MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXVIII, Mathematics No. 2, 1953.

On some properties of the non-linear differential equations of the "Parametric excitation"

By

Masaya YAMAGUTI

To my Teacher, Toshizô MATSUMOTO
on the occasion of his 63rd birthday

(Received July 8, 1953)

Introduction. The author and his collaborator S. Mizohata[1] have obtained a proof for the existence of the periodic solutions of the non-linear differential equations of the following type under comparatively weak conditions,

$$\ddot{x}+f(x)\dot{x}+g(x)=p(t).$$

This problem has been raised from the researches of the non-linear vibrations in the field of engineerings. In this report, we shall discuss the wider class of problem which contains the so-called "Parametric Excitation"[2] which has not yet been rigorously discussed. For example, one case is expressed by the equation,[3]

$$\ddot{x}+\beta_0\dot{x}+(p_0^2+\alpha_0\cos 2\omega t)x+\gamma_0 x^3=p_0\cos(\omega t+\varphi)$$

on which we shall have the following conclusion in this report: This equation has at least one periodic solution having such property that

$$-x(t)=x\left(t+\frac{\pi}{\omega}\right) \quad \text{as} \quad \beta_0>0, \quad \gamma_0>0.$$

We shall describe the obtained results as two theorems I, II and add several examples.

Now we shall consider the following differential equation

$$(1) \qquad \ddot{x}+f(x)\dot{x}+g(x,t)=p(t),$$

where the functions $f(x)$, $g(x, t)$ satisfy the Lipschitz condition[4] with respect to x, and $g(x, t)$ has a continuous partial derivative $g_t(x, t)$ and $g(x, t)$, $p(t)$ are continuous periodic functions of t having the period ω, and $p(t)$ satisfies the condition $\int_0^\omega p(t)dt=0$. We put the functions $F(x)$, $G(x, t)$, $P(t)$ as follows:

$$F(x)=\int_0^x f(x)dx, \quad G(x, t)=\int_0^x g(x, t)dx, \quad P(t)=\int_0^t p(t)dt.$$

Since the function $P(t)$ is bounded by the assumption, we put

$$P_0=\max_{0\leq t<\infty} |P(t)|$$

THEOREM I Hypotheses:

(α) $F(x) sgn\, x\rightarrow\infty$ as $|x|\rightarrow\infty$

(β) $g(x, t)\, sgn\, x\geqq k_0>0,$ for $|x|>\xi_0$

(γ) $|F(x)|>\frac{1}{k_1}\left|\frac{G_t(x, t)}{g(x, t)}\right|,$ for $|x|>\xi_1$

where k_0, ξ_0, k_1, ξ_1 are positive constants and $0<k_1<1$.
Conclusion: For any x_0, $\dot{x}_0$ the solution $x(t)$ of (1) for which $x(0)=x_0$, $\dot{x}(0)=\dot{x}_0$ satisfies

$$|x(t)|<B, \quad |\dot{x}(t)|<B \quad \text{for} \quad t>t_0(x_0, \dot{x}_0)$$

where B is a constant independent of x_0, $\dot{x}_0$ and there is at least one periodic solution with period ω among such solutions.
PROOF (i) We arrange the constants ξ_2, m, ξ_3, F_0 and g_0 for later use. When we suppose that $y=\dot{x}+F(x)-P(t)$, we can write (1) as follows:

$$(2) \qquad \begin{cases} \dot{x}=y-F(x)+P(t) \\ \dot{y}=-g(x, t). \end{cases}$$

Next we define the function $P(x, y, t)$ as follows:

$$(3) \qquad P(x, y, t)=\frac{y^2}{2}+G(x, t).$$

Then we can say that the plane curve given by the equation (4) below is simple on the plane $t=t$ for sufficiently large value of C (for example $C\geqq\max_{\substack{|x|\leq\xi_0 \\ 0\leq t\leq\omega}} |G(x, t)|$)

$$(4) \qquad P(x, y, t)=C.$$

Let us consider the derivative $\frac{dP}{dt}$ along the trajectory of the solution of (2).

