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Kelvin principle and some inequalities in the theory of functions I

By

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1. Introduction. Recently Z. Nehari[8] has, by means of Dirichlet principle, obtained various inequalities of function theory and potential theory which may be reduced to statements regarding the properties of harmonic functions with constant boundary values, that is, functions obtainable from the Green's function. While his method is very useful to deduce some inequalities important in the theory of conformal maps, it seems difficult to derive, by this method, several inequalities which may be reduced to statements regarding the properties of harmonic functions with a vanishing normal derivative on some of boundary components of a given domain.

It is the aim of the present paper to show that the inequalities of this type can be deduced from the classical Kelvin principle[7,10]. Although some of results obtained in this paper are not new, the method used there will suggest a more or less systematic treatment of the inequalities of this type.

2. Kelvin principle and a monotonic functional. Let q be any vector function defined in a given domain D, satisfying the following conditions;

$$(1)\qquad \begin{aligned} &\operatorname{div} q=0 \quad \text{in } D \\ &qn=f(s) \quad \text{on } C \text{ (boundary of } D), \end{aligned}$$

n being the unit vector in the direction of outward normal and $f(s)$ a function of arc-length s defined on C satisfying the condition $\int_C f(s)ds=0$. Under the latter condition there exists a harmonic function ϕ in D, satisfying the condition

$$\frac{\partial\phi}{\partial n}=f(s) \quad \text{on } C,$$

only up to an additive constant. We have then

$$\iint_D |\operatorname{grad} \phi|^2 d\tau \leqq \iint_D |q|^2 d\tau \qquad (d\tau = dxdy)\,. \tag{2}$$

This is the result of Kelvin (Thomson) principle.

Remark.[10] The minimum property of ϕ in question holds good allowing a more general class of vector functions q. They satisfy (1) except along a closed or open curve in the domain D along which q and qn must have one-sided limits and qn must be continuous.

The domain we shall consider will be assumed to be bounded by a finite number of closed analytic curves and they will be embedded in a given closed Riemann surface R of finite genus. The symbol $S(z)$ will be used to denote a *singularity function* with the following properties: $S(z)$ is real, harmonic, and single-valued on R, with the possible exception of a finite number of points at which $S(z)$ has specified singularities.

The following result indicates a monotonic functional associated with $S(z)$.

THEOREM I. *Let D and D_1 be two domains embedded in R such that $D \subset D_1$ and that $D_1 - D$ contains no singularities of $S(z)$, and let C and C_1 denote the boundaries of D and D_1, respectively. Let further $p(z)$ denote the function which has a vanishing normal derivative on C and is such that $p(z) + S(z)$ is harmonic in D. If $p_1(z)$ denotes the corresponding function associated with D_1, then*

$$\int_C p(z) \frac{\partial S}{\partial n} ds \geqq \int_{C_1} p_1(z) \frac{\partial S}{\partial n} ds\,, \tag{3}$$

where the differentiation is performed with respect to the outer normal.

Proof. In the above principle we put

$$q = \left[\frac{\partial(p+S)}{\partial x},\ \frac{\partial(p+S)}{\partial y}\right] \quad \text{in } D$$

$$= \left[\frac{\partial S}{\partial x},\ \frac{\partial S}{\partial y}\right] \quad \text{in } D_1 - D\,,$$

and

$$\phi = P_1 + S \quad \text{in } D_1.$$

Using the fact that

$$\operatorname{div} q = 0 \quad \text{in } D \text{ and in } D_1 - D, \qquad qn = \frac{\partial S}{\partial n} = \frac{\partial \phi}{\partial n} \qquad \text{on } C_1,$$

and

$$\frac{\partial p}{\partial n}=0 \quad \text{on } C,$$

and considering the additional remark, we can apply the above principle in this case. Therefore it holds that

$$(4) \qquad (p_1+S,\ p_1+S)_{D_1} \leqq (p+S,\ p+S)_D + (S,\ S)_{D_1-D},$$

where $(u, u)_D=\iint_D (u_x^2+u_y^2)d\tau$.

