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On transformations of differential equations

By

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In the first two sections we consider the system of ordinary differential equations

$$\frac{dy_i}{dx}=f_i(x, y_1, y_2, \cdots, y_n) \qquad (i=1, 2, \cdots, n) \tag{1}$$

where $f_i(x, y_1, y_2, \cdots, y_n)$ are defined and continuous in a region

$$E_{n+1}\ :\ 0\leqq x\leqq a\ ,\quad |y_i|<+\infty \qquad (i=1, 2, \cdots, n).$$

Let us consider $(y_1, y_2, \cdots, y_n)$ as a vector y, then $(f_1, f_2, \cdots, f_n)$ defines a vector-function of (x, y), conveniently written $f(x, y)$. Thus (1) assumes the simple form

$$\frac{dy}{dx}=f(x, y). \tag{2}$$

In § 3, the differential equation of the second order is investigated as a special case of (1).

§ 1. Transformations of (1)

Let $f(t)$ be the greatest value of 1, t and $\max\limits_{\substack{0\leqq x\leqq a\\ |y|\leqq t}}|f(x, y)|$, where $|y|=\sqrt{y_1^2+y_2^2+\cdots+y_n^2}$ and $|f|=\sqrt{f_1^2+f_2^2+\cdots+f_n^2}$, then $f(t)$ is a positive continuous function of t, not less than unity, in $0\leqq t<+\infty$. Now for a given positive constant σ, consider the function $\lambda(r)$ defined by the relation

$$\frac{1}{\{\lambda(r)\}^{\sigma}}=\int_r^{r+1}\frac{dt}{\{f(t)\}^2},$$

then $\lambda(r)$ is a continuous function of r in $0\leqq r<+\infty$, $\lambda(r)\geqq 1$ in $0\leqq r<+\infty$ and $\lim\limits_{r\to+\infty}\lambda(r)=+\infty$. And evidently $\lambda(r)$ has the continuous derivative

$$\lambda'(r)=\frac{1}{\sigma}\{\lambda(r)\}^{1+\sigma}[\{f(r)\}^{-2}-\{f(r+1)\}^{-2}] \quad (\geqq 0).$$

Next put

$$\rho(r)=r\lambda(r)$$

then $\rho(r)$ and its derivative $\rho'(r)$ are also continuous functions of r in $0\leqq r<+\infty$ where $\rho(0)=0$, $\rho(r)>0$ for $r>0$, $\lim_{r\to+\infty}\rho(r)=+\infty$ and $\rho'(r)=\lambda(r)+r\lambda'(r)\geqq 1$. Therefore there exists the inverse function of $\rho(r)$, written $r(\rho)$, which is a continuous function of ρ in $0\leqq\rho<+\infty$ and whose derivative $r'(\rho)$ is also continuous since $r'(\rho)=1/\rho'(r)$ holds. Thus we have

$$r(0)=0, \quad r(\rho)>0 \quad \text{for} \quad \rho>0, \quad \lim_{r\to+\infty} r(\rho)=+\infty$$

and $r'(\rho)>0$ for $0\leqq\rho<+\infty$.

Now consider a mapping from $(y_1, y_2, \cdots, y_n)$-space onto $(\eta_1, \eta_2, \cdots, \eta_n)$-space, represented by

$$\eta_i=\frac{y_i}{|y|}\rho(|y|)=y_i\lambda(|y|) \qquad (i=1, 2, \cdots, n). \tag{3}$$

Since $|\eta|=\rho(|y|)$, (3) yields immediately the inverse

$$y_i=\frac{\eta_i}{|\eta|}r(|\eta|) \qquad (i=1, 2, \cdots, n). \tag{4}$$

Thus (3) maps topologically the whole $(y_1, y_2, \cdots, y_n)$-space onto the whole $(\eta_1, \eta_2, \cdots, \eta_n)$-space. Now we have for the partial derivatives,

$$\frac{\partial\eta_i}{\partial y_i}=\lambda(|y|)+\frac{y_i^2}{|y|}\lambda'(|y|) \qquad (i=1, 2, \cdots, n)$$

and

$$\frac{\partial\eta_i}{\partial y_j}=\frac{y_i y_j}{|y|}\lambda'(|y|) \qquad (i\neq j;\ i, j=1, 2, \cdots, n)$$

and they are continuous functions of y and moreover we have

$$\frac{\partial(\eta_1, \eta_2, \cdots, \eta_n)}{\partial(y_1, y_2, \cdots, y_n)}=\lambda^{n-1}\{\lambda+|y|\lambda'\}>0.$$

Hence $\partial y_i/\partial\eta_j$ $(i, j=1, 2, \cdots, n)$ are also continuous functions of η.

