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Automorphism-groups of differential fields and group-varieties

By

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Recently E. R. Kolchin [1] has developed a beautiful Galois theory of differential fields. The Galois group is equipped with an algebraico-geometric structure. His main theorem asserts Galois correspondence between the intermediate differential fields and the algebraic subgroups (i. e. the closed subgroups in Zariski topology) of the automorphism group. His results show also that the groups of strong isomorphisms in his sense have, if irreducible, almost all properties of group variety in the sense of A. Weil. Combining Kolchin's results with Weil's method of construction of group varieties ([4]), and using an idea of Nakano [5], we shall show in this note that Kolchin's irreducible groups are, as he conjectured, group varieties in Weil's sense. Further we shall add some remarks on the specialization and the solvability in the whole (eventually reducible) group.

1. Let g/F be a strongly normal extension (cf. [1]). Thus F is of characteristic O, and g is finitely generated over F (not only in differential sense, but also in algebraic sense). g and F have the same constant field C, which is assumed to be algebraically closed. Let G^* and G be the group of the strong isomorphisms and the automorphism group, respectively. For the present we shall assume that G^* is irreducible. All the fields to be considered are contained in a universal extension g^* with constant field C^*. g and C^* are linearly disjoint over C, and so specializations over C and those over g are equivalent for constants.

2. Let $\bar{\sigma}$ be a generic element of G^*. $C_{\bar{\sigma}}$ is finitely generated over C, and we can set $C_{\bar{\sigma}}=C(\gamma_{\bar{\sigma}})$, $(\gamma_{\bar{\sigma}})=(\gamma_{\bar{\sigma}1}, \cdots, \gamma_{\bar{\sigma}m})$. Moreover we can, and shall, take $(\gamma_{\bar{\sigma}})$ so that the affine variety defined over C with generic point $(\gamma_{\bar{\sigma}})$ may be (everywhere) normal. Let $g=$

$F(\eta_1, \cdots, \eta_n)$.

We have

(1) $\bar{\sigma}\eta = A(\eta, \gamma_{\bar{\sigma}})/B(\eta, \gamma_{\bar{\sigma}}) = R(\eta, \gamma_{\bar{\sigma}})$,

(2) $\gamma_{\bar{\sigma}} = C(\eta, \bar{\sigma}\eta)/D(\eta, \bar{\sigma}\eta) = S(\eta, \bar{\sigma}\eta)$,

(3) $\eta = E(\bar{\sigma}\eta, \gamma_{\bar{\sigma}})/F(\bar{\sigma}\eta, \gamma_{\bar{\sigma}}) = T(\bar{\sigma}\eta, \gamma_{\bar{\sigma}})$,

where $A, \cdots, F$ are polynomials with coefficients in F, $R = A/B$, $S = C/D$, $T = E/F$. (Strictly speaking, we must write $\bar{\sigma}\eta_i = R_i(\eta, \gamma_{\bar{\sigma}})$, $1 \leq i \leq n$, and so on. The above are, therefore, symbolic notations, but no confusion will occur.)

Lemma 1. There is a polynomial $M(X) \in C[X_1, \cdots, X_m]$ such that

(a) $M(\gamma_{\bar{\sigma}}) \neq 0$,
(b) if $(\gamma_{\bar{\sigma}}) \to (\gamma')$ over C and if $M(\gamma') \neq 0$, then $\bar{\sigma}$ has a uniquely determined specialization σ over that specialization and we have

(4) $\sigma\eta = R(\eta, \gamma')$, (5) $\gamma' = S(\eta, \sigma\eta)$, (6) $\eta = T(\sigma\eta, \gamma')$.

Proof. We have only to take M so that $M(\gamma') \neq 0$ may mean $\Pi_i B_i(\eta, \gamma') \cdot \Pi_j D_j(\eta, R(\eta, \gamma')) \cdot \Pi_k F_k(R(\eta, \gamma'), \gamma') \neq 0$. For a detailed proof see [1], Ch. II, prop. 9.

Lemma 2. There is a polynomial $P(Y) \in g[Y_1, \cdots Y_n]$ with the following properties:

(a) $P(\bar{\sigma}\eta) \neq 0$,
(b) if $\sigma \in G^*$, $P(\sigma\eta) \neq 0$ then $(\gamma_{\bar{\sigma}})$ has a unique specialization (γ_σ) over $\bar{\sigma} \to \sigma$ with reference to g such that the γ_σ are constants and that $M(\gamma_\sigma) \neq 0$. (Therefore σ and γ_σ are related by (4), (5), (6).)