From the Green's theorem and the assumptions, the left-hand side of (4) equals to

$$\int_{C_1} (p_1+S)\frac{\partial S}{\partial n}ds$$

and the right-hand side of (4) equals to

$$\int_C (p+S)\frac{\partial S}{\partial n}ds+\int_{C_1} S\frac{\partial S}{\partial n}ds-\int_C S\frac{\partial S}{\partial n}ds.$$

Thus we obtain the required result

$$\int_{C_1} p_1\frac{\partial S}{\partial n}ds \leqq \int_C p\frac{\partial S}{\partial n}ds. \qquad \text{Q.E.D.}$$

3. Conformal mapping on radial slit domain. As the first application of Theorem I, consider the Neumann function $N(z, \zeta)$ of a finite plane domain D defined by the following properties:

(i) $N(z, \zeta)$ is harmonic in D, except at the point $z=\zeta \in D$;

(ii) $N(z, \zeta)+\log|z-\zeta|$ is harmonic at $z=\zeta$;

(iii) $\partial N(z, \zeta)/\partial n=-2\pi/L$ for $z \in C$, L=total length of C.

In the Theorem I we put

$$p(z)=N(z, \zeta)-N(z, \eta) \qquad (\zeta, \eta \in D)$$

and

$$S(z)=\log|z-\zeta|-\log|z-\eta|.$$

Obviously $p(z)$ and $S(z)$ satisfy the assumptions of Theorem I in the case where R is the whole plane. And further consider the analytic functions $q(z)$ and $\sigma(z)$ such that

$$p(z)=Re\{q(z)\} \qquad \text{and} \qquad S(z)=Re\{\sigma(z)\}.$$

From the relation

$$\frac{\partial(p+S)}{\partial n}ds=Re\left\{\frac{1}{i}(q'(z)+\sigma'(z))dz\right\} \qquad \text{on } C$$

and $Im\{q(z)\}=$const. on each boundary component of C because of $\partial p/\partial n=0$ on C, it follows that

$$\int_C p\frac{\partial S}{\partial n}ds=\int_C p\frac{\partial(p+S)}{\partial n}ds$$

$$=Re\left[\frac{1}{i}\int_C p\{q'(z)+\sigma'(z)\}\,dz\right]$$

$$=Re\left[\frac{1}{i}\int_C q(z)\{q'(z)+\sigma'(z)\}\,dz\right] \tag{5}$$

$$=Re\left[\frac{-1}{i}\int_C q'(z)\{q(z)+\sigma(z)\}\,dz\right], \quad \text{(by integration by parts)}$$

$$=2\pi Re\,[q(\zeta)+\sigma(\zeta)-q(\eta)-\sigma(\eta)], \quad \text{(by residue theorem).}$$

It is well known[1] that the analytic function

$$Q(z)=\exp\{-q(z)\} \tag{6}$$

conformally maps the domain D onto the whole plane slit along radial segments directed towards the origin such that the point $z=\zeta$ corresponds to the origin and the point $z=\eta$ to infinity. Thus, putting

$$Q(z)=A_0(z-\zeta)+\cdots\cdots \text{ in the vicinity of } z=\zeta$$
$$=A_\infty(z-\eta)^{-1}+\cdots \text{ in the vicinity of } z=\eta,$$

the above integral equals to

$$2\pi Re\left[\log\left(\frac{z-\zeta}{Q(z)\,(z-\eta)}\right)_{z=\zeta}-\log\left(\frac{z-\zeta}{Q(z)\,(z-\eta)}\right)_{z=\eta}\right]$$

$$=2\pi\log\left(\left|\frac{A_\infty}{A_0}\right||\zeta-\eta|^{-2}\right).$$

Hence we obtain from Theorem I the following

COROLLARY 1. *Let $Q(z)$ be an analytic function which maps a finite domain D onto the whole plane slit along the radial segments directed towards the origin, and $Q(\zeta)=0$ $(\zeta\in D)$ and has the expansion in the vicinity of $z=\eta$ $(\eta\in D)$*

$$Q(z)=\frac{1}{z-\eta}+\cdots.$$

Then $|Q'(\zeta)|$ is a monotone increasing domain functional.

