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Note on the boundedness of solutions of a system of differential equations

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In the foregoing papers(1) we have discussed the existence of a periodic solution of the non-linear differential equation and we have obtained a sufficient condition(2) for the *boundedness* of solutions in order to use Massera's theorem.(3) That condition is analogous to Okamura's theorem(4) for the possibility of the continuation of solutions. In this paper we will obtain necessary and sufficient conditions for the boundedness of solutions of such a type that we have discussed formerly on the evaluation of the derivatives of solutions of a differential equation of the second order.(5)

Now we consider a system of differential equations,

$$\frac{dy_i}{dx}=f_i(x, y_1, \cdots, y_n) \quad (i=1, 2, \cdots, n), \tag{1}$$

where $f_i(x, y_1, \cdots, y_n)$ are *continuous* in the domain

$$\Delta_1: \ 0\leqq x<\infty, \qquad -\infty<y_i<+\infty \ (i=1, 2, \cdots, n).$$

And we discuss the boundedness for $0\leqq x<\infty$ of solutions starting from $x=0$.

To simplify our statements, here we give at first a list of the domains and their symbols used in the following.

(1) Yoshizawa; "*On the non-linear differential equation*", These Memoirs, Vol. 28 (1953), pp. 133-141 and "*Note on the existence theorem of a periodic solution of the non-linear differential equation*", *ibid.*, pp. 153-159.

(2) Yoshizawa; *ibid.*, p.153.

(3) Massera; "*The existence of periodic solutions of systems of differential equations*", Duke Math. Journal, Vol. 17 (1950), pp. 457-475.

(4) Okamura; Functional Equations (in Japanese), Vol. 32 (1942) or Yoshizawa; *ibid.*, p. 139.

(5) Yoshizawa; "*On the evaluation of the derivatives of solutions of $y''=f(x, y, y')$*", These Memoirs, Vol. 28 (1953), pp.27-32.

$$\Delta_2:\ 0\leqq x<\infty,\quad \sum_{i=1}^{n} y_i^2\leqq\beta^2,$$

$$\Delta_3:\ x=0,\quad \sum_{i=1}^{n} y_i^2\leqq\alpha^2,$$

$$\Delta_4:\ 0\leqq x<\infty,\quad \sum_{i=1}^{n} y_i^2=\beta^2,$$

$$\Delta_5:\ 0\leqq x<\infty,\quad \sum_{i=1}^{n} y_i^2\leqq\gamma^2,$$

$$\Delta_6:\ 0\leqq x<\infty,\quad \sum_{i=1}^{n} y_i^2=\gamma^2.$$

Theorem 1. *In order that, given a positive number α and for a suitable positive number β ($>\alpha$), for any solution $y_i=y_i(x)$ ($i=1, 2, \cdots, n$) of* (1) *such as*

$$(2)\qquad y_1(0)^2+y_2(0)^2+\cdots+y_n(0)^2\leqq\alpha^2;$$

we have for $0\leqq x<\infty$

$$(3)\qquad y_1(x)^2+y_2(x)^2+\cdots+y_n(x)^2<\beta^2,$$

it is necessary and sufficient that there exists a non-negative continuous function $\Phi(x, y_1, \cdots, y_n)$ satisfying the following conditions in the domain Δ_2 in Δ_1; namely

1° *$\Phi(x, y_1, \cdots, y_n)=0$, provided $(x, y_1, \cdots, y_n)\in\Delta_3$,*

2° *$\Phi(x, y_1, \cdots, y_n)>0$, provided $(x, y_1, \cdots, y_n)\in\Delta_4$,*

3° *$\Phi(x, y_1, \cdots, y_n)$ satisfies locally the Lipschitz condition with regard to $(y_1, \cdots, y_n)$ in Δ_2 (as the necessary condition, there is such a function that satisfies non-locally the Lipschitz condition) and we have for all points in the interior of Δ_2*

$$(4)\qquad \overline{\lim_{h\to 0}}\ \frac{1}{h}\left\{\Phi(x+h, y_1+hf_1, \cdots, y_n+hf_n)-\Phi(x, y_1, \cdots, y_n)\right\}\leqq 0.$$