Proof. Writing $M(C(\eta, Y)/D(\eta, Y)) = H(Y)/K(Y)$, where H, $K \in g[Y]$ and K is a power product of the D_i, we take

$$P(Y) = H(Y) \cdot \Pi_i D_i(\eta, Y).$$

If $P(\sigma\eta) \neq 0$, set $\gamma_\sigma = S(\eta, \sigma\eta)$. We see $M(\gamma_\sigma) \neq 0$, and (γ_σ) is a unique specialization of $(\gamma_{\bar{\sigma}})$ over $\bar{\sigma} \to \sigma$. Differerentiating (1), we have

$$0 = [\partial_i C(\eta, \bar{\sigma}\eta) \cdot D(\eta, \bar{\sigma}\eta) - C(\eta, \bar{\sigma}\eta) \cdot \partial_i D(\eta, \bar{\sigma}\eta)]/D(\eta, \bar{\sigma}\eta)^2.$$

Specializing $\bar{\sigma}$ to σ, we have $\partial_i \gamma_\sigma = 0$ for all ∂_i, and this completes the proof.

3. We shall denote by $\mathfrak{B}$ and $\mathfrak{N}$, resp., the frontier defined by $M(X)=0$ on the affine variety V and the algebraic set defined by $P(\sigma\eta)=0$ in G^*.

Proposition 1. For any differential polynomial $Q(Y)\epsilon g\{Y\}=g\{Y_1, \cdots, Y_n\}$ such that $Q(\bar{\sigma}\eta)\neq 0$, and for any $\sigma\epsilon G^*$, there exists a $\rho\epsilon G$ such that $Q(\rho\sigma\eta)\neq 0$.

Proof. If it were false, there would be some Q and some σ with the following property:

$$\text{(A)}\quad Q(\bar{\sigma}\eta)\neq 0, \text{ and } \forall\rho\epsilon G\ Q(\rho\sigma\eta)=0.$$

Let Q_0 be one with the least number of terms among such Q, and let one of the coefficients of Q_0 be equal to 1. We have $Q_0^\rho(\tau\sigma\eta)=\rho[Q_0(\rho^{-1}\tau\sigma\eta)]=0$ for some $\sigma\epsilon G^*$ and for all ρ, $\tau\epsilon G$. As $Q^0-Q_0^\rho$ has fewer terms than Q_0, we must have $Q_0(\bar{\sigma}\eta)-Q_0^\rho(\bar{\sigma}\eta)=0$. Then we have $Q_0^\rho=Q_0$. For otherwise we can choose $\xi\epsilon g$ so that $Q_0-\xi[Q_0-Q_0^\rho]$ may have fewer terms than Q_0, and this dif. polynomial has property (A), contradiction. Therefore $Q_0^\rho=Q_0$ for all $\rho\epsilon G$. But this means $Q_0\epsilon F\{Y\}$. Then $Q_0(\rho\sigma\eta)=\rho\sigma(Q_0(\eta))=0$, $\therefore Q_0(\eta)=0$, $\therefore Q_0(\bar{\sigma}\eta)=\bar{\sigma}\ Q_0(\eta)]=0$, contradiction. *Q.E.D.*

Proposition 2. There exist a finite number of automorphisms $\rho_1, \cdots, \rho_n\epsilon G$ such that $\forall\sigma\epsilon G^*\ \exists i\ \rho_i\sigma\notin\mathfrak{N}$.

Proof. By prop. 1 we have $\cap_{\rho\epsilon G}\rho^{-1}\mathfrak{N}=\phi$. By the compactness of G^* in Zariski topology (see the end of the proof of [1], Ch. II, prop. 13) there exist $\rho_1, \cdots, \rho_N\epsilon G$ such that $\cap_{i=1}^N\rho_i^{-1}\mathfrak{N}=\phi$. *Q.E.D.*

Lemma 3. $\rho\epsilon G, \sigma\epsilon G^* \Rightarrow C_{\rho\sigma}=C_\sigma$. Especially $\rho\bar{\sigma}$ is a generic element of G^*.

Proof. Transforming the formula $g\cup\sigma g=g\cup C_\sigma$ by ρ, we have $g\cup\rho\sigma g=g\cup C_\sigma$. Taking account of [1], Ch I, prop. 3, Cor. 5 this means $C_{\rho\sigma}=C_\sigma$. As dim $\sigma=\partial^0C_\sigma/C$, the lemma is proved.

4. By Lemma 1, we can write every point of $V-\mathfrak{B}$ in the form (γ_σ), $\sigma\epsilon G^*$. ρ_i being the same as in prop. 2, we shall write $\rho_i\sigma=\sigma_i$. Put $V_i=V$, $\mathfrak{B}_i=\mathfrak{B}$ $(1\leq i\leq n)$. By lemma 3, the subvariety $T_{ji}=\{(\gamma_{\bar{\sigma}_i})\times(\gamma_{\bar{\sigma}_j}), C\}$ of $V_i\times V_j$ is a birational correspondence between V_i and V_j. The following lemma shows that we can define an abstract variety V over C (with universal domain C^*) by $V=[V_i;\ \mathfrak{B}_i;T_{ji}]$.