In the special case[6] where the domain D is the circle $|z|<R$,

$$(7) \qquad Q(z)=\frac{1}{\eta-\zeta}\,\frac{1-|\eta/R|^2}{1-(\bar{\zeta}\eta/R^2)}\,\frac{z-\zeta}{z-\eta}\,\frac{1-(\bar{\zeta}z/R^2)}{1-(\bar{\eta}z/R^2)}.$$

Using Corollary 1 and considering the limiting case $R\to\infty$ in (7), we obtain the following

COROLLARY 2. *Let $Q(z)$ $(Q(\zeta)=0)$ be a normalized radial slit mapping function. Then it holds that*

$$(8) \qquad |Q'(\zeta)|\leqq\frac{1}{|\zeta-\eta|^2}.$$

By means of a linear transformation $z'=(z-\zeta)/(z-\eta)$, we easily obtain from Corollary 2 the following well known results[5),9)].

COROLLARY 3. *Let D be a domain containing $z=0$ and $z=\infty$, and $Q(z)$ be a radial slit mapping function such that $Q(0)=0$ and $\lim_{z\to\infty} Q(z)/z=1$. Then it holds that*

$$(9) \qquad |Q'(0)|\leqq 1$$

and is a monotone increasing domain functional.

In the special case where the domain D is the whole plane, $Q(z)=z$ and $Q'(0)=1$. Therefore (9) is easily obtained.

This Corollary is equivalent to the following result:

Let $f(z)$ $(f(\zeta)=0)$ be univalent in a plane domain D and $f(z)=(z-\eta)^{-1}+\cdots$ in the vicinity of $z=\eta$ $(\eta\in D)$. And further let $Q(z)$ be the normalized radial slit mapping function of Corollary 1. Then it follows that

$$(10) \qquad |f'(\zeta)|\geqq|Q'(\zeta)|.$$

4. Bergman kernel function. The second application of Theorem I concerns the Bergman kernel function $K_0(z,\zeta)$ of a plane domain D associated with the class of functions $f(z)$ which are regular analytic, $\iint_D|f(z)|^2d\tau<\infty$ and have single-valued integrals $\int^z f(z)\,dz$. It is well known[1)] that

$$(11) \qquad K_0(z,\zeta)=\frac{2}{\pi}\frac{\partial^2 N(z,\zeta)}{\partial z\partial\bar{\zeta}},$$

$N(z,\zeta)$ being the Neumann function of D and the differential operators $\partial/\partial z$ and $\partial/\partial\bar{z}$ are defined by

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\qquad \frac{\partial}{\partial\bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right),\qquad z=x+iy.$$

Further we introduce the function

$$(12)\qquad L_0(z,\zeta)=-\frac{2}{\pi}\frac{\partial^2 N(z,\zeta)}{\partial z\partial\zeta}=\frac{1}{\pi(z-\zeta)^2}+l_0(z,\zeta)$$

which plays an important role in the theory of conformal mapping. Both $K_0(z,\zeta)$ and the function $l_0(z,\zeta)$ defined in (12) are regular in D.

If $a_1,\cdots,a_n$ are complex numbers, then

$$(13)\qquad p(z)=Re\left\{\sum_{\nu=1}^{n}a_\nu\frac{\partial N(z,\zeta_\nu)}{\partial\zeta_\nu}\right\}$$

has a vanishing normal derivative on C by the property of the Neumann function and may be identified with $p(z)$ of Theorem I. The corresponding singularity function is

$$(14)\qquad S(z)=Re\left\{\sum_{\nu=1}^{n}a_\nu\frac{\partial\log|z-\zeta_\nu|}{\partial\zeta_\nu}\right\}=\frac{-1}{2}Re\left\{\sum_{\nu=1}^{n}\frac{a_\nu}{z-\zeta_\nu}\right\},$$

and $p(z)+S(z)$ is harmonic in D. If $q(z,\zeta)$ denotes the analytic function of z for which $Re\{q(z,\zeta)\}=N(z,\zeta)$, we easily obtain

$$(15)\qquad p(z)=\frac{1}{2}Re\left\{\sum_{\nu=1}^{n}\left[a_\nu\frac{\partial q(z,\zeta_\nu)}{\partial\zeta_\nu}+\bar{a}_\nu\frac{\partial q(z,\zeta_\nu)}{\partial\bar{\zeta}_\nu}\right]\right\}.$$

