prime divisor of zero in $\mathfrak{o}''$, then there exists only one prime divisor $\bar{\mathfrak{q}}$ of zero in $\bar{\mathfrak{o}}$ such that $\bar{\mathfrak{q}} \supseteq \mathfrak{q}'' \bar{\mathfrak{o}}$, where $\bar{\mathfrak{o}}$ denotes the completion of $\mathfrak{o}''$.

Proof. Let $\bar{\mathfrak{J}}$ be the completion of $\mathfrak{r}[a'_0]$. Then it is evident that $\bar{\mathfrak{o}}$ is a subring of the total quotient ring of $\bar{\mathfrak{J}}$. Therefore we have only to prove the following.

LEMMA H. If $\mathfrak{q}'$ is a prime divisor of zero in $\mathfrak{o}[a'_0]$, then there exists only one prime divisor $\bar{\mathfrak{q}}'$ of zero in $\bar{\mathfrak{J}}$ such that $\bar{\mathfrak{q}}' \cap \mathfrak{o}[a'_0] = \mathfrak{q}'$.

Proof. Let $g(x) = f_1(x) \cdots f_n(x)$ with irreducible monic polynomials $f_i(x)$ over $\mathfrak{o}$. Then since $\mathfrak{o}$ is separably algebraically closed in $\mathfrak{r}$, we see that $f_i(x)$ are irreducible also over $\bar{\mathfrak{r}}$. This proves our assertion.

LEMMA I. Let the prime divisors of $z\mathfrak{o}''$ be $\mathfrak{p}_1, \cdots, \mathfrak{p}_t$. Let S be the complementary set of $z\mathfrak{r}$ with respect to $\mathfrak{r}$. Then $\mathfrak{o}''_S$ is a principal ideal ring and whose maximal ideals are $\mathfrak{p}_1 \mathfrak{o}''_S, \cdots, \mathfrak{p}_t \mathfrak{o}''_S$.

Proof. Let $\mathfrak{p}^*$ be a maximal ideal of $\mathfrak{o}''_S$. Set $\mathfrak{p}' = \mathfrak{p}^* \cap \mathfrak{o}$, $\mathfrak{p}'' = \mathfrak{p}^* \cap \mathfrak{o}''$ and $\tilde{\mathfrak{p}} = \mathfrak{p}^* \cap \tilde{\mathfrak{o}}$. By the condition 1) in our theorem and since $\mathfrak{o}''/\mathfrak{p}_i$ is a finite integral extension of $\mathfrak{r}/z\mathfrak{r}$, we see that $\bar{\mathfrak{p}}_i = \mathfrak{p}_i \bar{\mathfrak{o}}$ is a prime ideal. Further every prime divisor of zero in $\bar{\mathfrak{o}}$ is con-

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4) C. Chevalley, On the theory of local rings, Ann. of Math., 44 (1943), § II, Lemma 9.

tained in one of $\mathfrak{p}_i$[5]. Now if $\mathfrak{p}'=(0)$, then we see that $\mathfrak{p}''$ is a prime divisor of zero. Therefore $\mathfrak{p}''$ is contained in one of $\mathfrak{p}''_i=\bar{\mathfrak{p}}_i \cap \mathfrak{o}''$, which is a contradiction to our assumption that $\mathfrak{p}^*$ is maximal. Therefore $\mathfrak{p}''$ is one of $\mathfrak{p}''_i$. This shows $\mathfrak{o}''_S$ has no maximal ideal other than $\mathfrak{p}''_i\mathfrak{o}''_S$. Let S'' be the intersection of complementary sets of $\mathfrak{p}''_1, \cdots, \mathfrak{p}''_t$ with respect to $\mathfrak{o}''$. Then $\mathfrak{o}''_S=\mathfrak{o}''_{S''}$. Let w_i be an element of $\mathfrak{p}_i$ which is not in $\mathfrak{p}_i^{(2)}$ and not in every $\mathfrak{p}_j$ other than $\mathfrak{p}_i$ and take an element b_i of $w_i\bar{\mathfrak{o}} : \mathfrak{p}_i$ which is in none of $\mathfrak{p}_j$. Then since $w_i\mathfrak{o}'' \supseteq b_i\mathfrak{p}_i\mathfrak{o}''$ and $b_i \in S$, we see that $\mathfrak{p}_i\mathfrak{o}''_S=w_i\mathfrak{o}''_S$. Therefore we have only to show that $\mathfrak{p}_i\mathfrak{o}''=\mathfrak{p}''_i$ but this follows easily from that $\mathfrak{p}_i\bar{\mathfrak{o}}=\mathfrak{p}''_i\bar{\mathfrak{o}}=\bar{\mathfrak{p}}_i$ and from that $\mathfrak{o}''/z\mathfrak{o}''$ is Noetherian by the corollary to Lemma C.