Lemma 4. If $(\gamma_{\sigma'})\epsilon V_i-\mathfrak{B}_i$, $(\gamma_{\tau'})\epsilon V_j-\mathfrak{B}_j$, and if $(\gamma_{\sigma'})\times(\gamma_{\tau'})\epsilon T_{ji}$, then T_{ji} is biregular at $(\gamma_{\sigma'})\times(\gamma_{\tau'})$, and we can write $\sigma'=\rho_i\sigma=\sigma_i, \tau'=\rho_j\sigma=\sigma_j$ for some $\sigma\epsilon G^*$. Moreover, we have $(\gamma_{\bar{\sigma}_i}, \gamma_{\bar{\sigma}_j}, \bar{\sigma})\rightarrow$

$(\gamma_{\sigma i}, \gamma_{\sigma j}, \sigma)$.

Proof. Set $\rho_j\rho_i^{-1}=\rho_{ji}$. By (1) we have $\bar{\sigma}_i\eta=A(\eta, \gamma_{\bar{\sigma}i})/B(\eta, \gamma_{\bar{\sigma}i})$. It follows $\bar{\sigma}_j\eta=\rho_{ji}\bar{\sigma}_i\eta=A(\rho_{ji}\eta, \gamma_{\bar{\sigma}i})/B(\rho_{ji}\eta, \gamma_{\bar{\sigma}i})$. Now $(\gamma_{\sigma'})\notin\mathfrak{B}$ implies $B(\eta, \gamma_{\sigma'})\neq 0$, $\therefore B(\rho_{ji}\eta, \gamma_{\sigma'})=\rho_{ji}[B(\eta, \gamma_{\sigma'})]\neq 0$, $\rho_{ji}\sigma'\eta=A(\rho_{ji}\eta, \gamma_{\sigma'})/B(\rho_{ji}\eta, \gamma_{\sigma'})$. Therefore $\bar{\sigma}_j\eta$ has a unique specialization $\rho_{ji}\sigma'\eta$ over $(\gamma_{\bar{\sigma}i})\rightarrow(\gamma_{\sigma'})$ with reference to g. On the other hand $(\gamma_{\bar{\sigma}i}, \gamma_{\bar{\sigma}j})\rightarrow(\gamma_{\sigma'}, \gamma_{\tau'})/c$, and $(\gamma_{\tau'})\notin\mathfrak{B}$, so we have $(\gamma_{\bar{\sigma}i}, \gamma_{\bar{\sigma}j}, \bar{\sigma}_j)\rightarrow(\gamma_{\sigma'}, \gamma_{\tau'}, \tau')/g$. Hence $\rho_{ji}\sigma'=\tau'$. Then we have, by (2) and (5), $(\gamma_{\bar{\sigma}j})=S(\eta, \bar{\sigma}_j\eta)$, $(\gamma_{\tau'})=S(\eta, \rho_{ji}\sigma'\eta)$, and $(\gamma_{\bar{\sigma}j})$ has a unique specialization $(\gamma_{\tau'})$, therefore is finite, over $(\gamma_{\bar{\sigma}i})\rightarrow(\gamma_{\sigma'})/c$.

As V_i is normal, this implies that the projection of T_{ji} on V_i is regular at $(\gamma_{\sigma'})$. Similarly the projection on V_j is regular at $(\gamma_{\tau'})$. The first part of the lemma is proved.

Setting $\rho_i^{-1}\sigma'=\sigma$, we have $\sigma'=\rho_i\sigma=\sigma_i$, $\tau'=\rho_{ji}\sigma'=\rho_j\sigma=\sigma_j$. We see also that $\bar{\sigma}\eta=\rho_i^{-1}\bar{\sigma}_i\eta=R(\rho_i^{-1}\eta, \gamma_{\bar{\sigma}i})$ has a unique specialization $\sigma\eta=\rho_i^{-1}\sigma'\eta=R(\rho_i^{-1}\eta, \gamma_{\sigma'})$ over $(\gamma_{\bar{\sigma}i})\rightarrow(\gamma_{\sigma'})$. *Q.E.D.*

By this lemma, the above mentioned Variety $\boldsymbol{V}$ is well defined, and, every point of $\boldsymbol{V}$ can be written in the form $\boldsymbol{P}(\sigma)$. The representatives of $\boldsymbol{P}(\sigma)$ are those (γ_{σ_i}) which do not lie in $\mathfrak{B}$. By prop. 2, there exists a $\boldsymbol{P}(\sigma)\in\boldsymbol{V}$ for every $\sigma\in G^*$. Thus there is a one-to-one correspondence between G^* and $\boldsymbol{V}$.