The expression in brackets of (15) has the following properties: (i) it is single-valued in D, since the periods of $q(z,\zeta)$ are pure imaginary, (ii) it has a constant imaginary part on each boundary component by the properties of Neumann function, (iii) its partial derivative $\partial/\partial z$ has the principal part such as $-a_\nu/(z-\zeta_\nu)^2$.

We obtain, in the same manner as in (5),

$$\begin{aligned}
\int_C p(z)\frac{\partial S}{\partial n}ds&=\int_C p(z)\frac{\partial(S+p)}{\partial n}ds\\
&=\int_C p(z)\frac{\partial}{\partial n}Re\left\{\frac{1}{2}\sum_{\nu=1}^{n}\left[a_\nu\frac{\partial q(z,\zeta_\nu)}{\partial\zeta_\nu}+\bar{a}_\nu\frac{\partial q(z,\zeta_\nu)}{\partial\bar{\zeta}_\nu}-\frac{a_\nu}{z-\zeta_\nu}\right]\right\}ds\\
&=\int_C p(z)\frac{\partial}{\partial s}Re\left\{\frac{1}{2i}\sum_{\nu=1}^{n}\left[a_\nu\frac{\partial q(z,\zeta_\nu)}{\partial\zeta_\nu}+\bar{a}_\nu\frac{\partial q(z,\zeta_\nu)}{\partial\bar{\zeta}_\nu}-\frac{a_\nu}{z-\zeta_\nu}\right]\right\}ds\\
&=Re\left\{\frac{1}{2i}\int_C\frac{1}{2}\sum_{\nu=1}^{n}\left[a_\nu\frac{\partial q(z,\zeta_\nu)}{\partial\zeta_\nu}+\bar{a}_\nu\frac{\partial q(z,\zeta_\nu)}{\partial\bar{\zeta}_\nu}\right]\right.\\
&\qquad\left.\times\sum_{\nu=1}^{n}\left[a_\nu\frac{\partial^2 q(z,\zeta_\nu)}{\partial z\partial\zeta_\nu}+\bar{a}_\nu\frac{\partial^2 q(z,\zeta_\nu)}{\partial z\partial\bar{\zeta}_\nu}+\frac{a_\nu}{(z-\zeta_\nu)^2}\right]dz\right\}
\end{aligned}$$

(by (i) and (ii))

$$= Re\left\{\frac{-1}{4i}\int_C \sum_{\nu=1}^n \left[a_\nu \frac{\partial^2 q(z,\zeta_\nu)}{\partial z \partial \zeta_\nu} + \bar{a}_\nu \frac{\partial^2 q(z,\zeta_\nu)}{\partial z \partial \bar{\zeta}_\nu}\right]\right.$$

$$\left.\times \sum_{\nu=1}^n \left[a_\nu \frac{\partial q(z,\zeta_\nu)}{\partial \zeta_\nu} + \bar{a}_\nu \frac{\partial q(z,\zeta_\nu)}{\partial \bar{\zeta}_\nu} - \frac{a_\nu}{z-\zeta_\nu}\right] dz\right\}$$

(by integration by parts and (i))

$$= \frac{\pi}{2} Re\left\{\sum_{\nu,\mu=1}^n a_\mu a_\nu \left[\frac{\partial^2 q(\zeta_\mu,\zeta_\nu)}{\partial \zeta_\mu \partial \zeta_\nu} + \frac{1}{(\zeta_\mu-\zeta_\nu)^2}\right] + \sum_{\nu,\mu=1}^n a_\mu \bar{a}_\nu \frac{\partial^2 q(\zeta_\mu,\zeta_\nu)}{\partial \zeta_\mu \partial \bar{\zeta}_\nu}\right\}$$