Now by Lemma I, we see that $\mathfrak{o}''_S$ is a direct sum of principal ideal rings $\mathfrak{o}_1, \cdots, \mathfrak{o}_v$, where v is the number of prime divisors of zero in $\mathfrak{o}''$. Let e_i be the identity of $\mathfrak{o}_i$. Then we can write $e_i= a_i/d_i$ $(a_i \in \mathfrak{o}'', d_i \in \mathfrak{r})$. Set $d=d_0^2\Pi_{i=1}^{v} d_i$ and let $\bar{\bar{\mathfrak{o}}}$ be the integral closure of $\mathfrak{o}$ in its total quotient ring. Then

LEMMA J. $d\bar{\bar{\mathfrak{o}}} \subseteq \bar{\mathfrak{r}}[a'_0]$.

Proof. By Lemma G, we see that $\bar{\bar{\mathfrak{o}}}e_1, \cdots, \bar{\bar{\mathfrak{o}}}e_v$ are integrity domains. Therefore $d_0\mathfrak{o}e_i \subseteq \bar{\mathfrak{r}}[a'_0]e_i$. Therefore $d_0^2 d_i\bar{\bar{\mathfrak{o}}}e_i \subseteq \mathfrak{r}[a'_0]$, which proves our assertion.

Now, since d is in $\mathfrak{r}$, by condition 2) in our theorem and by Lemma 6 in [L.R.][6], we see that $\mathfrak{o}$ is an integrity domain, which contradicts to that $\mathfrak{o}''$ is not a integrity domain. Thus $\mathfrak{o}$ must coincides with $\mathfrak{r}$.

§ 2. Weierstrassean rings.

Definition. A Henselian regular local ring $\mathfrak{r}$ is called a Weierstrassean ring if there exists a set of classes of Henselian regular local rings $G_0, G_1, \cdots, G_n, \cdots$, satisfying the following conditions:

1) $\mathfrak{r}$ is contained in one of G_n.

2) Every member of G_n is of dimension n.

3) If $\mathfrak{p}$ is a minimal prime ideal of a member $\mathfrak{r}_n$ of G_n, then there exists a member $\mathfrak{r}_{n-1}$ of G_{n-1} such that $\mathfrak{r}_n/\mathfrak{p}$ is isomorphic to a finite integral extension of $\mathfrak{r}_{n-1}$, provided that $n \geq 1$.

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5) See the proof of Lemma 5 in [L.R.] or the same in Zariski's paper "Analytical irreducibility of normal varieties", Ann. of Math., 49 (1948).

6) Cf. O. Zariski, Sur la normalité analytique des variété normales, Ann. Inst. Fourier, 2 (1950).

4) If $\mathfrak{r}_n$ is a member of G_n, then every almost finite integral extension of $\mathfrak{r}_n$ is a finite $\mathfrak{r}_n$-module.

THEOREM 2. *If* $\mathfrak{r}$ *is a Weierstrassean ring, then* $\mathfrak{r}$ *is algebraically closed in its completion* $\bar{\mathfrak{r}}$. *Further if* $\mathfrak{p}$ *is a prime ideal of* $\mathfrak{r}$ *then* $\mathfrak{p}\bar{\mathfrak{r}}$ *is a prime ideal.*

Proof. This is an immediate consequence of Theorem 1 if we observe that $\mathfrak{r}/\mathfrak{p}$ is a finite integral extension of a Weierstrassean ring[7].

Corollary 1. Let a local integrity domain $\mathfrak{o}$ be a finite integral extension of a Weierstrassean ring $\mathfrak{r}$. Then the completion $\bar{\mathfrak{o}}$ of $\mathfrak{o}$ is an integrity domain. If moreover $\mathfrak{o}$ is integrally closed, then $\bar{\mathfrak{o}}$ is integrally closed, providéd that $\mathfrak{o}$ is separable over $\mathfrak{r}$.

Proof. The first half is evident. The last half can be proved by the same way as in the proof of Theorem 5 in [L.R.][8].

Corollary 2. Let $\mathfrak{o}$ be a local ring such that 1) $\mathfrak{o}$ contains a Weierstrassean ring $\mathfrak{r}$ and is a finite $\mathfrak{r}$-module and 2) the zero ideal of $\mathfrak{o}$ is a primary ideal. Then the zero ideal of the completion $\bar{\mathfrak{o}}$ is a primary ideal.