(1) is transformed by (3), x being unchanged, to the system

$$\frac{d\eta_i}{dx}=\lambda(|y|)f_i(x, y)+\frac{y_i\lambda'(|y|)}{|y|}(y_1f_1+y_2f_2+\cdots+y_nf_n)$$
$$(i=1, 2, \cdots, n).$$

Consider its second members as functions of $(x, \eta_1, \eta_2, \cdots, \eta_n)$, written $g_i(x, \eta_1, \eta_2, \cdots, \eta_n)$ $(i=1, 2, \cdots, n)$, then we obtain a system of n equations in the n unknowns $\eta_1, \eta_2, \cdots, \eta_n$:

$$\text{(5)} \qquad \frac{d\eta_i}{dx}=g_i(x, \eta_1, \eta_2, \cdots, \eta_n) \qquad (i=1, 2, \cdots, n)$$

where g_i are continuous functions in the region $[0\leqq x\leqq a, |\eta|<+\infty]$. Since

$$f(r)\leqq\{\lambda(r)\}^{\sigma/2}\leqq\{\lambda(r)\}^{\sigma}$$

and

$$\lambda'(r)<\frac{1}{\sigma}\frac{\{\lambda(r)\}^{1+\sigma}}{\{f(r)\}^2},$$

we have

$$\begin{aligned}|g(x, \eta)|&\leqq|f(x, y)|\{\lambda(|y|)+|y|\lambda'(|y|)\}\\ &\leqq f(|y|)\{\lambda(|y|)+f(|y|)\lambda'(|y|)\}\\ &<(1+1/\sigma)\{\lambda(|y|)\}^{1+\sigma},\end{aligned}$$

and finally

$$\frac{|g(x, \eta)|}{|\eta|^{1+\sigma}}<\frac{1+1/\sigma}{|y|^{1+\sigma}}.$$

Consequently we have

$$\text{(6)} \qquad \lim_{|\eta|\to+\infty}\frac{|g(x, \eta)|}{|\eta|^{1+\sigma}}=0,$$

uniformly for x in $0\leqq x\leqq a$.

Now consider the second mapping effected by

$$\text{(7)} \qquad \begin{cases} Y_i=\dfrac{2\eta_i}{1+|\eta|^2} & (i=1, 2, \cdots, n), \\ Y_{n+1}=1-\dfrac{2}{1+|\eta|^2} \end{cases}$$

which maps topologically the whole $(\eta_1, \eta_2, \cdots, \eta_n)$-space, the point at infinity $|\eta|=+\infty$ being added, onto the whole unit sphere in $(Y_1, Y_2, \cdots, Y_{n+1})$-space:

$$Y_1^2+Y_2^2+\cdots+Y_{n+1}^2=1$$

whose pole $(0, 0, \cdots, 0, 1)$ is the image of the point at infinity $|\eta|=+\infty$. The inverse of (7) is given by

$$(8)\qquad \eta_i=\frac{Y_i}{1-Y_{n+1}}\qquad (i=1, 2, \cdots, n).$$

The system (5) is transformed by (7), x unchanged, to a system of the form

$$(9)\qquad \begin{cases} \dfrac{dY_i}{dx}=h_i(x, Y_1, Y_2, \cdots, Y_{n+1}) \qquad (i=1, 2, \cdots, n+1), \\ Y_1^2+Y_2^2+\cdots+Y_{n+1}^2=1\,, \end{cases}$$

where

$$h_i(x, Y_1, Y_2, \cdots, Y_{n+1})=\sum_{j=1}^{n}\frac{\partial Y_i}{\partial \eta_j}g_j(x, \eta)\qquad (i=1, 2, \cdots, n+1)$$