$$(5) \qquad \frac{d}{dt}P(x(t), y(t), t)=-g(x, t)\{F(x)-P(t)\}+G_t(x, t).$$

Put $\frac{dP}{dt}=-\varphi(x,t)$. By the hypothesis ($\alpha$), we find a constant ξ_2 ($>\max(\xi_0,\xi_1)$) that satisfies the condition (6)

$$(6) \qquad (1-k_1)F(x)\,sgn\,x > P_0+\frac{\varepsilon}{k_0} \quad \text{for} \quad |x|>\xi_2,$$

where ε is an arbitrary positive constant.
On the other hand we may write $\varphi(x,t)$ as follows:

$$\varphi(x,t)=g(x,t)\{(1-k_1)F(x)-P(t)\}+g(x,t)\left\{k_1F(x)-\frac{G_t(x,t)}{g(x,t)}\right\}.$$

From (6) and (γ) we have

$$|\varphi(x,t)|>|g(x,t)|\{|(1-k_1)F(x)|-P_0\}+|g(x,t)|\left\{|k_1F(x)|-\left|\frac{G_t(x,t)}{g(x,t)}\right|\right\}$$

$$>k_0\frac{\varepsilon}{k_0}+0. \quad \text{or} \quad |x|>\xi_2$$

Therefore we can say

$$(7) \qquad \dot{P}(x(t),y(t),t)<-\varepsilon, \qquad \text{for} \quad |x(t)|>\xi_2 \quad 0\leqq t\leqq\omega,$$

and we put

$$(8) \qquad m=\max_{\substack{|x|>\xi_2\\ 0\leqq t\leqq\omega}}|\dot{P}(x(t),y(t),t)|$$

and we define ξ_3

$$(9) \qquad \xi_3>\left(1+\frac{4m}{\varepsilon}\right)\xi_2.$$

Now we put F_0, g_0 as follows:

$$(10) \qquad F_0=\max_{\substack{|x|\leqq\xi_3\\ 0\leqq t\leqq\omega}}|F(x)-P(t)|, \qquad (11) \qquad g_0=\max_{\substack{|x|\leqq\xi_3\\ 0\leqq t\leqq\omega}}|g(x,t)|$$

(ii) We shall consider several domains in (x,y,t)-space.

$$(12) \quad \begin{cases} \mathfrak{E}: & 0\leqq t\leqq\omega \\ \mathfrak{A}: & |x|\leqq\xi_2, \quad 0\leqq t\leqq\omega \\ \mathfrak{L}: & |x|\leqq\xi_3, \quad \text{,,} \\ \mathfrak{L}_+: & -\xi_2\leqq x\leqq\xi_3, \quad \text{,,} \\ \mathfrak{L}_-: & -\xi_3\leqq x\leqq\xi_2, \quad \text{,,} \\ \mathfrak{B}_+: & \xi_2\leqq x\leqq\xi_3, \quad \text{,,} \\ \mathfrak{B}_-: & -\xi_3\leqq x\leqq-\xi_2, \quad \text{,,} \\ \bar{\mathfrak{P}}: & y>F_0, \\ \underline{\mathfrak{P}}: & y<-F_0, \end{cases} \qquad \begin{cases} \bar{\mathfrak{L}}: & \mathfrak{L}\cap\bar{\mathfrak{P}} \\ \underline{\mathfrak{L}}: & \mathfrak{L}\cap\underline{\mathfrak{P}} \\ \bar{\mathfrak{L}}_+: & \mathfrak{L}_+\cap\bar{\mathfrak{P}} \\ \underline{\mathfrak{L}}_+: & \mathfrak{L}_+\cap\underline{\mathfrak{P}} \\ \bar{\mathfrak{L}}_-: & \mathfrak{L}_-\cap\bar{\mathfrak{P}} \\ \underline{\mathfrak{L}}_-: & \mathfrak{L}_-\cap\underline{\mathfrak{P}}. \end{cases}$$

Next we denote by D_C the domain contained in $\mathfrak{E}$ enclosed by the surface (4) which itself is denoted by S_C, and denote by $S_C(t)$ the section curve of S_C by the plane $t=t$ and denote by $D_C(t)$ the section domain of 2 dimension of D_C cut by the plane $t=t$.

Lemma 1. If we choose C of (4) sufficiently large, we can say that the trajectory of the solution of (2) which entered in $\mathfrak{L}$ after the crossing of S_C must pass through only $\bar{\mathfrak{L}}$ (or $\underline{\mathfrak{L}}$) so long as it stays in $\mathfrak{L}$, and we may suppose that the time spent on this passing of $\mathfrak{L}$ may be as short as we hope.