Proof. Let $\mathfrak{p}$ be the prime divisor of zero in $\mathfrak{o}$. Then by Theorem 2 we see that $\mathfrak{p}\bar{\mathfrak{o}}$ is a prime ideal. Therefore $\mathfrak{p}\bar{\mathfrak{o}}$ is the radial of $\bar{\mathfrak{o}}$. Now since $\mathfrak{r}$ has no zero divisor in $\mathfrak{o}$, we see that the completion $\bar{\mathfrak{r}}$ of $\mathfrak{r}$ has no zero divisor in $\bar{\mathfrak{o}}$ and $\bar{\mathfrak{o}}$ is a finite $\bar{\mathfrak{r}}$-module. Therefore the zero ideal of $\bar{\mathfrak{o}}$ has no imbedded prime divisor. This proves our assertion.

§ 3. Examples (I).

Definition. A local ring $\mathfrak{o}$ is said to be of finitely generated type over a local ring $\mathfrak{r}$ if 1) there exists a finitely generated ring $\mathfrak{J}$ over $\mathfrak{r}$, 2) $\mathfrak{o}$ is a quotient ring of $\mathfrak{J}$ with respect to a maximal ideal $\mathfrak{m}$ of $\mathfrak{J}$ and 3) $\mathfrak{r} \subseteq \mathfrak{o}$ and $\mathfrak{m}$ contains every non-unit of $\mathfrak{r}$. This $\mathfrak{J}$ is called the generating ring of $\mathfrak{o}$ over $\mathfrak{r}$.

Definition. A local ring $\mathfrak{o}$ is said to be of finite type over a local ring $\mathfrak{r}$ if 1) $\mathfrak{o}$ is of finitely generated type over $\mathfrak{r}$ and 2) its generating ring $\mathfrak{J}$ over $\mathfrak{r}$ can be chosen so that $\mathfrak{J}$ is integral over $\mathfrak{r}$.

THEOREM 3. *Let* $\mathfrak{r}$ *be a regular local ring of finitely generated*

7) Cf. Corollary 4 to Lemma 3 in [H.R. II].

8) The proof was not correct because we did not prove that the completion of $\mathfrak{o}$ is an integrity domain. But here we proved this last fact, therefore we can use its proof. In the following sections we see that the theorem is true.

As for the method of the proof, cf, Zariski, l.c, note 6).

type over a field and let $\mathfrak{r}^*$ *be its Henselization. Then* $\mathfrak{r}^*$ *is a Weierstrassean ring.*

Proof. For condition 3) in the definition of Weierstrassean ring, observe that if $\mathfrak{p}^*$ is a proper ideal of $\mathfrak{r}^*$ then $\mathfrak{p}^* \cap \mathfrak{r} \neq (0)$. Condition 4) follows easily from the well known theorem of F. K. Schmidt[9].

Corollary 1. Assume that an integrally closed local integrity domain $\mathfrak{o}$ is of finitely generated type over a field K, then the completion $\bar{\mathfrak{o}}$ of $\mathfrak{o}$ is an integrally closed integrity domain.

Proof. It is evident that $\mathfrak{o}$ is of finite type over a regular local ring $\mathfrak{r}$ which is of finitely generated type over K. Then the Henselization $\mathfrak{o}^*$ of $\mathfrak{o}$ is an integrally closed finite integral extension of the Henselization $\mathfrak{r}^*$ of $\mathfrak{r}$. Therefore the completion of $\mathfrak{o}^*$ is an integrally closed integrity domain, by Theorem 5 in [L.R.]. Since $\mathfrak{o}^*$ contains $\mathfrak{o}$ as a dense subspace[10], we see our statement.

Corollary 2. Let a local integrity domain $\mathfrak{o}$ be of finitely generated type over a field K and let $\tilde{\mathfrak{o}}$ be the integral closure of $\mathfrak{o}$ in its quotient field. Then the completion of $\tilde{\mathfrak{o}}$ is the integral closure of the completion $\bar{\mathfrak{o}}$ of $\mathfrak{o}$ in its total quotient ring. Therefore the number of maximal ideals of $\tilde{\mathfrak{o}}$ is equal to the number of prime divisors of zero in $\bar{\mathfrak{o}}$.

4. Examples (II).

THEOREM 4. *Let* $\mathfrak{r}$ *be a regular local ring which satisfies the following conditions:*

1) $\mathfrak{r}$ *contains a complete Noetherian local integrity domain* $\mathfrak{o}$.

2) $\mathfrak{r}$ *is of finite type over a quotient ring* $\mathfrak{o}_{\mathfrak{p}}$ *of* $\mathfrak{o}$ *with respect to its prime ideal* $\mathfrak{p}$.