In the above proof, we saw that $\bar{\sigma}$ has a unique specialization σ over $(\gamma_{\bar{\sigma}_i})\rightarrow(\gamma_{\sigma_i})/C$ (or what is the same, over $\boldsymbol{P}(\bar{\sigma})\rightarrow\boldsymbol{P}(\tau)$). Similarly, if $\boldsymbol{P}(\sigma)\rightarrow\boldsymbol{P}(\tau)/C$ then σ is uniquely specialized to τ over that specialization.

Conversely, let $\sigma,\tau\in G^*$, $\sigma\rightarrow\tau$. By prop. 2, we have $\rho_j\tau\notin\mathfrak{B}$ for some j. As dim $\rho_j=0$, ρ_j and σ are independent, $\rho_j\sigma\rightarrow\rho_j\tau$, whence $\rho_j\sigma\notin\mathfrak{B}$. Then (γ_{σ_j}) and (γ_{τ_j}) have a sense and by (5)

$$(\gamma_{\sigma_j})\rightarrow(\gamma_{\tau_j})/C, \ i.\ e.\ \boldsymbol{P}(\sigma)\rightarrow\boldsymbol{P}(\tau)/C.$$

By the last remark we have $(\boldsymbol{P}(\sigma), \sigma)\rightarrow(\boldsymbol{P}(\tau), \tau)$. Now $\boldsymbol{P}(\sigma)$ has no other specialization over $\sigma\rightarrow\tau$. For if $(\sigma, \boldsymbol{P}(\sigma))\rightarrow(\tau, \boldsymbol{P}(\tau'))$, then τ' is the unique specialization of σ over $\boldsymbol{P}(\sigma)\rightarrow\boldsymbol{P}(\tau')$, $\therefore \tau=\tau'$. Therefore $\sigma\rightarrow\tau$ in G^* and $\boldsymbol{P}(\sigma)\rightarrow\boldsymbol{P}(\tau)/C$ in $\boldsymbol{V}$ are completely equivalent.

5. **Lemma 5.** The multiplication in $\boldsymbol{V}$ given by $\boldsymbol{P}(\sigma)\cdot\boldsymbol{P}(\tau)=\boldsymbol{P}(\sigma\tau)$ is a law of composition in Weil's sense, by means of which $\boldsymbol{V}$ becomes a group variety.

Proof. Let $\bar{\sigma}$, $\bar{\tau}$ be two independent generic element of G^*. By

(1) we have

$$\bar\tau\eta=R(\eta,\gamma_{\bar\tau}),\quad \therefore\ \bar\sigma\bar\tau\eta=R(\bar\sigma\eta,\gamma_{\bar\tau}),$$

$$\therefore\ C(\boldsymbol{P}(\bar\sigma\bar\tau))=C_{\bar\sigma\bar\tau}\subset C(\gamma_{\bar\sigma}\ \gamma_{\bar\tau})=C(\boldsymbol{P}(\bar\sigma),\boldsymbol{P}(\bar\tau)).$$

Therefore there is a function Ψ, defined over C on $V\times V$, with values in V, such that $\boldsymbol{P}(\bar\sigma\bar\tau)=\Psi(\boldsymbol{P}(\bar\sigma),\boldsymbol{P}(\bar\tau))$. Let $(\gamma_{\bar\tau_i})\to(\gamma_{\tau_i})$ $\notin\mathfrak{B}$. We can write $\rho_j^{-1}\eta=W(\eta)$, where W is a rational function with coefficients in F. Then we have

$$\bar\tau_j\eta=R(\eta,\gamma_{\bar\tau_j}),$$

$$(7)\quad \bar\sigma\bar\tau\eta=\bar\sigma\rho_i^{-1}\bar\tau_j\eta=R(\bar\sigma\rho_j^{-1}\eta,\gamma_{\bar\tau_j})=R(W(\bar\sigma\eta),\gamma_{\bar\tau_j}).$$

Similarly, as $(\gamma_{\tau_j})\notin\mathfrak{B}$, we have

$$(8)\quad \sigma\tau\eta=R(W(\sigma\eta),\gamma_{\tau_j}),$$

where $R=A/B$ and $B(W(\sigma\eta),\gamma_{\tau_j})=\sigma\rho_j^{-1}[B(\eta,\gamma_{\tau_j})]\neq 0$. On the other hand, by $N°4$, $\sigma\eta$ is a unique specialization of $\bar\sigma\eta$ over $\boldsymbol{P}(\bar\sigma)\to\boldsymbol{P}(\sigma)$.