Suppose that the trajectory $(x(t), y(t), t)$ of (2) has entered in crossing S_C at t_0, then we have from (2),

$$y(t)-y(t_0)=\int_{t_0}^{t}\dot{y}dt=-\int_{t_0}^{t}g(x, t)dt.$$

Since $(x(t), y(t), t)\in\mathfrak{L}$, we have $|y(t)-y(t_0)|\leqq g_0\omega$, or

(13) $\quad |y(t_0)|-g_0\omega\leqq|y(t)|\leqq|y(t_0)|+g_0\omega, \quad$ for $\quad (x(t), y(t), t)\in\mathfrak{L}$.

Because of the assumption that $(x(t_0), y(t_0))\in S_C(t_0)\cap\mathfrak{L}$, we can establish the following inequality by taking C sufficiently large:

$$F_0<|y(t_0)|-g_0\omega$$

and then we can suppose $(x(t), y(t), t)\in\bar{\mathfrak{L}}$ (or $\underline{\mathfrak{L}}$) for $t_0\leqq t\leqq\omega$ so long as it stays in such domain $\mathfrak{L}$. Obviously, from the first equation of (2) we have

$$(14)\qquad \begin{cases}\dot{x}>y-F_0>0, & \text{for} \quad (x(t), y(t), t)\in\bar{\mathfrak{L}}\\ \dot{x}<y+F_0<0, & \text{for} \quad (x(t), y(t), t)\in\underline{\mathfrak{L}}.\end{cases}$$

Then the solution $x(t)$ of (1) is monotone with respect to t in such domain. Suppose that the solution has entered in $\mathfrak{L}$ at $t=t_0$ and it stays in $\mathfrak{L}$ (i.e. in $\bar{\mathfrak{L}}$ or $\underline{\mathfrak{L}}$) until $t_0+\tau$, then we have

$$(15)\qquad \tau=\int_{x(t_0)}^{x(t_0+\tau)}\frac{dx}{y(t)-F(x)+P(t)}.$$

Now we consider the only case where the solution passes through $\bar{\mathfrak{L}}$. (Same discussion may be possible for $\underline{\mathfrak{L}}$.) By (13) we have

$$(16)\qquad \tau\leqq\int_{x(t_0)}^{x(t_0+\tau)}\frac{dx}{|y(t)|-F_0}\leqq\frac{2\xi_3}{|y(t_0)|-g_0\omega-F_0},$$

$$(x(t), y(t), t)\in\bar{\mathfrak{L}},\ t_0\leqq t\leqq t_0+\tau.$$

Then we conclude that τ may be supposed as small as we hope by increasing C. (i.e. by increasing $|y(t_0)|$) Q.E.D.

Lemma 2. If we choose C of (4) sufficiently large, we can say that $P(x(t), y(t), t)$ decreases when the trajectory of (2) traverses $\mathfrak{L}_+$ (or $\mathfrak{L}_-$) after the crossing of S_C (or before the crossing).

As we have seen in the proof of lemma 1, the trajectory (2) staying in $\mathfrak{L}_+$ (or $\mathfrak{L}_-$) must pass through only $\bar{\mathfrak{L}}_+$ or $\underline{\mathfrak{L}}_+$ (or, $\bar{\mathfrak{L}}_-$ or $\underline{\mathfrak{L}}_-$.) Here we prove lemma 2 in the case where the trajectory passes through $\bar{\mathfrak{L}}_+$. (The same discussion may be applied on the other cases.) Now $x(t)$ is monotone increasing with respect to t while it stays in $\bar{\mathfrak{L}}_+$ (as we have seen in the proof of lemma 1).

We shall denote by τ the time spent to pass $\mathfrak{A}$ and denote by τ' the time spent to pass $\mathfrak{B}_+$ and we assume that the trajectory $(x(t), y(t)\ t)$ crosses S_C at $t=t_0$ and departs from $\bar{\mathfrak{L}}_+$ at $t=t_0+\tau+\tau'$ after traversing $\bar{\mathfrak{L}}_+$, then we have

$$\tau=\int_{x(t_0)}^{x(t_0+\tau)}\frac{dx}{\dot{x}}\leqq\int_{x(t_0)}^{x(t_0+\tau)}\frac{dx}{|y(t)|-F_0}\leqq\frac{2\xi_2}{|y(t_0)|-\omega g_0-F_0},$$

$$\tau'=\int_{x(t_0+\tau)}^{x(t_0+\tau+\tau')}\frac{dx}{\dot{x}}\geqq\int_{x(t_0+\tau)}^{x(t_0+\tau+\tau')}\frac{dx}{|y(t)|+F_0}\geqq\frac{\xi_3-\xi_2}{|y(t_0)|+\omega g_0+F_0}.$$