and

$$(10)\qquad Y_1h_1+Y_2h_2+\cdots+Y_{n+1}h_{n+1}=0.$$

The second members of the former equations of (9) are not defined for $Y_{n+1}=1$, though it seems clear that, for any fixed value of σ such that $0<\sigma\leqq 1$, they converge uniformly to zero as $Y_{n+1}\to 1-0$. And therefore, if we put

$$h_i(x, 0, 0, \cdots, 0, 1)=0\qquad (i=1, 2, \cdots, n+1),$$

h_i are continuous functions on the whole surface

$$S_{n+1}\,:\, 0\leqq x\leqq a,\qquad Y_1^2+Y_2^2+\cdots+Y_{n+1}^2=1$$

in $(x, Y_1, Y_2, \cdots, Y_{n+1})$-space. Finally, consider the product of the mappings (3) and (7),

$$(11)\begin{cases} Y_i=\dfrac{2y_i\lambda(|y|)}{1+\{\rho(|y|)\}^2}\equiv Y_i(y_1, y_2, \cdots, y_n)\qquad (i=1, 2, \cdots, n) \\ Y_{n+1}=1-\dfrac{2}{1+\{\rho(|y|)\}^2}\equiv Y_{n+1}(y_1, y_2, \cdots, y_n) \end{cases}$$

which maps topologically the whole $(y_1, y_2, \cdots, y_n)$-space, the point at infinity $|y|=+\infty$ being added, onto the whole unit sphere in $(Y_1, Y_2, \cdots, Y_{n+1})$-space. Then we have the following

Theorem 1. (1) *is transformed by means of* (11), *x unchanged, to* (9) *whose second members are continuous on the surface* S_{n+1} *in* $(x, Y_1, Y_2, \cdots, Y_{n+1})$-*space. The segment*

$$L\,:\, 0\leqq x\leqq a,\qquad Y_1=Y_2=\cdots=Y_n=0,\qquad Y_{n+1}=1$$

may be regarded as the image of $|y|=+\infty$.

§ 2. Applications

Theorem 2.[1] *A necessary and sufficient condition for every solution of* (1) *to have an end point, whose x-coordinate is equal to a, is that there exists a positive continuous function* $\varphi(x, y_1, y_2, \cdots, y_n)$, *defined in* E_{n+1} *with the first partial derivatives which is also continuous in the interior of* E_{n+1}, *and that* φ *converges uniformly in* $0 \leqq x \leqq a$ *to zero as* $|y| \to +\infty$, *and moreover that, in the interior of* E_{n+1}, *we have*

$$\frac{\partial \varphi}{\partial x}+\frac{\partial \varphi}{\partial y_1} f_1+\frac{\partial \varphi}{\partial y_2} f_2+\cdots+\frac{\partial \varphi}{\partial y_n} f_n \geqq 0 . \tag{12}$$

Proof. Consider a region

$$R_{n+2}: 0 \leqq x \leqq a, \quad |Y_i| \leqq b \quad (i=1, 2, \cdots, n+1)$$

where b is such a constant as $b>1$. For every point $P(x, Y_1, Y_2, \cdots, Y_{n+1})$ in R_{n+2}, let $P_0(x, Y_{01}, Y_{02}, \cdots, Y_{0n+1})$ denote the point in which the ray issuing from the point $(x, 0, 0, \cdots, 0)$ and passing through P cuts S_{n+1} and put

$$\frac{Y_1}{Y_{01}}=\frac{Y_2}{Y_{02}}=\cdots=\frac{Y_{n+1}}{Y_{0n+1}}=\delta .$$

Then if we define

$$h_i^*(x, Y_1, Y_2, \cdots, Y_{n+1})=\delta h_i(x, Y_{01}, Y_{02}, \cdots, Y_{0n+1})$$

h_i^* are continuous functions in R_{n+2}. Evidently we get

$$h_i^*=h_i \quad \text{on } S_{n+1}$$

and

$$Y_1 h_1^*+Y_2 h_2^*+\cdots+Y_{n+1} h_{n+1}^*=0 \quad \text{in } R_{n+2}.$$

Now in R_{n+2}, consider the system

$$\frac{dY_i}{dx}=h^*(x, Y_1, Y_2, \cdots, Y_{n+1}) \qquad (i=1, 2, \cdots, n+1) \tag{13}$$

which is an extension of the system (9). It is not difficult to show that a necessary and sufficient condition for every solution of (1) to have an end point on $x=a$ is that the segment L is the unique solution of (13) (or (9)) arriving at the point $N(a, 0, 0, \cdots, 0, 1)$.