Then the Henselization $\mathfrak{r}^*$ *of* $\mathfrak{r}$ *is a Weierstrassen ring.*

Proof. For the condition 4) in the definition of Weierstrassean ring, see Theorem 7 in Appendix. We prove the condition 3): We set $\mathfrak{q} = \mathfrak{p}^* \cap \mathfrak{o}$, where $\mathfrak{p}^*$ is a minimal prime ideal of $\mathfrak{r}^*$. Then $\mathfrak{q}$ is a prime ideal of $\mathfrak{o}$ contained in $\mathfrak{p}$. Then we can find a complete regular local ring $\mathfrak{o}'$ such that $\mathfrak{o}/\mathfrak{q}$ is a finite $\mathfrak{o}'$-module ($\mathfrak{o}' \subseteq \mathfrak{o}/\mathfrak{q}$). For the unequal characteristic case we choose $\mathfrak{o}'$ so that it is

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9) F. K. Schmidt, Über die Erhaltung der Kettensätze der Idealtheorie bei beliebigen endlichen Körpererweiterungen, Math. Zeit., 41 (1936).

10) See Theorem 3 in [H.R. II].

11) Cf. I. S. Cohen, On the structure and ideal theory of complete local rings, Trans. Amer. Math. Soc., 59 (1946).

unramified[11]. Now we set $\mathfrak{p}' = (\mathfrak{p}/\mathfrak{q}) \cap \mathfrak{o}'$. Then $\mathfrak{o}'_{\mathfrak{p}'}$ is a regular local ring[12] and evidently $\mathfrak{r}/\mathfrak{p}^* \cap \mathfrak{r}$ is of finite type over $\mathfrak{o}'_{\mathfrak{p}'}$. Therefore $\mathfrak{r}^*/\mathfrak{p}^*$ is a finite module over the Henselization of $\mathfrak{o}'_{\mathfrak{p}'}$[13].

Corollary 1. If an integrally closed local integrity domain $\mathfrak{o}$ is a quotient ring of a complete Noetherian local ring $\mathfrak{r}$ with respect to a prime ideal of $\mathfrak{r}$, then the completion of $\mathfrak{o}$ is an integrity domain.

Corollary 2. If an integrally closed local integrity domain $\mathfrak{o}$ is a geometric local ring (in generalized sense as was defined in [L.R.]), then the completion of $\mathfrak{o}$ is an integrity domain. For geometric local integrity domain, similar assertion as in the last of Corollary 2 to Theorem 3 holds also (see [L.R.]).

Corollary 3. Let $\mathfrak{o}$ be an integrally closed geometric local integrity domain and let $\mathfrak{o}^*$ be its Henselization. If $\mathfrak{p}^*$ is a prime ideal of $\mathfrak{o}^*$, then $\mathfrak{p}^*\bar{\mathfrak{o}}$ is a prime ideal, where $\bar{\mathfrak{o}}$ denotes the completion of $\mathfrak{o}$.

Remark. This Corollary 3 shows in particular the following: Let V be an algebraic variety which is normal at a point P and let W be a subvariety of V which goes through P. Then the sheets of W at P can be defined already in the Henselization of the local ring $Q_V(P)$ of P on V.

§ 5. Examples (III).

Let $z_1, \cdots, z_n$ be n variables and let C be the complex number field. Let $\mathfrak{o}_n$ be the ring of convergent power series in $z_1, \cdots, z_n$ with coefficients in C (i.e., the ring of analytic functions at the origin $\mathbf{O}$ of complex n-space) and let $\bar{\mathfrak{o}}_n$ be the ring of formal power series in $z_1, \cdots, z_n$ with coefficients in C. Then as is well known, $\mathfrak{o}_n$ is a regular local ring and $\bar{\mathfrak{o}}$ is its completion.

THEOREM 5. *$\mathfrak{o}_n$ is a Weierstrassean ring.*

Proof. As is well known, in $\mathfrak{o}_n$ the Weierstrass preparation theorem holds.[14] Therefore we have only to show that $\mathfrak{o}_n$ is Henselian. Though it may be easy to see this fact by virtue of Theorem 4 in [H.R. I], we prove this in the following form:

$\mathfrak{o}_n$ is algebraically closed in $\bar{\mathfrak{o}}_n$,[14]

making no use of Theorem 1.

12) Cf. l.c. note 11).

13) See § 5 in [H.R. II].

14) We can prove the same even when C is an arbitrary field with an Archimedian or non-Archimedian valuation.