Proof. It is clear that the condition is sufficient. Now we suppose that, for any solution of (1) satisfying (2), the inequality (3) holds for $0\leqq x<\infty$. Let Q be a point $(x^Q, y_1^Q, \cdots, y_n^Q)$ in Δ_2 and P be a point $(0, y_1^P, \cdots, y_n^P)$ in Δ_3. And we construct the Okamura's D-function $D(P, Q)$[6] with respect to the interval $0\leqq x\leqq x^Q$. We

(6) Hayashi and Yoshizawa; "*New treatise of solutions of a system of ordinary differential equations and its application to the uniqueness theorems*", These Memoirs, Vol. 26 (1951), pp. 225-233.

indicate by $\delta(Q)$ the minimum of $D(P, Q)$ when P displaces in $\varDelta_3$. Here if we put

$$\Phi(x, y_1, \cdots, y_n)=\delta(Q) \quad \text{with} \quad Q=(x, y_1, \cdots, y_n),$$

we can see that this $\Phi(x, y_1, \cdots, y_n)$ is the desired function by the properties of D-function.

In Theorem 1 we have obtained a necessary and sufficient condition for the boundedness of solutions by using the Okamura's D-function, and yet such a condition may also be obtain by the consideration of the *distance* between a point and the solutions as follows; namely

Theorem 2. *In order that the proposition in Theorem 1 may be verified, it is necessary and sufficient that, for a suitable positive constant* γ $(>\alpha)$, *there exists a non-negative function* $\Phi(x, y_1, \cdots, y_n)$ *of* $(x, y_1, \cdots, y_n)$ *satisfying the following conditions in the domain* $\varDelta_5$; *namely*

1° $\Phi(x, y_1, \cdots, y_n)>0$, *provided* $(x, y_1, \cdots, y_n)\in\varDelta_3$,

2° $\Phi(x, y_1, \cdots, y_n)=0$, *provided* $(x, y_1, \cdots, y_n)\in\varDelta_6$,

3° *for any solution of* (1), $y_i=y_i(x)$ $(i=1, 2, \cdots, n)$, *the function* $\Phi(x, y_1(x), \cdots, y_n(x))$ *is the non-decreasing function of* x (*so long as it has a sense*).

Proof. Now we shall show that the condition is necessary. Suppose that there exists β for α given and then we take a positive number γ such as $\gamma>\beta$. Here we consider all the solutions of (1) starting from P in the domain $\varDelta_5$ and indicate it by $\mathfrak{Y}_P$. Now we take a point Q such that $x^Q\geqq x^P$, $\sum_{i=1}^{n}(y_i^Q)^2=\gamma^2$ and consider the section of the solutions belonging to $\mathfrak{Y}_P$ by $x=x^Q$ which lie in $\varDelta_5$ for $x^P\leqq x\leqq x^Q$. If this section is an empty set, we put

$$D(P, Q)=\sqrt{\sum_{i=1}^{n}(y_i^Q-y_i^P)^2}$$

and if otherwise, it is clearly a bounded closed set and hence we can consider on the plane $x=x^Q$ the distance between this closed set and Q, which we denote by $D(P, Q)$. Now we indicate by $\delta(P)$ the infimum of $D(P, Q)$ when Q displaces in $\varDelta_6$, yet satisfying $x^Q\geqq x^P$. Hence if we put

$$\Phi(x, y_1, \cdots, y_n)=\delta(P) \quad \text{with} \quad P=(x, y_1, \cdots, y_n),$$

this $\Phi(x, y_1, \cdots, y_n)$ is the desired function.

Remark. Since by $\delta(P) \geqq \gamma - \beta > 0$ we have for $P \in \Delta_3$

$$\inf_{P \in \Delta_3} \delta(P) > 0,$$

we can replace the condition 1° and 2° by the condition

$$\inf_{(x, y_1, \cdots, y_n) \in \Delta_3} \Phi(x, y_1, \cdots, y_n) > \sup_{(x, y_1, \cdots, y_n) \in \Delta_6} \Phi(x, y_1, \cdots, y_n).$$

The continuity of $\Phi(x, y_1, \cdots, y_n)$ cannot be obtained in case of Theorem 2; to be continuous it is sufficient to see that Φ satisfies the Lipschitz condition, but it is impossible in general.

But in the case where $\Phi(x, y_1, \cdots, y_n)$ has the continuous first partial derivatives, the condition 3° is clearly replaced by the inequality

$$\frac{\partial \Phi}{\partial x} + \sum_{i=1}^{n} \frac{\partial \Phi}{\partial y_i} f_i(x, y_1, \cdots, y_n) \geqq 0,$$

and therefore it is useful in important practical cases.