Therefore

$$\frac{\tau}{\tau'}\leqq\frac{2\xi_2}{\xi_3-\xi_2}\cdot\frac{|y(t_0)|+\omega g_0+F_0}{|y(t_0)|-\omega g_0-F_0}.$$

Since $|y(t_0)|\to\infty$ when $C\to\infty$, we have

$$\frac{\tau}{\tau'}\leqq\frac{2\xi_2}{\xi_3-\xi_2}(1+\eta),$$

where η is small for large value of C. By the determination of (9), we have

$$\frac{2\xi_2}{\xi_3-\xi_2}<\frac{\varepsilon}{2m}.$$

Putting $\eta\leqq\frac{1}{2}$ we have

$$\frac{\tau}{\tau'}\leqq\frac{3}{4}\cdot\frac{\varepsilon}{m}. \tag{17}$$

On the other hand, let us calculate the variation δP of $P(x(t), y(t), t)$ during the passage of the trajectory from t_0 to $t_0+\tau+\tau'$.

$$\delta P=\int_{t_0}^{t_0+\tau}\dot{P}\,dt+\int_{t_0+\tau}^{t_0+\tau+\tau'}\dot{P}\,dt$$

$$\delta P\leqq m\tau-\varepsilon\tau'<-m\tau'\cdot\frac{\varepsilon}{4m}. \tag{18}$$

Q.E.D.

(iii) Now we shall show that the trajectory $(x(t), y(t), t)$ of (2) started from $S_C(0)$ at $t=0$, must enter into $D_C(\omega)$ at $t=\omega$. (Of course we have to increase the value of C.)

1° The trajectory of the solution of (2) started from $S_C(0)\cap\mathfrak{L}$ must depart from $\mathfrak{L}$ within one period after passing $\mathfrak{L}$ because of lemma 1.

2° Such a trajectory must pass through only $\bar{\mathfrak{L}}$ or $\underline{\mathfrak{L}}$ because of lemma 1.

3° The value of $P(x, y, t)$ along such a trajectory must decrease when it has passed $\mathfrak{L}_{\underset{(-)}{+}}$. (lemma 2) Therefore such a trajectory started from $S_C(0)\cap\mathfrak{L}$ must enter into the interior of D_C after passing $\mathfrak{L}_+$.

4° Therefore of all trajectories of the solutions of (2) started from $S_C(0)$, there must be for each at least one time point at which they enter into the part of D_C belonging to the exterior of $\mathfrak{L}$.

5° Now we should suppose that one solution started from $S_C(0)$ does not enter into the interior of $D_C(\omega)$ at $t=\omega$. Of this solution, t_0 is a time point assured by 4°. Then there is a time point τ as follows:

$$\begin{aligned}&t_0<\tau\leqq\omega, \qquad (x(\tau), y(\tau))\in S_C(\tau),\\ &(x(\tau+\eta), y(\tau+\eta))\bar{\in} D_C(\tau+\eta), \quad \eta: \text{small}.\end{aligned} \tag{19}$$

If there are many such time points, we take the least one of them. Therefore $(x(t), y(t), t)$ must go out of D_C at τ, from that we have

$$\dot{P}(x(\tau), y(\tau), \tau)>0$$

and this occurs only when $|x(\tau)|<\xi_2$, then we have

$$(x(\tau), y(\tau), \tau)\in\mathfrak{A}\cap S_C(\tau). \tag{20}$$

Since $(x(t_0), y(t_0), t_0)\bar{\in}\mathfrak{L}$, then $(x(t), y(t), t)$ passes through $\mathfrak{B}_+$ (or $\mathfrak{B}_-$) during the interval $t_0\leqq t\leqq\tau$. Therefore by lemma 2, P decreases. On the other hand, the value of P before its entering into $\mathfrak{B}_+$ (or $\mathfrak{B}_-$) is smaller or equal to C (τ is the least time point which satisfies (19)). Consequently it contradicts to (19) that along the trajectory $P(x, y, t)$ decreases from the value of P at t_0:

$$P(x(t_0), y(t_0), t_0) \leqq C.$$

Hence all the solutions of (2) started from $S_C(0)$ arrive at the interior of $D_C(\omega)$ at $t=\omega$.