That $\mathfrak{o}_1$ is Henselian is easy by virtue of Theorem 4 in [H.R. I]. Assume that $\mathfrak{F}$ is an almost finite integrally closed integral extension of $\mathfrak{o}_1$ which is contained in $\bar{\mathfrak{o}}_1$. Then since $\mathfrak{F}$ is a valuation ring, it is a dense subspace of $\bar{\mathfrak{o}}_1$. Since $\mathfrak{o}_1$ is of characteristic 0, $\mathfrak{F}$ is a finite $\mathfrak{o}_1$-module. Therefore $\mathfrak{o}_1$ is a dense and closed subspace of $\mathfrak{F}$, which shows that $\mathfrak{o}_1=\mathfrak{F}$. Therefore $\mathfrak{o}_1$ is algebraically closed in $\bar{\mathfrak{o}}_1$. Now, we prove our assertion by induction on the dimension n of $\mathfrak{o}_n$: Assume that $\mathfrak{o}_{n-1}$ is algebraically closed in its completion.

Assume that an element y of $\bar{\mathfrak{o}}_n$ satisfies an irreducible monic polynomial $f(x)=x^r+c_1x^{r-1}+\cdots+c_r$ $(c_i \in \mathfrak{o}_n)$ over $\mathfrak{o}_n$. Since $y \in \bar{\mathfrak{o}}_n$, we can write $y=\sum a_{i_1,\dots,i_n} z_1^{i_1}\cdots z_n^{i_n}$. On the other hand, since $c_i \in \mathfrak{o}_n$, $c_1, \cdots, c_r$ are bounded in a sphere S_ρ of radius ρ with the orgin O as the center, we can find a positive value M such that $|c_j(z)| < M-1$ if $z \in S_\rho$ (for any $j=1, \cdots, r$). We consider an algebraic function Y: $Y^r+c_1Y^{r-1}+\cdots+c_r=0$. Then we see that $|Y(z)|<M$ for any $z \in S_\rho$. Now we consider a new system of local coordinates $z_1, z_2, \cdots, z_{n-1}, z'_n=z_n-(\lambda_1 z_1+\cdots+\lambda_{n-1}z_{n-1})$ with a general system of complex numbers $\lambda_1, \cdots, \lambda_{n-1}$ with conditions $|\lambda_1|\leq 1, \cdots, |\lambda_{n-1}|\leq 1$. We denote by S' the sphere of radius ρ/n with center 0 in this new coordinate system. Then evidently if $z' \in S'$ then $|Y(z')|<M$. Let Y' be the function on the hyperplane $z'_n=0$, reduced from Y. Then we have $|Y'(z')|<M$ if $z' \in S'$ and $z'_n=0$.

On the other hand, we represent y as a power series in $z_1, \cdots, z_{n-1}, z'_n$. Then we have $y=y_0+z'_nP'$, where

$$y_0=\sum_{(j_1,\cdots,j_{n-1})}\Big(\sum_{i_1+i_n=j_1,\cdots,i_{n-1}+i_n=j_{n-1}} a_{i_1,\dots,i_n}\lambda_1^{i_n}\cdots\lambda_{n-1}^{i_n}\Big)z_1^{j_1}\cdots z_{n-1}^{j_{n-1}}$$

and P' is a power series.

Since $\mathfrak{o}_n/z'_n\mathfrak{o}_n=\mathfrak{o}_{n-1}$ and since y is integral over $\mathfrak{o}_n$, we have y_0 is integral over $\mathfrak{o}_{n-1}$. By our induction assumption, y_0 is an analytic function. Therefore y_0 must be a branch of Y'. Therefore $|y_0(z')| < M$ for any $z' \in S'$. This shows, by Cauchy's evaluation formular,

$$\Big|\sum_{i_k+i_n=j_k} a_{i_1,\dots,i_n}\lambda_1^{i_n}\cdots\lambda_{n-1}^{i_n}\Big|<\frac{M}{(\rho/n)^{\Sigma j_s}}$$

and therefore

$|a_{i_1,\dots,i_n}|<M/(\rho/n)^{n\Sigma i_s}$. Thus we see that y is convergent, i.e., $y \in \mathfrak{o}_n$.

Remark. The last half of Theorem 2, when it is applied for

$\mathfrak{o}_n$, speaks that the notion of algebroid varieties (in Chevalley's sense) covers that of analytic varieties.

§ 6. **Examples (IV).**

Let $\mathfrak{v}$ be a discrete special valuation ring with prime element p. Assume that $\mathfrak{v}$ is of characteristic 0. Throughout this section p and $\mathfrak{v}$ maintain these meanings.