From this and from (7), (8) we see that $\sigma\tau\eta$ is a unique specialization of $\bar\sigma\bar\tau\eta$ over $(\boldsymbol{P}(\bar\sigma),\boldsymbol{P}(\bar\tau))\to(\boldsymbol{P}(\sigma),\boldsymbol{P}(\tau))/g$. Hence $\boldsymbol{P}(\bar\sigma\bar\tau)$ is uniquely specialized to $\boldsymbol{P}(\sigma\tau)$ over $(\boldsymbol{P}(\bar\sigma),\boldsymbol{P}(\bar\tau))\to(\boldsymbol{P}(\sigma),\boldsymbol{P}(\tau))/C$. Since the product of two normal varieties is again normal, $V\times V$ is normal. Therefore $\Psi(\boldsymbol{P}(\sigma),\boldsymbol{P}(\tau))$ is defined and is equal to $\boldsymbol{P}(\sigma\tau)$. Thus Ψ is everywhere defined, and of course associative.

Next we consider inverse elements. As $C_{\bar\sigma}=C_{\bar\sigma}^{-1}$, we can define a function Φ over C by

$$\boldsymbol{P}(\bar\sigma^{-1})=\Phi\boldsymbol{P}(\bar\sigma)).$$

Let $(\gamma_{\sigma_i})\notin\mathfrak{B}$. We can write $\rho_i\eta=U(\eta)$, where U is a rational function with coefficients in F. By (3) we have $\eta=(\bar\sigma_i\eta,\gamma_{\bar\sigma_i})$, $\therefore\ \bar\sigma_i^{-1}\eta=T(\eta,\gamma_{\bar\sigma_i})$, $\bar\sigma^{-1}\eta=\bar\sigma_i^{-1}(\rho_i\eta)=\bar\sigma_i^{-1}(U(\eta))=U(\bar\sigma_i^{-1}\eta)=U[T(\eta,\gamma_{\bar\sigma_i})]$. By $(\gamma_{\sigma_i})\notin\mathfrak{B}$ we have also $\sigma^{-1}\eta=U[T(\eta,\gamma_{\sigma_i})]$, and we see as above that $\bar\sigma^{-1}$ and $\boldsymbol{P}(\bar\sigma^{-1})$ are uniquely specialized over $\boldsymbol{P}(\bar\sigma)\to\boldsymbol{P}(\sigma)/g$ to σ^{-1} and $\boldsymbol{P}(\sigma^{-1})$ respectively. $\Phi(\boldsymbol{P}(\sigma))$ is then defined and equal to $\boldsymbol{P}(\sigma^{-1})$.

Thus our V satisfies all the conditions of a group variety, and the lemma is proved.

6. Now we abandon our assumption that G^* is irreducible. Our result, then, can be summarized in the following.

Theorem. If g/F is a strongly normal extension, the component of the identity G_0^* of the strong isomorphism group G^* is represented faithfully by an (abstract) group variety V. V, together with its law of composition, is defined over C, and C^* is its universal domain. In this representation, the irreducible sets of G_0^* correspond to the Subvarieties of V defined over C. If $\sigma \in G_0^*$ is represented by $\boldsymbol{P}(\sigma) \in V$ then $C(\boldsymbol{P}(\sigma)) = C_\sigma$. Hence the automorphisms in G_0^* are represented by the subgroup $V(c)$ consisting of the C-rational Points of V.

Remark. If this group variety V is complete (i. e. an abelian variety), then by a theorem of Matsusaka we can take, as a model of G_0^*, an abelian variety in a projective space, which is birationally equivalent over C to V ([6]).

7. An application. In the rest of this note we shall identify G_0^* with V, σ with $\boldsymbol{P}(\sigma)$.

Proposition 3. If $\tau \in G$ (not necessarily $\tau \in G_0 = G_0^* \cap G$), then the correspondence $V \ni \sigma \to \tau^{-1}\sigma\tau \in V$ is an everywhere biregular birational correspondence from V to V.

Proof. If $\tau \in G$, $\sigma \in G^*$ then $\sigma\tau g = \sigma g$, $g \cup \sigma\tau g = g \cup \sigma g = g \cup C_\sigma$. Transforming by τ^{-1}, we have $g \cup \tau^{-1}\sigma\tau g = g \cup C_\sigma$, $\therefore C_\sigma = C_{\tau^{-1}\sigma\tau}$. Now let $\bar{\sigma}$ be a generic element of G_0^*. $\bar{\sigma} \times \tau^{-1}\bar{\sigma}\tau$ has a locus $\boldsymbol{T}_\tau$ on $V \times V$ over C. We shall prove that the birational correspondence $\boldsymbol{T}_\tau$ is everywhere biregular and $\boldsymbol{T}_\tau(\sigma) = \tau^{-1}\sigma\tau$.