6° Then also the solutions of (2) started from $D_C(0)$ arrive at the interior $D_C(\omega)$, where $D_C(0)$ is congruent to $D_C(\omega)$. Considering the mapping which maps $D_C(0)$ to $D_C(\omega)$, we conclude that there is at least one fixed point in $D_C(0)$, and this process is repeated in all intervals $(n\omega, (n+1)\omega)$. Then there is at least one periodic solution of (2) of period ω. This solution $x(t)$ is a solution of (1). Then also we can conclude, by similar discussion as in the former paper[1], that for all solutions, we have

$$|x(t)| < B, \quad |\dot{x}(t)| < B \qquad t > t_0(x(0), \dot{x}(0))$$

Q.E.D.

THEOREM II If we add one more hypothesis (δ) below to (α), (β), (γ) of theorem I, then the differential equation (1) has at least one periodic solution of period ω such that $x(t) = -x\left(t+\frac{\omega}{2}\right)$.

$$(\delta) \qquad \begin{aligned} & f(x) = f(-x), \qquad -g(x, t) = g(-x, t), \\ & g(x, t) = g\left(x, t+\frac{\omega}{2}\right), \qquad p(t) = -p\left(t+\frac{\omega}{2}\right). \end{aligned}$$

PROOF We shall write (1) as follows:

$$(21) \qquad \begin{cases} \dot{x} = y - F(x) + \bar{P}(t) \\ \dot{y} = -g(x, t) \end{cases}$$

where (22)
$$\bar{P}(t) = P(t) - \frac{1}{2} P\left(\frac{\omega}{2}\right),$$

then $\bar{P}(t)$ satisfies $\bar{P}(t) = -\bar{P}\left(t+\frac{\omega}{2}\right)$ by the last condition of (δ). Now we shall consider the equation (21) with the time interval $0 \leqq t \leqq \frac{\omega}{2}$ only. The domain $D_C(0)$ is congruent to $D_C\left(\frac{\omega}{2}\right)$ by the condition (δ). Then we consider the mapping T which maps $(x(0), y(0))$ on $\left(x\left(\frac{\omega}{2}\right), y\left(\frac{\omega}{2}\right)\right)$ and T is supposed to be determined by the following formula:

$$T:\quad \begin{cases} x\left(\frac{\omega}{2}\right)=M(x(0),y(0)) \\ y\left(\frac{\omega}{2}\right)=N(x(0),y(0)). \end{cases}$$

Hence by the same discussion as in the proof of theorem I, we can say

$$T(\bar{D}_C(0)) \in D_C\left(\frac{\omega}{2}\right);$$

here $D_C(0)$ is congruent to $D_C\left(\frac{\omega}{2}\right)$ ($\bar{D}_C(0)$ is the closure of $D_C(0)$). Next we consider another mapping U:

$$U:\quad \begin{cases} \bar{x}(0)=-x\left(\frac{\omega}{2}\right) \\ \bar{y}(0)=-y\left(\frac{\omega}{2}\right) \end{cases}$$

and by the symmetry conditions (δ) of $S_C(0)$ and $S_C\left(\frac{\omega}{2}\right)$, we can say that if $\left(x\left(\frac{\omega}{2}\right), y\left(\frac{\omega}{2}\right)\right) \in D_C\left(\frac{\omega}{2}\right)$, then $(\bar{x}(0), \bar{y}(0)) \in D_C\left(\frac{\omega}{2}\right)$.

Then we consider the product mapping UT. Since U and T are topological, UT is a topological mapping by which $(x(0), y(0))$ is mapped to $(\bar{x}(0), \bar{y}(0))$, and we can say

$$UT(\bar{D}_C(0)) \subset D_C\left(\frac{\omega}{2}\right). \tag{23}$$

Therefore we conclude that there is at least one fixed point in $\bar{D}_C(0)$. Then there is a solution $(x_0(t), y_0(t))$ of (21) such that

$$\begin{cases} \bar{x}_0(0)=x_0(0)=-x_0\left(\frac{\omega}{2}\right) \\ \bar{y}_0(0)=y_0(0)=-y_0\left(\frac{\omega}{2}\right). \end{cases} \tag{24}$$

Next we consider the behaviour of $(\bar{x}_0(\tau), \bar{y}_0(\tau))$ in the interval $0\leqq\tau\leqq\frac{\omega}{2}$ instead of the behaviour of $(x_0(t), y_0(t))$ in the interval $\frac{\omega}{2}\leqq t\leqq\omega$, where $\bar{x}_0(\tau)=-x_0\left(\tau+\frac{\omega}{2}\right)$, $\bar{y}_0(\tau)=-y_0\left(\tau+\frac{\omega}{2}\right)$.