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1) H. Okamura, Functional Equations (in Japanese), Vol. 32 (1942), pp. 21–27.

Let P be a variable point $(x, Y_1, Y_2, \cdots, Y_{n+1})$ in R_{n+2}, then we can define the D-function $D(P, N)$[2] with regard to (13). If we put

$$\psi(x, Y_1, Y_2, \cdots, Y_{n+1})=D(P, N),$$

ψ can be replaced by a function $\psi_1(x, Y_1, Y_2, \cdots, Y_{n+1})$ which has continuous first partial derivatives in the interior of R_{n+2} and the same properties as those of ψ in R_{n+2}.[3] Put

$$\varphi(x, y_1, y_2, \cdots, y_n)=\psi_1(x, Y_1(y_1, y_2, \cdots, y_n), \cdots, Y_{n+1}(y_1, y_2, \cdots, y_n)),$$

then, if L is the unique solution of (13) arriving at the point N, φ possesses the properties required in the theorem. Therefore the condition is necessary.

It is easy to show that the condition is sufficient.

Example 1. If $f=O(|y|)$ as $|y|\to+\infty$, i.e., if there be such a positive number k that for $0\leqq x\leqq a$ and $|y|\geqq r_0$, r_0 being a positive number, we have

$$\frac{|f|}{|y|}\leqq k,$$

put

$$\varphi(x, y_1, y_2, \cdots, y_n)=|y|^{-1/k}e^{nx}.$$

We have proved in § 1 that, concerning the second members of the system (5), we have $g(x, \eta)=o(|\eta|^{1+\sigma})$ as $\eta\to+\infty$ (cf. (6)). Now the above example shows that in general $g(x, \eta)\fallingdotseq O(|\eta|)$ as $|\eta|\to+\infty$. For, if $g(x, \eta)=O(|\eta|)$, every solution of (5) has an end point on $x=a$ and therefore every solution of (1) has also an end point on $x=a$.

Example 2. If there be a positive continuous function $\phi(u)$ of u for $u\geqq 0$, $\int_0^{+\infty}\frac{du}{\phi(u)}=+\infty$ and $|f|\leqq\phi(|y|)$, then put

$$\varphi(x, y_1, y_2, \cdots, y_n)=e^{nx-\int_0^{|y|}\frac{du}{\phi(u)}}.$$

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2) H. Okamura, "*Sur l'Unicité des Solutions d'un Système d'Équations différentielles ordinaires*", Mem. Coll. Sci. Kyoto Univ. **A. 23** (1941), pp. 225–231; H. Okamura, "*Sur une sorte de distance relative à un système différentiel*", Proc. Physico-Math. Soc. of Japan, 3rd series, Vol. **25** (1943), pp. 514–523; K. Hayashi and T. Yoshizawa, "*New Treatise of Solutions of a System of Ordinary Differential Equations and its Application to the Uniqueness Theorems*", Mem. Coll. Sci. Kyoto Univ. **A. 26** (1951), pp. 225–233.

3) H. Okamura, "*Condition nécessaire et suffisante remplie par les Équations différentielles ordinaires sans points de Peano*", Mem. Coll. Sci. Kyoto Univ. **A, 24** (1942), pp. 24–27.

Theorem 3. *A necessary and sufficient condition for every solution of* (1) *to have an end point, whose x-coordinate is equal to a, is that, given any positive number* α, *there exists a positive number* $\beta(\alpha)$ *such that, for any solution* $y=y(x)$ *of* (1) *through a point* (x_0, y_0) *arbitrary in* E_{n+1}, *provided that* $|y(x_0)|\leqq\alpha$, *we have* $|y(x)|<\beta(\alpha)$ *so long as* $y=y(x)$ *lies in* E_{n+1} *for* $x_0\leqq x\leqq a$.

Proof. Since the sufficiency of the condition is easily verified, we will prove only its necessity.