(by (iii) and residue theorem)

$$= \frac{\pi}{2} Re\left\{\sum_{\nu,\mu=1}^n a_\mu a_\nu \left[-\pi L_0(\zeta_\mu,\zeta_\nu) + \frac{1}{(\zeta_\mu-\zeta_\nu)^2}\right]\right.$$

$$\left. + \pi \sum_{\nu,\mu=1}^n a_\mu \bar{a}_\nu K_0(\zeta_\mu,\zeta_\nu)\right\},$$

because of the relation

$$\frac{\partial q(z,\zeta)}{\partial z} = 2\frac{\partial N(z,\zeta)}{\partial z}.$$

By Theorem I we obtain the following

COROLLARY 4. *The quantity*

$$(16) \qquad Re\left\{\sum_{\nu,\mu=1}^n a_\mu \bar{a}_\nu K_0(\zeta_\mu,\zeta_\nu) - \sum_{\nu,\mu=1}^n a_\mu a_\nu l_0(\zeta_\mu,\zeta_\nu)\right\}$$

is a monotone decreasing domain functional.

The analogous result for the Bergman kernel function $K(z,\zeta)$ for which $K(z,\zeta) = (-2/\pi)\cdot[\partial^2 G(z,\zeta)/\partial z \partial \bar{\zeta}]$, $G(z,\zeta)$ being the Green's function of D, was derived by Bergman and Schiffer[2] from the Hadamard's variation formula, and recently by Nehari[8] by means of Dirichlet principle.

5. Modification of Theorem I. In this section we shall modify Theorem I and derive a theorem which is useful in the treatment of extremal problems in the theory of conformal mapping of multiply-connected domains.

THEOREM II. *Let R, D, D_1, C, C_1 and $S(z)$ have the same meaning as in Theorem I, and moreover D and D_1 have a common boundary component γ. Let $p(z)$ denote the function satisfying the following conditions*: (i) $p(z) = \text{const.}$ *on* γ, (ii) $\partial p/\partial n = 0$ *on* $\Gamma = C - \gamma$, (iii) $p+S$ *is harmonic in* D, (iv) $\int_\beta [\partial(p+S)/\partial n] ds = 0$ *for any closed contour* β *in* D. *If $p_1(z)$ is the corresponding function associated with D_1, then*

$$\int_{\Gamma} p(z)\frac{\partial S(z)}{\partial n}ds-\int_{\gamma} S(z)\frac{\partial p(z)}{\partial n}ds$$
$$(17)\qquad \geqq \int_{\Gamma_1} p_1(z)\frac{\partial S(z)}{\partial n}ds-\int_{\gamma} S(z)\frac{\partial p_1(z)}{\partial n}ds\,,$$

where $\Gamma_1=C_1-\gamma$.

