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On a two-dimensional projectively connected space in the wide sense with torsion

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§ **1.** Consider a two-dimensional space R_2 whose moving point is designed by parameters u and v. Suppose that a curve C drawn on R_2 is developed on a curve Γ of a three-dimensional projective space S_3 by means of the projective connexion $\Gamma_{\alpha r}^{\beta}$ (α, $\beta=0, 1, 2, 3$; $r=1, 2$), the moving frame of reference $[A A_1 A_2 A_3]$ along to Γ being defined by

$$dA_\alpha=\Gamma_{\alpha r}^{\beta}du^r A_\beta, \tag{1·1}$$

$$\begin{pmatrix} A_0\equiv A\,; & u^1=u, & u^2=v \\ \alpha,\ \beta=0, 1, 2, 3\,; & & r=1, 2 \end{pmatrix}.$$

We shall denote hereafter by α, β, $\cdots$ the suffixes which take the values 0, 1, 2, 3; by $a, b,\cdots i, j,\cdots r, s,\cdots$ those which take the values 1, 2, except for § 2.

We use the following notations:

$$H_{rs}=\frac{1}{2}(\Gamma_{rs}^{3}+\Gamma_{sr}^{3}), \tag{1·2}$$

$$H=det\,|H_{rs}|\,, \tag{1·3}$$

$$H_{rst}=\frac{1}{3}(\varDelta_r H_{st}+\varDelta_s H_{tr}+\varDelta_t H_{rs}), \tag{1·4}$$

where
$$\varDelta_t H_{rs}=\frac{\partial H_{rs}}{\partial u^t}-\Gamma_{rt}^{a}H_{as}-\Gamma_{st}^{a}H_{ra}\,,$$

$$H_r=\frac{1}{4}H_{rst}H^{st}, \tag{1·5}$$

where H^{st} is the normalized cofactor of the element H_{ts} in H,

$$K_{rst}=H_{rst}-H_r H_{st}-H_s H_{tr}-H_t H_{rs}\,. \tag{1·6}$$

Let $R_{\alpha rs}^{\beta}$ be the curvature tensor relative to the connexion $\Gamma_{\alpha r}^{\beta}$, that is

$$R^{\beta}_{\alpha rs}=\frac{\partial\Gamma^{\beta}_{\alpha r}}{\partial u^{s}}-\frac{\partial\Gamma^{\beta}_{\alpha s}}{\partial u^{r}}+\Gamma^{\sigma}_{\alpha r}\Gamma^{\beta}_{\sigma s}-\Gamma^{\sigma}_{\alpha s}\Gamma^{\beta}_{\sigma r}\,. \tag{1·7}$$

If we suppose that $H\neq 0$, we can make without loss of generality

$$\Gamma^{\beta}_{0r}=\delta^{\beta}_{r},\qquad \Gamma^{a}_{ar}=0,\qquad \Gamma^{3}_{3r}=0, \tag{1·8}$$

$$H=1,\qquad H_{r}=0, \tag{1·9}$$

$$K_{rst}=H_{rst}\,. \tag{1·10}$$

Moreover, if we take the asymptotic parameters u and v, we get

$$H_{11}=H_{22}=0,\qquad H_{12}=H_{21}=\pm 1\,. \tag{1·11}$$

§ **2.** Suppose now that the curves on the n-dimensional space R_n with the dominant projective connexion are developed into a curve on a projective space S_N. We have the following theorem concerning the existence of a surface V_n which lies in S_N and has a contact of ν-th order with the development of R_n.*)

THEOREM. *The necessary and sufficient conditions for the existence of a surface V_n having a contact of ν-th order are expressed by*

$$D_{i_\lambda}D_{i_{\lambda-1}}\cdots D_{i_1}R^{p}_{0lm}=0\quad\text{for}\quad 0\leq\lambda\leq\nu-3, \tag{2·1}$$

$$\begin{pmatrix} i,\,l,\,m=1,\,2,\,\cdots,\,n\,; \\ p=n+1,\,\cdots\cdots,\,N \end{pmatrix}.$$

If these conditions are satisfied, and the equation of V_n is denoted by

$$z^{p}=\frac{1}{2}H^{p}_{ij}z^{i}z^{j}+\cdots+\frac{1}{\nu!}H^{p}_{i_1\cdots i_\nu}z^{i_1}\cdots z^{i_\nu}+\cdots\,, \tag{2·2}$$

the coefficients $H^{p}_{i_1\cdots i_\lambda}$ $(\lambda=2,\,\cdots,\,\nu)$ are determined by

$$\begin{aligned} H^{p}_{i_1\cdots i_\lambda}=\Delta_{i_\lambda}H^{p}_{i_1\cdots i_{\lambda-1}}&+\sum_{s=2}^{\lambda-3}(s-1)\sum_{j}H^{p}_{j_1\cdots j_s}H^{q}_{j_{s+1}\cdots j_{\lambda-1}}\Gamma^{0}_{qi_\lambda}\\ &+\sum_{j}(\lambda-3)H^{p}_{j_1\cdots j_{\lambda-2}}\Gamma^{0}_{j_{\lambda-1}i_\lambda}\\ &-\sum_{s=2}^{\lambda-2}\sum_{j}H^{q}_{j_1\cdots j_s}H^{p}_{j_{s+1}\cdots j_{\lambda-1}a}\Gamma^{a}_{qi_\lambda}\,,\\ &(q:n+1\to N,\quad a:1\to n) \end{aligned} \tag{2·3}$$

and such $H^{p}_{i_1\cdots i_\lambda}$ $(\lambda=2,\,\cdots,\,\nu-1)$ are symmetric relative to $i_1\cdots i_\lambda$, and moreover we have