At first consider the region

$$E_{n+1}^{\alpha}\ :\ 0\leqq x\leqq a, \quad |y|\leqq\alpha$$

which is a bounded closed region. Let S_{n+1}^{α} be the image of E_{n+1}^{α} under the mapping (11), then S_{n+1}^{α} is a bounded closed set, and hence, the set of all the points, which are on any solutions of (9) going to the right from any points in S_{n+1}^{α}, is a closed set.[4] Under the assumption of the theorem, this set has no point common with the segment L. Consequently Y_{n+1}-coordinate of every point in this set is smaller than a positive number $\gamma(\alpha)(<1)$. Hence we consider such a positive number $\beta(\alpha)$ that $\gamma=1-(1/\{1+\rho(\beta)\}^2)$. Then, for any point (x, y) in the inverse image of this set under (11) (i.e., in the set of all points which are on any solutions of (1) going to the right from any points in E_{n+1}^{α}), we get $|y|<\beta(\alpha)$.

Corollary. *The content of Theorem 2 is also verified when we suppose* φ *to be defined merely for the region* $[0\leqq x\leqq a,\ |y|\geqq r_0]$, r_0 *being a positive constant.*

Remark. Conditions for every solution of (1) to have an end point on $x=0$ may be obtained in the same way and however, for instance, the inequality (12) may be replaced by the following

$$\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial y_1}f_1+\frac{\partial\varphi}{\partial y_2}f_2+\cdots+\frac{\partial\varphi}{\partial y_n}f_n\leqq 0\,. \tag{14}$$

§ 3. Differential equation of the second order

In this section we will investigate the differential equation

$$\frac{d^2y}{dx^2}=f\Big(x,\ y,\ \frac{dy}{dx}\Big), \tag{15}$$

as a special case of (1).

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4) It may be easily verified by means of the equicontinuity of solutions of (9): the equicontinuity is oweing to the boundedness of h_i.

Let D be a bounded closed region $[0 \leqq x \leqq a,\ \underline{w}(x) \leqq y \leqq \overline{w}(x)]$ in xy-plane, where $\underline{w}(x)$, $\overline{w}(x)$ and their derivatives are continuous in $0 \leqq x \leqq a$ and $\underline{w}(x) < \overline{w}(x)$ in $0 < x < a$. And let D^* be a three dimensional region of the point (x, y, z), where $(x, y) \in D$ and $-\infty < z < +\infty$. Moreover, suppose $f(x, y, z)$, defined and continuous in D^*.

Now consider the system of differential equations

$$\frac{dy}{dx} = z, \quad \frac{dz}{dx} = f(x, y, z) \tag{16}$$

which is equivalent to (15), then it becomes by the vector notation

$$\frac{d\mathfrak{y}}{dx} = \mathfrak{F}(x, \mathfrak{y}),$$

where $\mathfrak{y} = (y, z)$ and $\mathfrak{F}(x, \mathfrak{y}) = (z, f(x, y, z))$. Of course, we can apply the mapping (11) to (16), but this time, we proceed by means of a mapping from (y, z)-space into (η, ζ)-space represented by

$$\begin{cases} \eta = Y(y, z) \equiv y\displaystyle\int_{z^2}^{+\infty} \frac{dt}{\{f(t)\}^3}, \\ \zeta = Z(z) \equiv \begin{cases} \displaystyle\int_0^z \frac{dt}{\{f(t)\}^2} & \text{for } z \geqq 0, \\ -\displaystyle\int_0^{|z|} \frac{dt}{\{f(t)\}^2} & \text{for } z < 0, \end{cases} \end{cases} \tag{17}$$

where $f(t)$ is the greatest value of 1, t and $\max\limits_{\substack{(x,y)\in D \\ |z| \leqq t}} |f(x, y, z)|$ and then $f(t)$ is a positive continuous function of t. Clearly (17) may be solved for y and z. Since

$$\frac{\partial Y}{\partial y} = \int_{z^2}^{+\infty} \frac{dt}{\{f(t)\}^3} > 0, \quad \frac{\partial Y}{\partial z} = -2yz\frac{1}{\{f(z^2)\}^3}$$

and

$$\frac{dZ}{dz} = \frac{1}{\{f(|z|)\}^2} > 0,$$

these derivatives are continuous in D^*.