Proof. We put

$$u=p_1+S \quad \text{in } D_1,$$

and

$$v=\begin{cases} p+S & \text{in } D \\ S & \text{in } D_1-D. \end{cases}$$

Using the notation of Dirichlet integral

$$(h,k)_D=\iint_D (h_x k_x+h_y k_y)d\tau\,,$$

we easily obtain

$$(v,v)_{D_1}=(u,u)_{D_1}-2(u,u-v)_{D_1}+(u-v,u-v)_{D_1}\,,$$

and therefore

$$(18)\qquad (v,v)_{D_1}\geqq (u,u)_{D_1}-2(u,u-v)_{D_1}\,.$$

From Green's theorem

$$\begin{aligned}
(u,u)_{D_1}&=\int_{C_1}(p_1+S)\frac{\partial(p_1+S)}{\partial n}ds\\
&=\int_{\gamma}(p_1+S)\frac{\partial(p_1+S)}{\partial n}ds+\int_{\Gamma_1}(p_1+S)\frac{\partial S}{\partial n}ds\\
(19)\qquad &\qquad\qquad\qquad\qquad\qquad (\partial p_1/\partial n=0 \text{ on } \Gamma_1)\\
&=\int_{\gamma}p_1\frac{\partial(p_1+S)}{\partial n}ds+\int_{\gamma}S\frac{\partial p_1}{\partial n}ds+\int_{\Gamma_1}p_1\frac{\partial S}{\partial n}ds+\int_{C_1}S\frac{\partial S}{\partial n}ds\\
&=\int_{\gamma}S\frac{\partial p_1}{\partial n}ds+\int_{\Gamma_1}p_1\frac{\partial S}{\partial n}ds+\int_{C_1}S\frac{\partial S}{\partial n}ds\\
&\qquad\qquad\qquad\qquad (p_1=\text{const. on } \gamma \text{ and (iv)}).
\end{aligned}$$

Similarly

$$(20)\qquad (v,v)_{D_1}=(v,v)_D+(v,v)_{D_1-D}$$
$$=\int_{\gamma}S\frac{\partial p}{\partial n}ds+\int_{\Gamma}p\frac{\partial S}{\partial n}ds+\int_{C}S\frac{\partial S}{\partial n}ds+\int_{C_1}S\frac{\partial S}{\partial n}ds-\int_{C}S\frac{\partial S}{\partial n}ds\,,$$

and

(21) $$(u, u-v)_{D_1}=(u, u-v)_D+(u, u-v)_{D_1-D}$$

$$=\int_C (p_1+S)\frac{\partial(p_1-p)}{\partial n}ds+\int_{\Gamma_1}(p_1+S)\frac{\partial p_1}{\partial n}ds-\int_{\Gamma}(p_1+S)\frac{\partial p_1}{\partial n}ds$$

$$=\int_\gamma (p_1+S)\frac{\partial(p_1-p)}{\partial n}ds\,.$$

On the other hand, from (iv) of the assumptions

$$\int_C \frac{\partial(p+S)}{\partial n}ds=\int_{C_1}\frac{\partial(p_1+S)}{\partial n}ds \quad (=0)$$

and from the assumption that $S(z)$ is harmonic in D_1-D

$$\int_C \frac{\partial S}{\partial n}ds=\int_{C_1}\frac{\partial S}{\partial n}ds\,.$$

From both relations it holds that

$$\int_C \frac{\partial p}{\partial n}ds=\int_{C_1}\frac{\partial p_1}{\partial n}ds$$

and therefore

(22) $$\int_\gamma \frac{\partial p}{\partial n}ds=\int_\gamma \frac{\partial p_1}{\partial n}ds \qquad \text{(by (ii))}.$$

Using (i) and (22) we obtain the equality

(23) $$(u, u-v)_{D_1}=\int_\gamma S\frac{\partial p_1}{\partial n}ds-\int_\gamma S\frac{\partial p}{\partial n}ds\,.$$

From (18), (19), (20) and (23) we obtain the required result (17). Q.E.D.

6. Bounded radial slit mapping. To illustrate the application of Theorem II, consider the case of a finite multiply-connected plane domain D and of the analytic function $F(z, \zeta)$ which maps D onto the unit circle slit along radial segments directed towards the origin. $F(z, \zeta)$ may be so normalized as to satisfy $F(\zeta, \zeta)=0$ for $\zeta \in D$ and we may require that the outer boundary γ of D be transformed into the unit circumference. The argument of $(z-\zeta)^{-1}F(z, \zeta)$ returns to its initial value if z describes any boundary component of D, and $|F(z, \zeta)|$ is constant on γ and $\arg F(z, \zeta)=\text{const.}$ on $\Gamma(=C-\gamma)$. We may therefore set $p(z)=\log|F(z, \zeta)|$ and $S(z)=-\log|z-\zeta|$ and apply Theorem II. It follows that