Definition. Let $\mathfrak{o}$ be a local integrity domain with maximal ideal $\mathfrak{m}$. Assume that $\mathfrak{o}$ is of finitely generated type over the valuation ring $\mathfrak{v}$. Then we call a system $x_1, \cdots, x_n$ of elements of $\mathfrak{o}$ a system of parameters of $\mathfrak{o}$ if 1) $(p, x_1, \cdots, x_n)\mathfrak{o}$ is a primary ideal belonging to $\mathfrak{m}$ and 2) $n+1=\dim \mathfrak{o}$.

Remark. Existence of system of parameters is evident. Further, if $(x_1, \cdots, x_n)$ is a system of parameters of $\mathfrak{o}$, then 1) $\mathfrak{o}$ is algebraic over $\mathfrak{r}=\mathfrak{v}[x_1, \cdots, x_n]_{(\mathfrak{v}, x_1, \cdots, x_n)}$ and 2) $\mathfrak{r}$ is a subspace of $\mathfrak{o}$.

Definition. Such a ring as $\mathfrak{r}$ in above remark is called a nucleous $\mathfrak{o}$.

LEMMA 1. Let $\mathfrak{o}$ be a local integrity domain which is of finitely generated type over $\mathfrak{v}$. Let $\mathfrak{r}$ be a nucleous of $\mathfrak{o}$. Assume that the Henselization $\mathfrak{r}^*$ of $\mathfrak{r}$ is algebraically closed in its completion $\bar{\mathfrak{r}}$, then there exists a local integrity domain $\mathfrak{o}'$ such that $\mathfrak{o}'$ is of finite type over both $\mathfrak{o}$ and $\mathfrak{r}$, further, the completion of $\mathfrak{o}$ contains no nilpotent element and the integral closure $\tilde{\mathfrak{o}}$ of $\mathfrak{o}$ in its quotient field K is a finite $\mathfrak{o}$-module.

Proof. Let R be the quotient field of $\mathfrak{r}$ and let L be a finite normal extension of R containing K. Let $\mathfrak{J}$ be the totality of $\mathfrak{r}$-integers in L and set $\mathfrak{J}'=\mathfrak{o}[\mathfrak{J}]$. Let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$ and let $\mathfrak{m}'$ be an arbitrary maximal ideal of $\mathfrak{J}'$ and set $\mathfrak{m}=\mathfrak{m}'\cap\mathfrak{J}$. Let $\mathfrak{h}'$ be the decomposition ring of $\mathfrak{m}$ and set $\mathfrak{h}=\mathfrak{h}'_{\mathfrak{m}\cap\mathfrak{h}'}$. Then $\mathfrak{h}$ is a regular local ring with maximal ideal $(p, x_1, \cdots, x_n)\mathfrak{h}$ $(x_i \in \mathfrak{r})$ therefore $\mathfrak{h}$ is a subspace of $\mathfrak{J}'_{\mathfrak{m}'}$. Now we prove that $\mathfrak{J}'_{\mathfrak{m}'}$ is integral over $\mathfrak{h}$. Let a be an element of $\mathfrak{J}'_{\mathfrak{m}'}$ and let $f(x)$ be an irreducible primitive polynomial over $\mathfrak{h}$ which is satisfied by a. Since L is normal over the quotient field of $\mathfrak{h}$, $f(x)$ is irreducible over the Henselization $\mathfrak{r}^*$ of $\mathfrak{h}$. Since $\mathfrak{r}^*$ is algebraically closed in its completion $\bar{\mathfrak{r}}$, $f(x)$ is irreducible over $\bar{\mathfrak{r}}$. On the other hand, the completion of $\mathfrak{J}'_{\mathfrak{m}'}$ is a finite $\bar{\mathfrak{r}}$-module. Therefore a is integral over $\bar{\mathfrak{r}}$. Assume that $g(x)$ is a monic polynomial over $\bar{\mathfrak{r}}$ satisfied by a. Then since $f(x)$ is irreducible, $f(x)\bar{\mathfrak{r}}[x]$ is a prime ideal and therefore $f(x)$ is a factor of $g(x)$. This is impossible unless $f(x)$ is monic. Therefore a must be integral over $\mathfrak{h}$. This shows the first half of our lemma. Now let S be the totality of elements of $\mathfrak{J}$ which is unit in $\mathfrak{J}'$. Then above proof shows that $\mathfrak{J}_S$ contains $\mathfrak{J}'$. Therefore the completion of $\mathfrak{J}'$ contains no nilpotent element. Since $\mathfrak{J}'$ is a finite $\mathfrak{o}$-module, $\mathfrak{o}$ is a subspace of $\mathfrak{J}'$ and therefore the completion of $\mathfrak{o}$ contains no nilpotent element, therefore $\bar{\mathfrak{o}}$ is a finite $\mathfrak{o}$-module[15].