Now put $b = \displaystyle\int_0^{+\infty} \frac{dt}{\{f(t)\}^2}$ (>0), $\overline{W}(x, \zeta) = Y(\overline{w}(x), Z^{-1}(\zeta))$ and $\underline{W}(x, \zeta) = Y(\underline{w}(x), Z^{-1}(\zeta))$, $Z^{-1}(\zeta)$ being the inverse function of $Z(z)$, $Z(z)$ converges to b or $-b$ respectively as $z \to +\infty$ or $-\infty$

and $\overline{W}(x, \zeta)$ and $\underline{W}(x, \zeta)$ converge uniformly for x in $0 \leqq x \leqq a$ to zero as $\zeta \to \pm b$. If we put $\overline{W}(x, b) = \overline{W}(x, -b) = \underline{W}(x, b) = \underline{W}(x, -b) = 0$, $\overline{W}(x, \zeta)$ and $\underline{W}(x, \zeta)$ are continuous in $[0 \leqq x \leqq a, -b \leqq \zeta \leqq b]$ and we get $\underline{W}(x, \zeta) < \overline{W}(x, \zeta)$ in $[0 < x < a, -b < \zeta < b]$. Their partial derivatives $\overline{W}_x$, $\overline{W}_\zeta$, $\underline{W}_x$ and $\underline{W}_\zeta$ are also continuous.

Therefore, the mapping (17), x unchanged, maps topologically the region D, the points at infinity $|z| = +\infty$ being added, onto the bounded closed region

$$\mathfrak{D} : 0 \leqq x \leqq a, \quad -b \leqq \zeta \leqq b, \quad \underline{W}(x, \zeta) \leqq \eta \leqq \overline{W}(x, \zeta)$$

in (x, η, ζ)-space. The segment

$$L_1 : 0 \leqq x \leqq a, \quad \eta = 0, \quad \zeta = b$$

may be regarded as the image of $z = +\infty$ and the segment

$$L_2 : 0 \leqq x \leqq a, \quad \eta = 0, \quad \zeta = -b$$

as the image of $z = -\infty$.

The system (16) is transformed by (17), x being unchanged, to the system

$$\frac{d\eta}{dx} = z \int_{z^2}^{+\infty} \frac{dt}{\{f(t)\}^3} - 2yz \frac{f(x, y, z)}{\{f(z^2)\}^3},$$

$$\frac{d\zeta}{dx} = \frac{f(x, y, z)}{\{f(|z|\}^2}.$$

Consider its second members as functions of (x, η, ζ), written $g_1(x, \eta, \zeta)$ and $g_2(x, \eta, \zeta)$ respectively, they are defined and continuous in $\mathfrak{D}$, except on the segments L_1 and L_2. Since they converge uniformly to zero as $\zeta \to \pm b$, put $g_1(x, 0, b) = g_1(x, 0, -b) = g_2(x, 0, b) = g_2(x, 0, -b) = 0$. Then g_1 and g_2 are continuous functions in $\mathfrak{D}$ and we obtain the following system

$$(18) \qquad \begin{cases} \dfrac{d\eta}{dx} = g_1(x, \eta, \zeta) \\[2ex] \dfrac{d\zeta}{dx} = g_2(x, \eta, \zeta) \end{cases}$$

whose second members are continuous on the bounded closed region $\mathfrak{D}$ in (x, η, ζ)-space.

Theorem 4.[5] *A necessary and sufficient condition for any solution of* (15) *going to the right from any point in* D *to have, preserving the continuity of its derivative, an end point on the boundary of* D *is that there exists a positive continuous function* $\Phi(x, y, z)$ *defined in* D^* *as follows; namely* $\Phi(x, y, z)$ *converges uniformly for* $(x, y) \in D$ *to zero as* $z \to \pm\infty$ *and satisfies the Lipschitz condition with regard to* (y, z), *i.e., given any positive number* c, *there exists such a positive constant* K_c *that, if* $(x, y) \in D$, $(x, \bar{y}) \in D$, $|z| \leqq c$ *and* $|\bar{z}| \leqq c$, *we have*

$$|\Phi(x, y, z) - \Phi(x, \bar{y}, \bar{z})| \leqq K_c(|y-\bar{y}| + |z-\bar{z}|). \tag{19}$$