The condition mentioned above are those for the solutions starting from $x=0$ and yet we can obtain a necessary and sufficient condition in order that, given a positive number α, there exists a suitable positive number β depending only on α such that, for any solution of (1) satisfying at *every* $x=x_0$

$$\sum_{i=1}^{n} y_i(x_0)^2 \leqq \alpha^2,$$

we have always for $x_0 \leqq x < \infty$

$$\sum_{i=1}^{n} y_i(x)^2 < \beta^2.$$

If we change the notion of the distance, a necessary and sufficient condition in order that the solutions of (1) are bounded as x increases is mentioned as follows by the method in Theorem 2; namely

Theorem 3. *In order that, for any solution of* (1), $y_i = y_i(x)$ $(i=1, 2, \cdots, n)$, *passing through any point* P, *there exists* $\alpha(P)$ *such that* $\sum_{i=1}^{n} y_i(x)^2 < \alpha^2(P)$ *for* $x^P \leqq x < \infty$, *it is necessary and sufficient that there exists a positive function* $\Phi(x, y_1, \cdots, y_n)$ *of* $(x, y_1, \cdots, y_n)$ *satisfying the following conditions in* Δ_1; *namely*

1° $\Phi(x, y_1, \cdots, y_n) > 0,$

2° $\Phi(x, y_1, \cdots, y_n)$ *tends to zero uniformly for* x, *as* $\sum_{i=1}^{n} y_i^2 \to \infty$,

3° *for any solution of* (1), $y_i = y_i(x)$ $(i=1, 2, \cdots, n)$, *the function* $\Phi(x, y_1(x), \cdots, y_n(x))$ *is the non-decreasing function of* x.

Proof. It is clear that the condition is sufficient. Now we show that the condition is necessary. Let P be a point in Δ_1 and $\mathfrak{Y}_P$ be the family of all the solutions passing through P. By the hypothesis, any solution belonging to $\mathfrak{Y}_P$ lies in the domain such as $\sum_{i=1}^{n} y_i^2 < \alpha^2(P)$, $0 \leqq x < \infty$, and hence the section of $\mathfrak{Y}_P$ by $x = x^Q$ $(x^P \leqq x^Q)$ is a bounded closed set. Now we consider the distances between the points of this set and a point Q on x-axis $(x^P \leqq x^Q)$ and we indicate the maximum distance by $D(P, Q)$. Clearly we have

$$0 \leqq D(P, Q) < \alpha(P).$$

Let $\gamma(P)$ be the supremum of $D(P, Q)$ when Q moves on x-axis, always satisfying $x^P \leqq x^Q$.

Now we put

$$\delta(P) = \frac{1}{1+\gamma^2(P)} \tag{5}$$

(this is the distance of points projected stereographically), then since $0 \leqq \gamma(P) \leqq \alpha(P)$, we have

$$0 < \frac{1}{1+\alpha^2(P)} \leqq \delta(P) \leqq 1.$$

And also if we have $\sum_{i=1}^{n} (y_i^P)^2 \geqq G^2$ for a sufficiently great positive number G independent of x^P, then we have

$$G \leqq D(P, Q)$$

for the point Q on x-axis, where $x^P = x^Q$, on the other hand we have

$$\gamma(P) \geqq D(P, Q),$$

hence

$$\gamma(P) \geqq G.$$

Therefore we can verify the inequality

$$\delta(P) < \varepsilon \quad (\varepsilon : \text{however small}) ;$$

namely $\delta(P)$ tends to zero uniformly for x as $\sum_{i=1}^{n}(y_i^P)^2\to\infty$. Moreover if P and Q $(x^P<x^Q)$ lie on the same solution of (1), we have

$$\delta(P)\leqq\delta(Q).$$

Now put

$$\Phi(x, y_1, \cdots, y_n)=\delta(P) \quad \text{with} \quad P=(x, y_1, \cdots, y_n),$$

and then $\Phi(x, y_1, \cdots, y_n)$ is the desired function.

Remark 1. If every solution of (1) is ***unique*** for the Cauchy-problem, "for any solution of (1)$\cdots$for $x^P\leqq x<\infty$" is replaced by "every solution of (1) is bounded as x increases".

Remark 2. In the case for the solutions starting from $x=0$, we can also discuss in the same way.

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