LEMMA 2. Let a regular local ring $\mathfrak{r}$ be of finitely generated type over the valuation ring $\mathfrak{v}$. Then the Henselization $\mathfrak{r}^*$ of $\mathfrak{r}$ is algebraically closed in its completion.

Proof. When $\mathfrak{r}$ is of dimension 1, we can prove by the same way as in the previous section. Therefore we can prove our assertion easily by induction on the dimension n of $\mathfrak{r}$, by virtue of Lemma 1 and Theorem 1.

THEOREM 6. *Let a regular local ring $\mathfrak{r}$ be of finitely generated type over the valuation ring $\mathfrak{v}$. Then the Henselization $\mathfrak{r}^*$ of $\mathfrak{r}$ is a Weierstrassean ring.*

Proof. We have only to show that if $\mathfrak{p}^*$ is a minimal prime ideal of $\mathfrak{r}^*$ then $\mathfrak{r}^*/\mathfrak{p}^*$ is a finite integral extension of a Henselian regular local ring $\mathfrak{r}'^*$ which is either a Henselization of a regular local ring of finitely generated type over $\mathfrak{v}$ or a Henselization of a geometric regular local ring. If $\mathfrak{p}^* \ni p$, then it is evident that the latter case holds. Therefore we assume that $\mathfrak{p}^* \not\ni p$. We set $\mathfrak{p}=\mathfrak{p}^* \cap \mathfrak{r}$. Let $x_1, \cdots, x_{n-1}$ be a system of parameters of $\mathfrak{r}/\mathfrak{p}$. Let $\mathfrak{r}'$ be the regular local ring $\mathfrak{v}[x_1, \cdots, x_{n-1}]_{(p, x_1, \cdots, x_{n-1})}$. Then by Lemma 1, we see that $\mathfrak{r}^*/\mathfrak{p}^*$ is a finite $\mathfrak{r}'^*$-module, where $\mathfrak{r}'^*$ is the Henselization of $\mathfrak{r}'$.

Corollary. If an integrally closed integrity domain $\mathfrak{o}$ is of finitely generated type over the valuation ring $\mathfrak{v}$, then the completion of $\mathfrak{o}$ is an integrally closed integrity domain. When we do not assume that $\mathfrak{o}$ is integrally closed, the similar assertion as in Corollary 2 to Theorem 3 holds also.

Appendix: Complete local integrity domains.

THEOREM 7[16]. *Let $\mathfrak{o}$ be a Noetherian complete local integrity domain. Then the integral closure $\bar{\mathfrak{o}}$ of $\mathfrak{o}$ in its quotient field is a*

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15) See Appendix below.

16) It was communicated to the writer that this was also proved independently by Mr. Y. Mori.

finite $\mathfrak{o}$*-module.*

For the proof, since it is known that $\mathfrak{o}$ contains a complete regular local ring $\mathfrak{r}$ such that $\mathfrak{o}$ is a finite $\mathfrak{r}$-module, our assertion is equivalent to the following

LEMMA 3. Let $\mathfrak{r}$ be a complete regular local ring and let a field L be a finite extension of the quotient field K of $\mathfrak{r}$. Then the totality $\mathfrak{o}$ of $\mathfrak{r}$-integers in L is a finite $\mathfrak{r}$-module.

Proof. If $\mathfrak{r}$ is of characteristic 0, then our assertion is evident since L is separable over K. Therefore we assume that $\mathfrak{r}$ is a formal power series ring $k\{x_1, \cdots, x_n\}$ over a field k of characteristic $p(\neq 0)$. We can take a purely inseparable extension L' of K so that LL' is separable over L'. Therefore we may treat only the case L is purely inseparable over K. Set $e=[L:K]$ and set $\mathfrak{k}=k^{1/e}$, $y_i=x_i^{1/e}$ $(i=1, \cdots, n)$ and $\bar{\mathfrak{r}}=\mathfrak{k}\{y_1, \cdots, y_n\}$. We may consider that $\bar{\mathfrak{r}}$ contains $\mathfrak{o}$ as a subring. An element a of $\mathfrak{o}$ can be expressed in $\bar{\mathfrak{r}}$ as a power series in $y_1, \cdots, y_n$ with coefficients in $\mathfrak{k}$. We use the term "leading form" of a in the sense of an element a of $\bar{\mathfrak{r}}$.

LEMMA K. Let $f_1, \cdots, f_m$ be leading forms of elements $a_1, \cdots, a_m$ of $\mathfrak{o}$ respectively. If $1, f_1, \cdots, f_m$ are linearly independent over $\mathfrak{r}$ then $1, a_1, \cdots, a_m$ are linearly independent over $\mathfrak{r}$.