And finally, for points of D^*, *we have*

$$\underline{D}^+_{[F]}\Phi(x, y, z) \geqq 0^{6)}. \tag{20}$$

Proof. Since g_1 and g_2 are continuous on the bounded closed region $\mathfrak{D}$, every solution of (18) has its end points on the boundary of $\mathfrak{D}$. Therefore a necessary and sufficient condition for any solution of (15) going to the right from any point in D to have, preserving the continuity of its derivative, an end point on the boundary of D is that the segment L_1 is the unique solution of (18) arriving at the point $A(a, 0, b)$ and the segment L_2 the unique solution arriving at the point $B(a, 0, -b)$.

Let P be a variable point (x, η, ζ) in $\mathfrak{D}$, then we can define two D-functions[7] $D(P, A)$ and $D(P, B)$. Now put

$$\Psi(x, \eta, \zeta) = \min\{D(P, A), D(P, B)\}$$

and

$$\Phi(x, y, z) = \Psi(x, Y(y, z), Z(z)).$$

Then, if L_1 is the unique solution of (18) arriving at A and L_2 is the unique solution arriving at B, Φ possesses the properties required in the theorem.

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5) H. Okamura, Functional Equations (in Japanese), Vol. **27** (1941), pp. 27–35; T. Yoshizawa, "*Note on the non-increasing solutions of* $y''=f(x, y, y')$", Mem. Coll. Sci. Kyoto Univ. **A. 27** (1952), p. 158, lemma 2.

6) (20) is Nagumo's notation. Cf. H. Okamura, "*Sur une sorte de distance relative à un système différentiel*", Proc. Physico-Math. Soc. of Japan, 3rd series, Vol. **25** (1943), pp. 520–521; T. Yoshizawa, "*On the Evaluation of the Derivatives of Solutions of* $y''=f(x, y, y')$", Mem. Coll. Sci. Kyoto Univ. **A. 28** (1953), p. 28.

7) Since $\mathfrak{D}$ is not a cuboid we need to define D-function by the Okamura's second method. Cf. H. Okamura, loc. cit. 6).

It is easy to show that the condition is sufficient.

Corollary. *In the condition of the theorem,* Φ *can be replaced by two functions* $\Phi_1(x, y, z)$ *and* $\Phi_2(x, y, z)$ *as follows; namely* $\Phi_1(x, y, z)$ *is defined in a region*

$$\Delta_1 : (x, y) \in D, \quad k_1 \leqq z < +\infty, \ k_1 : \text{constant}$$

and converges uniformly to zero as $z \to +\infty$, *and* $\Phi_2(x, y, z)$ *is defined in a region*

$$\Delta_2 : (x, y) \in D, \ -\infty < z \leqq k_2, \ k_2 : \text{constant}$$

and converges uniformly to zero as $z \to -\infty$.

For the proof, put

$$\Psi_1(x, \eta, \zeta) = D(P, A),$$

$$\Psi_2(x, \eta, \zeta) = D(P, B),$$

and then, put

$$\Phi_1(x, y, z) = \Psi_1(x, Y(y, z), Z(z)),$$

$$\Phi_2(x, y, z) = \Psi_2(x, Y(y, z), Z(z)).$$

Remark 1. A condition for any solution of (15) going to the left from any point in D, to have an end point on the boundary of D, may be obtained in the same way with only the modification that the inequality (20) shall be replaced by the following

(21) $$\overline{D}^{-}_{[\Gamma]}\Phi(x, y, z) \leqq 0.$$

Remark 2. $\Psi(x, \eta, \zeta)$ can be modified[8] to have bounded continuous partial derivatives in the interior of $\mathfrak{D}$. Therefore, $\Phi(x, y, z)$ can be also modified to have continuous partial derivatives in the interior of D^*. And then, in the interior of D^*, (20) reduces to

(22) $$\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} z + \frac{\partial \Phi}{\partial z} f(x, y, z) \geqq 0.$$

Example.[9] If there be a positive continuous function $\varphi(u)$ of u, defined for $u \geqq 0$, and $\int_0^\infty \frac{u\,du}{\varphi(u)} = +\infty$ and $|f(x, y, z)| \leqq \varphi(|z|)$, then put

8) H. Okamura, loc. cit. 3).