$$(24)\qquad \int_\Gamma p\frac{\partial S}{\partial n}ds=\int_\Gamma p\frac{\partial(p+S)}{\partial n}ds$$

$$=\int_\Gamma \log|F(z,\zeta)|\frac{\partial}{\partial n}\log\left|\frac{F(z,\zeta)}{z-\zeta}\right|ds$$

$$=Re\left\{\frac{1}{i}\int_\Gamma \log|F(z,\zeta)|\left[\log\frac{F(z,\zeta)}{z-\zeta}\right]'dz\right\}$$

$$=Re\left\{\frac{1}{i}\int_\Gamma \log F(z,\zeta)\left[\log\frac{F(z,\zeta)}{z-\zeta}\right]'dz\right\}$$

$$=Re\left\{\frac{-1}{i}\int_\Gamma \log\frac{F(z,\zeta)}{z-\zeta}[\log F(z,\zeta)]'\,dz\right\},$$

(by integration by parts).

Since $(\partial p/\partial n)ds=(1/i)[\log F(z,\zeta)]'dz$ and $p=0$ on γ,

$$(25)\qquad \int_\gamma S\frac{\partial p}{\partial n}ds=\int_\gamma (p+S)\frac{\partial p}{\partial n}ds$$

$$=\frac{1}{i}\int_\gamma (p+S)[\log F(z,\zeta)]'dz$$

$$=Re\left\{\frac{1}{i}\int_\gamma \log\frac{F(z,\zeta)}{z-\zeta}[\log F(z,\zeta)]'\,dz\right\}.$$

Therefore it follows from (24) and (25) that

$$\int_\Gamma p\frac{\partial S}{\partial n}ds-\int_\gamma S\frac{\partial p}{\partial n}ds=Re\left\{\frac{-1}{i}\int_c \log\frac{F(z,\zeta)}{z-\zeta}[\log F(z,\zeta)]'\,dz\right\}$$

$$=-2\pi\log|F'(\zeta,\zeta)| \qquad \text{(by residue theorem)}.$$

From Theorem II we obtain the following

COROLLARY 5. *Let $F(z,\zeta)$ be a normalized bounded radial slit mapping function of a finite multiply-connected domain D. Then it holds that $|F'(\zeta,\zeta)|$ increases if D increases, the outer boundary component of D being fixed.*

If the domain D is the unit circle $|z|<1$, $F(z,0)=z$ and $F'(0,0)=1$. Therefore we obtain the following

COROLLARY 6.[9] *Let D be a multiply-connected domain which is contained in the unit circle $|z|<1$ and contains the origin, and whose outer boundary component is the unit circumference. Then it holds that*

$$(26)\qquad |F'(0,0)|\leqq 1.$$

This is equivalent to the following well known theorem on bounded univalent functions: *Let G be a finite plane domain and let $f(z)$ be univalent, $|f(z)|\leqq 1$ in G and $f(\zeta)=0$. If, moreover, $f(z)$ maps the outer boundary component of G onto the unit circumference, then*

$$|f'(\zeta)|\geqq|F'(\zeta,\zeta)|, \tag{27}$$

where $F(z,\zeta)$ maps G onto the unit circle slit along radial segments directed towards the origin.

7. Circular ring with radial slits. Theorem II can slightly be modified as follows: *Let D, D_1 be two multiply-connected domains and $D\subset D_1$, and moreover have two common boundary components γ', γ''. Let $p(z)=\text{const.}$ on γ', γ'', $\partial p/\partial n=0$ on $\Gamma(C=\gamma'+\gamma''+\Gamma)$ and moreover $\int[\partial(p+S)/\partial n]\,ds=0$ for every closed contour in D. If $p_1(z)$ is the corresponding function associated with D_1, then*

$$\int_\Gamma p\frac{\partial S}{\partial n}ds-\int_{\gamma'+\gamma''}S\frac{\partial p}{\partial n}ds\geqq\int_{\Gamma_1}p_1\frac{\partial S}{\partial n}ds-\int_{\gamma'+\gamma''}S\frac{\partial p_1}{\partial n}ds\,, \tag{28}$$