$\tau\eta$ can be written in the form $\tau\eta_i = W_i(\eta)$, where the W are rational functions with coefficients in F. Let $\sigma \in G_0^*$, $(\gamma_{\sigma_i}) \notin \mathfrak{V}$. Then we have

$$\tau^{-1}\bar{\sigma}\tau\eta = \tau^{-1}\rho_i^{-1}\bar{\sigma}_i\tau\eta = W[R(\tau^{-1}\rho_i^{-1}\eta, \gamma_{\bar{\sigma}_i})],$$
$$\tau^{-1}\sigma\tau\eta = W[R(\tau^{-1}\rho_i^{-1}\eta, \gamma_{\sigma_i})].$$

From this we can see, as in other places, that $\tau^{-1}\sigma\tau (= \boldsymbol{P}(\tau^{-1}\sigma\tau))$ is a unique specialization of $\tau^{-1}\bar{\sigma}\tau$ over $\bar{\sigma} \to \sigma$, and that the projection of $\boldsymbol{T}_\tau$ on the first factor is regular at σ with $\boldsymbol{T}_\tau(\sigma) = \tau^{-1}\sigma\tau$. Replacing τ by τ^{-1}, and repeating the same argument, we see that $\boldsymbol{T}_\tau$ is everywhere biregular. *Q. E. D.*

Proposition 4. Let $\sigma_i \in G^*$, $1 \leqq i \leqq r$, and let $(\sigma_1, \cdots, \sigma_r) \to (\sigma_1', \cdots, \sigma_r')$. Let W (X) be any word (in r letters). Then $W(\sigma)$ has a unique specialization $W(\sigma')$ over $(\sigma) \to (\sigma')$. (Here the notation $\rho_i\sigma = \sigma_i$ is abandoned.)

Proof. If all σ_i lie in G_0^*, this is a property of a group variety. In the general case, let

$$G^*=\tau_1G_0^*\cup\cdots\cup\tau_tG_0^* \quad (\tau_i\in G)$$

be the decomposition of G^* into the components. We shall denote by $\boldsymbol{T}_i$ the transformation in G_0^* by τ_i: $\boldsymbol{T}_i(\sigma)=\tau_i^{-1}\sigma\tau_i$ $(\sigma\in G_0^*)$. If σ, $\sigma'\in G_0^*$, we have

$$(\tau_i\sigma)(\tau_j\sigma')=\tau_i\tau_j\boldsymbol{T}_j(\sigma)\sigma'$$
$$(\tau_i\sigma)^{-1}=\sigma^{-1}\tau_i^{-1}=\tau_i^{-1}\boldsymbol{T}_i^{-1}(\sigma^{-1}).$$

Thus for any word $W(X)$ we have $W(\sigma)=W(\tau)\cdot\Phi_W(\sigma)$, where we set $\sigma_i=\tau_{k(i)}\tilde{\sigma}_i$, and $W(\tau)$ denotes the group element obtained from $W(\sigma)$ by the substitution $\sigma_i\to\tau_{k(i)}$.

Φ_W is a function on $V=G_0^*$, with values in V, which is uniquely determined by W and (by lemma 5 and prop. 3) is defined everywhere. Now $\tilde{\sigma}_i=\tau_{k(i)}^{-1}\sigma_i$, and if $(\gamma_{\rho_s\sigma_i'})$ is a representative of $\boldsymbol{P}(\sigma_i')$, then

$$\tilde{\sigma}_i\eta_i=\tau^{-1}\rho_s^{-1}(\rho_s\sigma_i\eta_i)=R(\tau^{-1}\rho_s^{-1}\eta_i,\ \gamma_{\rho_s\sigma_i}) \quad (\tau=\tau_{k(i)}),$$
$$\tilde{\sigma}_i'\eta_i=R(\tau^{-1}\rho_s^{-1}\eta_i,\ \gamma_{\rho_s\sigma_i'}),\ \text{and}\ \tau^{-1}\rho_s^{-1}\eta_i\in g.$$

Hence we see that the $\tilde{\sigma}_i'$ are unique specializations of the $\tilde{\sigma}_i$ over $(\sigma)\to(\sigma')$. Then $(\sigma,\tilde{\sigma},\boldsymbol{\Phi}_W(\tilde{\sigma}))\to(\sigma',\tilde{\sigma}',\boldsymbol{\Phi}_W(\tilde{\sigma}'))$. Since $W(\tau)\in G$, we can see as above that $W(\tau)\cdot\boldsymbol{\Phi}_W(\sigma')=W(\sigma')$ is a unique specialization of $W(\tau)\cdot\boldsymbol{\Phi}_W(\tilde{\sigma})=W(\sigma)$ over $\Phi_W(\tilde{\sigma})\to\Phi_W(\tilde{\sigma}')$. *Q.E.D.*