Since the behaviour of $(x_0(t), y_0(t))$ is determined by (21) in the interval $\frac{\omega}{2}\leqq t\leqq\omega$, then we obtain the following equation for $(\bar{x}_0(t),\bar{y}_0(t))$

$$(25)\qquad \begin{cases}\dot{\bar{x}}(\tau)=y(\tau)-F(\bar{x}(\tau))+\bar{P}(\tau)\\ \dot{\bar{y}}(\tau)=-g(\bar{x}(\tau),\tau),\end{cases}\qquad 0\leqq\tau\leqq\frac{\omega}{2}.$$

This system is the same one as (21) in the interval $0\leqq t\leqq\frac{\omega}{2}$. Then $(\bar{x}_0(\tau),\bar{y}_0(\tau))$ which is $(\bar{x}_0(0),\bar{y}_0(0))$ at $\tau=0$, arrives at $(-\bar{x}_0(0),-\bar{y}_0(0))$ at $\tau=\frac{\omega}{2}$. In other words, the solution $(x_0(t),y_0(t))$ such as $x_0\left(\frac{\omega}{2}\right)=-x_0(0)$, $y_0\left(\frac{\omega}{2}\right)=-y_0(0)$ arrives at $(x_0(0),y_0(0))$ at $t=\omega$. By the assumption of the periodicity, this mapping will be repeated indefinitely. Consequently we can conclude that the system has at least one periodic solution which satisfies $x(t)=-x\left(t+\frac{\omega}{2}\right)$.

Q.E.D.

EXAMPLES

1. $\ddot{x}+2\beta_0\dot{x}+(p_0^2+\alpha_0\cos 2\omega t)x+\gamma_0x^3=p\cos(\omega t+\varphi)$, $\beta_0>0$, $\gamma_0>0$,

There is at least one periodic solution $x(t)=-x\left(t+\frac{\pi}{\omega}\right)$. Because the conditions (α), (β), (δ) of theorem II are obviously satisfied by this equation, and (γ) is fulfilled as follows:

$$\left|\frac{G_t(x,t)}{g(x,t)}\right|\leqq\frac{\omega d_0x^2}{\gamma_0|x^3|-(p_0+|\alpha_0|)|x|}\to 0\quad(|x|\to\infty).$$

2.[5] $\ddot{x}+b\dot{x}+x+(a-\varepsilon x)x\cos 2t+ex^3=0$, $b>0$, $e>0$, $\varepsilon>0$, $\frac{2be}{1+2b}>\varepsilon$.

We can say that the all solutions are bounded. Because the conditions (α), (β), are obviously satisfied, and (γ) is satisfied as follows:

$$\left|\frac{G_t(x,t)}{xg(x,t)}\right|\leqq\frac{\varepsilon\frac{x^4}{2}-ax^2}{ex^4-\varepsilon x^4-ax^2-x^2}\to\frac{\frac{\varepsilon}{2}}{e-\varepsilon}=\frac{\varepsilon}{2(e-\varepsilon)}<b\quad\text{as } |x|\to\infty.$$

3.[1] $\ddot{x}+f(x)\dot{x}+g(x)=p(t)$, where $f(x)=f(-x)$, $g(x)=-g(x)$, $p(t)=-p\left(t+\frac{\omega}{2}\right)$ and $\operatorname{sgn} xF(x)\to\infty$ as $x\to\infty$, $\operatorname{sgn}. g(x)\geqq k_0>0$, $|x|>\xi_0$. There is as least one solution such that $x(t)=-x\left(t+\frac{\omega}{2}\right)$. This is a special case of theorem II.

1) S. Mizohata and M. Yamaguti "On the existence of periodic solutions of the non-linear differential equation, $\ddot{x}+a(x)\cdot\dot{x}+\varphi(x)=p(t)$."
Mem. Coll. Sci. Univ. Kyoto Ser. Vol. xxv Mat. No. 2, 1952.

2) N. Minorsky "Parametric Excitations" Jour. Appl. Phy. Vol. 22 No. 1 p. 49.

3) Den Hartog "Mechanical Vibrations" 1946 3rd Ed. p. 408–411.

4) These conditions assure the unicity and the possibility of continuation of the solutions of (1).

5) N. Minorsky "Sur l'oscillateur non-linéaire de Mathieu."
Compt. Rend. des séances de L'Acad. Sci. t. 232 p. 2179–2180.