Proof. We consider linear combination $\sum a_i b_i$ $(b_i \in \mathfrak{r})$. Let g_i be the leading form of b_i for each i. Then it is evident that $f_i g_i$ is the leading form of $a_i b_i$. We may assume that $\deg(f_1 g_1) = \cdots = \deg(f_r g_r) = d$, $\deg(f_{r+j} g_{r+j}) > d$ $(j \geq 1)$. Then the leading form of $\sum a_i b_i$ is $\sum_{i=1}^{r} f_i g_i$ which is not in $\mathfrak{r}$ because $1, f_1, \cdots, f_r$ are linearly independent over $\mathfrak{r}$. Therefore $\sum a_i b_i$ is not in $\mathfrak{r}$ unless every b_i is zero, which proves our assertion.

Now we proceed the proof of Lemma 3. Since L is finite over K, there exists a set $f_1, \cdots, f_t$ of leading forms of elements $a_1, \cdots, a_t$ of $\mathfrak{o}$ such that $1, f_1, \cdots, f_t$ are linearly independent over $\mathfrak{r}$ and every leading form f of an arbitrary element a of $\mathfrak{o}$ is linearly dependent on $1, f_1, \cdots, f_t$ over $\mathfrak{r}$. Let $c_1, \cdots, c_u$ be the coefficients of $f_1, \cdots, f_t$. Then we see that if f is a leading form of an element a of $\mathfrak{o}$ then the coefficients of f are in $k(c_1, \cdots, c_u)$. Now let $d_0=1, d_1, \cdots, d_v$ be a linear basis for $k(c_1, \cdots, c_u)$ over k and let $m_0=1, m_1, \cdots, m_w$ be the totality of monomials of degree less than ne in $y_1, \cdots, y_n$. Then it is easy to see that if f is the leading form of an element a of $\mathfrak{o}$ then f is contained in the module generated by $d_0 m_0, \cdots, d_0 m_w, \cdots, d_v m_w$ over the form ring of $\mathfrak{r}$. Therefore there exists a finite set of leading forms $k_1, \cdots, k_s$ of elements $a_1, \cdots, a_s$ of $\mathfrak{o}$ such that every such f as above is contained in the module generated by $k_1, \cdots, k_s$ over the form ring of $\mathfrak{r}$. Then we have $\mathfrak{o}=\mathfrak{r}[a_1, \cdots, a_s]$ as follows:

LEMMA L. $\mathfrak{r}[a_1, \cdots, a_s]$ is a subspace of $\bar{\mathfrak{r}}$.

Proof. Let $\mathfrak{m}$ and $\bar{\mathfrak{m}}$ be the maximal ideals of $\mathfrak{r}[a_1, \cdots, a_s]$ and $\bar{\mathfrak{r}}$ respectively. Then since $\mathfrak{m}\subseteq\bar{\mathfrak{m}}$, we have $\mathfrak{m}^k\subseteq\bar{\mathfrak{m}}^k\cap\mathfrak{r}[a_1, \cdots, a_s]$ for every $k=1, 2 \cdots$. On other hand, since $\mathfrak{r}[a_1,\cdots, a_s]$ is complete, there exists $n(k)$ for each k such that $\bar{\mathfrak{m}}^{n(k)}\cap\mathfrak{r}[a_1, \cdots, a_s]\subseteq\mathfrak{m}^k$[17], which proves our assertion.

The above Lemma L shows that $\mathfrak{r}[a_1, \cdots, a_s]$ contains $\mathfrak{o}$. Since $a_1, \cdots, a_s$ are in $\mathfrak{o}$, we see that $\mathfrak{o}=\mathfrak{r}[a_1, \cdots, a_s]$.

Corollary 1. Assume that a local integrity domain $\mathfrak{o}$ is a quotient ring of a Noetherian complete local integrity domain $\mathfrak{r}$ with respect to its prime ideal $\mathfrak{p}$. Then any almost finite integral extension $\mathfrak{o}'$ of $\mathfrak{o}$ is a finite $\mathfrak{o}$-module.

Corollary 2. Let $\mathfrak{o}$ be a Noetherian local integrity domain. If the completion $\mathfrak{o}^*$ of $\mathfrak{o}$ contains no nilpotent element, then the integral closure $\bar{\mathfrak{o}}$ of $\mathfrak{o}$ in its quotient field is a finite $\mathfrak{o}$-module.

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17) Cf. Chevalley, l.c. note 4), Lemma 7 in § 1.