9) M. Nagumo, "*Über die Differentialgleichung* $y'' = f(x, y, y')$", Proc. Physico-Math. Soc. of Japan, 3rd series, Vol. **19** (1937), pp. 861-866.

$$\Phi_1(x, y, z) = e^{y - \int_0^z \frac{u du}{\varphi(u)}} \quad \text{for } z \geqq 0$$

and

$$\Phi_2(x, y, z) = e^{-y - \int_0^{|z|} \frac{u du}{\varphi(u)}} \quad \text{for } z \leqq 0 .$$

Theorem 5.[10] *A necessary and sufficient condition for any solution of* (15) *going to the right from any point in D to have, preserving the continuity of its derivative, an end point on the boundary of D is that, given any positive number α, there exists a positive number $\beta(\alpha)$ such that, for any solution $y=y(x)$ of* (15) *through a point (x_0, y_0) arbitrary in D, provided that $|y'(x)| \leqq \alpha$, we have $|y'(x)| < \beta(\alpha)$ as long as $y=y(x)$ lies in D for $x_0 \leqq x \leqq a$.*

For the proof proceed as in the proof of Theorem 3.

Theorem 6. *A necessary and sufficient condition for any solution of* (15) *going to the right from any interior point in D to reach, preserving the continuity of its derivative, at a point of the boundary of D is that there exists a positive continuous function $\Phi^*(x, y, z)$ defined in the interior of D^* as follows; namely $\Phi^*(x, y, z)$ has continuous partial derivatives in the interior of D^* and satisfies*

$$\frac{\partial \Phi^*}{\partial x} + \frac{\partial \Phi^*}{\partial y} z + \frac{\partial \Phi^*}{\partial z} f(x, y, z) \geqq 0 \tag{23}$$

and converges uniformly for $(x, y) \in D$ to zero as $z \to \pm \infty$.

Proof. Proceed as in the proof of Theorem 4. Since, this time, the inequality (20) is necessary only in the interior of D^*, it may be replaced by (23).

Remark. Theorems 4, 5 and 6 can be also verified when D is supposed merely as a bounded closed region in xy-plane.[11] If D is the region given in the beginning of this section we obtain the following

Theorem 7. *The condition of Theorem 6 is necessary and sufficient for any solution of* (15) *going to the right from any point in D to have, preserving the continuity of the derivative, an end point on the boundary of D.*

Proof. Since the necessity of the condition may be easily verified, we will prove only its sufficiency.

Suppose, on the contrary, that there exist a solution $y=y(x)$

10) M. Nagumo, loc. cit. 10); T. Yoshizawa, loc. cit. 6), p. 27.

11) T. Yoshizawa, loc. cit. 6), p. 30, foot notes 1).

of (15) going to the right from a point (x_0, y_0) in D whose image under (17), being a solution of (18), tends to a point (x_1, η_1, ζ_1) on the segments L_1 or L_2. If the condition in Theorem 6 holds, as $x \to x_1 - 0$ the solution $y = y(x)$ tends to a point $P_1(x_1, y_1)$ of the boundary of D and we get $\lim_{x \to x_1 - 0} y'(x) = +\infty$ (or $-\infty$). And, in any neighborhood of the point P_1 there is at least a boundary point $P_2(x_2, y_2)$ $(x_2 < x_1)$ of D on the solution $y = y(x)$. At the point P_2 the solution $y = y(x)$ has to be tangent to the boundary of D. On the other hand, there is such a constant K that $|\underline{w}'(x)|$, $|\overline{w}'(x)| < K$ and therefore $|y'(x_2)| < K$[12]. It contradicts the relation $\lim_{x \to x_1 - 0} y'(x) = +\infty$ (or $-\infty$). Hence the condition is sufficient.

Corollary. *In the conditions of Theorem 6 and 7, Φ^* can be replaced by two functions as in the corollary of Theorem 4.*

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12) The theorem may be proved when the curve $y = \overline{w}(x)$ (or the curve $y = \underline{w}(x)$) consists of a finite number of arcs of similar properties.