8. Proposition 5. If G^* is irreducible, then the commutator group $D(G^*)$ is also an irreducible group.

Proof. Let $\bar{\sigma}_1$, $\bar{\tau}_1$, $\bar{\sigma}_2$, $\bar{\tau}_2$, $\cdots$ be independent generic points of G^*. Set $\kappa_i=\bar{\sigma}_1\bar{\tau}_1\bar{\sigma}_1^{-1}\bar{\tau}_1^{-1}\cdots\bar{\sigma}_i\bar{\tau}_i\bar{\sigma}_i^{-1}\bar{\tau}_i^{-1}$. Then $\kappa_{i+1}\to\kappa_i$. Hence, if we denote by $\mathfrak{H}_i$ the irreducible set with generic point κ_i, we have

$$\mathfrak{H}_1\subseteq\mathfrak{H}_2\subseteq\cdots.$$

As dim $\mathfrak{H}_j\leq$dim G^*, we must have, for some i, $\mathfrak{H}_i=\mathfrak{H}_{i+1}$. Then $\mathfrak{H}_i=\mathfrak{H}_{i+1}=\mathfrak{H}_{i+2}=\cdots$, and $\mathfrak{H}_i$ is a group. By prop. 4 (the trivial case), $D(G^*)\subseteq\mathfrak{H}_i$. Now by [1], Ch. II, prop. 9, Cor. 1 we see that $\mathfrak{H}_i-D(G^*)$ is contained in a lower-dimensional bunch of subvarieties of $\mathfrak{H}_i$. By [1], Ch. II, prop. 15, $\mathfrak{H}_i=D(G^*)$. *Q.E.D.*
(This proposition, together with the proof, is Kolchin's. Cf. [2] §4. As the propositions used in the proof are valid for any group variety, this is a property of group varieties.)

Corollary 1. If G is irreducible, then the commutator group $D(G)$ is again irreducible.

Proof. We have only to show that $G\cap D(G^*)=D(G)$. Let $\sigma\in G$, $\sigma=W(\tau)$, $\tau_i\in G^*$, where W is a word of the form

$$W(X)=X_1X_2X_1^{-1}X_2^{-1}\cdots X_{2r-1}X_{2r}X_{2r-1}^{-1}X_{2r}^{-1}.$$

Specialize the τ_i simultaneously to some automorphismes τ_i'. By prop. 4, we have $\sigma=W(\tau')\in D(G)$. *Q. E. D.*

Corollary 2. If G^*(or G) is irreducible, and if it is solvable as an abstract group, then it is so as an algebraic group, i. e. it has a normal chain with abelian factor groups consisting of algebraic subgroups.

This is obvious by Cor. 1. Now we can remove the restriction to irreducible G. Our original proof of this was based upon the next proposition, but Kolchin pointed out the simple facts that G_0 and G/G_0 are solvable if G is solvable, and that any subgroup of the finite group G/G_0 corresponds to some algebraic subgroup of G. The above statement follows immediately.

Proposition 6. Let H_i, H_{i+1} be subgroups of G, and let $\mathfrak{H}_i$, $\mathfrak{H}_{i+1}$ be the smallest algebraic subgroups containing H_i and H_{i+1}, respectively. If H_{i+1} is a normal subgroup of H_i and if H_i/H_{i+1} is abelian (i.e. if $H_i\supset H_{i+1}\supset D(H_i)$), then we have also $\mathfrak{H}_i\supset\mathfrak{H}_{i+1}\supset D(\mathfrak{H}_i)$.

1). The case of a group variety. First we consider the case where G^* is irreducible (hence a group variety). The proof is valid for any algebraic subgroup of an (abstract) group variety.

We shall denote by the symbol *g.pr* the geometric projection, *i. e.* the operation of taking the projection of a variety or a bunch in a product space. Note that the geometric projection is identical with the set-theoretical projection when the projection is regular.

Let Γ be the graph of the function φ: $G\times G\ni a\times b\rightarrow aba^{-1}b^{-1}\in G$. Set $\mathcal{E}^*=g.pr_{12}[(G\times G\times\mathfrak{H}^*_{i+1})\cap\Gamma]$. If $h\in H_i$, we have

$$g.pr_1[(G\times h\times\mathfrak{H}^*_{i+1})\cap\Gamma]\supset H_i,\quad \therefore\ g.pr_1[(G\times h\times\mathfrak{H}^*_{i+1})\cap\Gamma]\supset\mathfrak{H}^*_i,$$
$$\therefore\ \mathfrak{H}^*_i\times H_i\subset\mathcal{E}^*.$$

Then if $h'\in\mathfrak{H}^*_i$, we have $g.pr_2[(h'\times G\times\mathfrak{H}^*_{i+1})\cap\Gamma]\supset H_i$, hence the left side contains $\mathfrak{H}_i^*$, $\mathfrak{H}_i^*\times\mathfrak{H}_i^*\subset\mathcal{E}^*$. Therefore for any a, $b\in\mathfrak{H}_i$ we have $aba^{-1}b^{-1}\in\mathfrak{H}_{i+1}$. Thus $\mathfrak{H}_{i+1}$ contains the commutator group of $\mathfrak{H}_i$, and so is a normal subgroup of $\mathfrak{H}_i$ with abelian factor group $\mathfrak{H}_i/\mathfrak{H}_{i+1}$.