$$(C_1=\gamma'+\gamma''+\Gamma_1).$$

As an application of the above result, we deal with an extremal problem on the analytic function $F_{\nu\mu}(z)$ which maps a schlicht domain D of connectivity $\geqq 3$ onto the circular ring $1<|w|<M$ slit along radial segments, where two boundary components C_ν, C_μ of D are transformed into the circumferences $|w|=1$ and $|w|=M$, respectively. We denote by D_1 the doubly-connected domain containing D which is bounded by C_ν and C_μ, and by $F_0(z)$ the function mapping D_1 onto the circular ring $1<|w|<M_0$. Clearly, $\log F_{\nu\mu}(z)-\log F_0(z)$ is regular and single-valued in D and we may apply the above result with $p(z)=\log|F_{\nu\mu}(z)|$, $S(z)=-\log|F_0(z)|$, $p_1(z)=\log|F_0(z)|$ and $\gamma'=C_\nu$, $\gamma''=C_\mu$. Then the left-hand side of (28) equals to

$$-\sum_{m\neq\nu,\mu}\int_{C_m}\log|F_{\nu\mu}|\frac{\partial}{\partial n}\log|F_0|ds+\int_{C_\nu+C_\mu}\log|F_0|\frac{\partial}{\partial n}\log|F_{\nu\mu}|\,ds$$

$$=-\sum_{m\neq\nu,\mu}\int_{C_m}\log|F_{\nu\mu}|\frac{\partial}{\partial n}\log|F_0|ds+\int_C\log|F_0|\frac{\partial}{\partial n}\log|F_{\nu\mu}|\,ds$$

$$\left(\frac{\partial}{\partial n}\log|F_{\nu\mu}|=0 \text{ on } C_m,\ m\neq\nu,\ \mu\right)$$

$$=-\sum_{m\neq\nu,\mu}\int_{C_m}\log|F_{\nu\mu}|\frac{\partial}{\partial n}\log|F_0|\,ds+\int_C\log|F_{\nu\mu}|\frac{\partial}{\partial n}\log|F_0|\,ds$$

(by Green's theorem)

$$=\int_{C_\nu+C_\mu}\log|F_{\nu\mu}|\frac{\partial}{\partial n}\log|F_0|\,ds=\log M\int_{C_\mu}\frac{\partial}{\partial n}\log|F_0|\,ds\,.$$

The right-hand side of (28) equals to

$$\int_{C_\mu+C_\nu}\log|F_0|\frac{\partial}{\partial n}\log|F_0|\,ds=\log M_0\int_{C_\mu}\frac{\partial}{\partial n}\log|F_0|\,ds\,.$$

Therefore it follows that

$$\log M\int_{C_\mu}\frac{\partial}{\partial n}\log|F_0|\,ds\geqq\log M_0\int_{C_\mu}\frac{\partial}{\partial n}\log|F_0|\,ds\,. \tag{29}$$

On the other hand,

$$\int_{C_\mu}\frac{\partial}{\partial n}\log|F_0|\,ds=\frac{1}{\log M_0}\int_{C_\nu+C_\mu}\log|F_0|\frac{\partial}{\partial n}\log|F_0|\,ds$$

$(|F_0|=1 \text{ on } C_\nu)$

$$=\frac{1}{\log M_0}(\log|F_0|,\ \log|F_0|)_{D_1}>0.$$

From (29) it follows that

$$M\geqq M_0. \tag{30}$$

From this result we obtain the following

COROLLARY 7.[3),4),9)] *Let D be a schlicht domain of connectivity $\geqq 3$ and let C_ν and C_μ denote two of its boundary components. If M_0 denotes the Riemann modulus of the doubly-connected domain D_1 bounded by C_ν and C_μ, then, within the conformal class of D, the problem $M_0=\max$ is solved by the domain whose boundary components other than C_ν and C_μ are transformed into radial slits directed towards the origin by the conformal mapping carrying D_1 into a circular ring about the origin.*

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