2). The general case. Let G^0 be the component of the identity of G. Set $H_i\cap G^0=H_i^0$, and let $\mathfrak{H}_i^0$ be the smallest algebraic subgroup containing H. The commutators of H_i^0 being in $H_{i+1}\cap G^0=H_{i+1}^0$, H_i^0/H_{i+1}^0 is abelian, hence $\mathfrak{H}_i^0/\mathfrak{H}_{i+1}^0$ is abelian by 1). H_i^0 is a normal subgroup of H_i of finite index. Let $\tau_jH_i^0(1\leq j\leq t_i)$ be the cosets of H_i. We can easily see that $\cup\,\tau_j\mathfrak{H}_i^0$ is the smallest algebraic set containing H_i, $\mathfrak{H}_i = \cup \tau_j \mathfrak{H}_i^0$. Considering the function ψ: $G^0 \ni a \to \tau^{-1} a \tau a^{-1} \in G^0$ (by prop. 3, this is an everywhere-defined (rational) function), we can see, by an analogous argument as in case 1), that $\tau^{-1} a \tau a^{-1} \in \mathfrak{H}_{i+1}^0$ for any $a \in \mathfrak{H}_i^0$, $\tau \in H_i$. We have also that $\tau \in H_i \Rightarrow \tau^{-1} H_{i+1}^0 \tau \subset G^0 \cap H_{i+1} = H_{i+1}^0$, $\therefore$ $\tau^{-1} H_{i+1}^0 \tau = H_{i+1}^0$, $\therefore$ $\tau^{-1} \mathfrak{H}_{i+1}^0 \tau = \mathfrak{H}_{i+1}^0$ for any $\tau \in H_i$.

Now let a, b be any two elements of $\mathfrak{H}_i$. We can write $a = \tau_\alpha a_1$, $b = \tau_\beta b_1$, $(a_1, b_1 \in \mathfrak{H}_i^0)$. Then we have

$$aba^{-1}b^{-1} = \tau_\alpha a_1 \tau_\beta b_1 a_1^{-1} \tau_\alpha^{-1} b_1^{-1} \tau_\beta^{-1}$$
$$= \tau_\alpha \tau_\beta \tau_\alpha^{-1} \tau_\beta^{-1} [\tau_\beta \{\tau_\alpha (\tau_\beta^{-1} a_1 \tau_\beta) b_1 a_1^{-1} \tau_\alpha^{-1}\} b_1^{-1} \tau_\beta^{-1}].$$

Since τ_α, $\tau_\beta \in H_i$, we have $\tau_\alpha \tau_\beta \tau_\alpha^{-1} \tau_\beta^{-1} \in H_{i+1} \subset \mathfrak{H}_{i+1}$. On the other hand, the elements [ ], { }, ( ) are in $\mathfrak{H}_i^0$. $\mathfrak{H}_i^0 / \mathfrak{H}_{i+1}^0$ being abelian we have

$$[\ ] = \tau_\beta \{\tau_\alpha (\tau_\beta^{-1} a_1 \tau_\beta) b_1 a_1^{-1} \tau_\alpha^{-1}\} b_1^{-1} \tau_\beta^{-1} \equiv \tau_\beta \{\tau_\alpha (\tau_\beta^{-1} a_1 \tau_\beta a_1^{-1}) b_1 \tau_\alpha^{-1}\} b_1^{-1} \tau_\beta^{-1}$$
$$\equiv \tau_\beta \{\tau_\alpha b_1 \tau_\alpha^{-1} b_1^{-1}\} \tau_\beta^{-1} \equiv \tau_\beta \tau_\beta^{-1} \equiv 1 \pmod{\mathfrak{H}_{i+1}^0}.$$

This completes the proof.

(In [3], Kolchin proved this proposition for the case of an algebraic matric group.)

Addendum. Kolchin gave also to prop. 1 a simpler proof as follows. There is a $\tau \in G$ such that $Q(\tau\eta) \neq 0$ because $Q(\bar{\sigma}\eta) \neq 0$. Let $\sigma_1 \in G$ be specialization of σ, and put $\rho = \tau \sigma_1^{-1}$. Then we have $\rho \in G$ $Q(\rho \sigma_1 \eta) \neq 0$. ρ and σ being independent, we have $\rho\sigma \to \rho\sigma_1$, therefore $Q(\rho\sigma\eta) \neq 0$. *Q.E.D.*

After this note was finished, we knew by his kind letter that our theorem and also the converse theorem (namely every group variety over a field of characteristic zero can be considered as the Galois group of a differential field) had been obtained by him with C. Chevalley. Professor Kolchin gave me also valuable criticisms on my proofs. I express here my heartfelt thanks for their kind appreciations to Prof. Kolchin and also to Profs. Chevalley and Akizuki.

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