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*) J. Kanitani: "Sur les surfaces osculatrices à un espace à connexion projective majorante," Mem. Col. of Sci., Univ. Kyoto, Ser. A. XXVI, No. 3, 1951.

$$(2\cdot4)\quad D_{i_\lambda}\cdots D_{i_1}R^p_{0lm}=(-1)^{\lambda-1}\{H^p_{i_1\cdots i_\lambda a}R^a_{0lm}+\sum_j H^p_{j_1\cdots j_{\lambda-1}a}R^a_{j_\lambda lm}$$
$$+\sum_{s=2}^{\lambda-1}\sum_j H^q_{j_1\cdots j_s}H^p_{j_{s+1}\cdots j_\lambda a}R^a_{qlm}$$
$$-(\lambda-1)H^p_{i_1\cdots i_\lambda}R^0_{0lm}-H^q_{i_1\cdots i_\lambda}R^p_{qlm}$$
$$-(\lambda-2)\sum_j H^p_{j_1\cdots j_{\lambda-1}}R^0_{j_\lambda lm}$$
$$-\sum_{s=2}^{\lambda-2}(s-1)\sum_j H^p_{j_1\cdots j_s}H^q_{j_{s+1}\cdots j_\lambda}R^0_{qlm}\}$$
$$\text{for}\quad 2\leq\lambda\leq\nu-2.$$

§ 3. Always, there is a surface V_2 having a contact of second order at the image point A of a point $P(uv)$ on R_2 with the development of any curve passing through the point $P(uv)$.

The condition for a contact of third order is

$$(3\cdot1)\qquad R^3_{0rs}\equiv0,$$

that is,

$$(3\cdot2)\qquad \Gamma^3_{rs}=\Gamma^3_{sr}=H_{rs}.$$

Then, such surface is expressed by

$$z^3=\frac{1}{2}H_{rs}z^rz^s+\frac{1}{6}H_{rst}z^rz^sz^t+\cdots.$$

Next, owing to $(1\cdot8)$, the conditions for a contact of 4-th order are

$$(3\cdot3)\qquad 0\equiv D_tR^3_{0rs}=H_{at}R^a_{0rs}-R^3_{trs}.$$

In $(3\cdot3)$, if we put $t=1$, $r=1$, $s=2$,

$$0=H_{21}R^2_{012}-R^3_{112}=2H_{21}\Gamma_{12}{}^2,$$

and if we put $t=2$, $r=1$, $s=2$,

$$0=2H_{12}\Gamma_{21}{}^1.$$

Consequently,

$$(3\cdot4)\qquad \Gamma_{12}{}^2=\Gamma_{21}{}^1=0.$$

Therefore,

$$(3\cdot5)\qquad \begin{cases} K_{111}=-2H_{12}\Gamma_{11}{}^2,\\ K_{112}=K_{121}=K_{211}=K_{221}=K_{212}=K_{122}=0,\\ K_{222}=-2H_{12}\Gamma_{22}{}^1.\end{cases}$$

For a contact of 5-th order, we have

$$(3\cdot6)\qquad 0\equiv D_j D_i R^{3}_{0rs} = -\{H_{ija}R^{a}_{0rs}+H_{ia}R^{a}_{jrs}+H_{ja}R^{a}_{irs} - H_{ij}(R^{0}_{0rs}+R^{3}_{3rs})\}.$$

Multiply the right hand side by H^{ij}, and sum up those expressions, then we get

$$0=R^{a}_{ars}-(R^{0}_{0rs}+R^{3}_{3rs}).$$

But, in general, we have

$$0=R^{\alpha}_{\alpha rs}=R^{0}_{0rs}+R^{a}_{ars}+R^{3}_{3rs}.$$

Hence,

$$(3\cdot7)\qquad R^{a}_{ars}=0,\qquad R^{0}_{0rs}+R^{3}_{3rs}=0.$$

Consequently, (3·6) becomes

$$(3\cdot8)\qquad 0=H_{ija}R^{a}_{0rs}+H_{ia}R^{a}_{jrs}+H_{ja}R^{a}_{irs}.$$

The coefficients H_{ijhk} are as follows:

$$(3\cdot9)\qquad \begin{cases} H_{1111}=\dfrac{\partial K_{111}}{\partial u}-3\Gamma_{11}{}^{1}K_{111}, \\ H_{1112}=\dfrac{\partial K_{111}}{\partial v}+3\Gamma_{22}{}^{2}K_{111}, \\ H_{1122}=2H_{12}(\Gamma_{22}{}^{1}\Gamma_{11}{}^{2}+\Gamma_{12}{}^{0}-H_{12}\Gamma_{32}{}^{2}) \\ \qquad\;\;=2H_{12}(\Gamma_{22}{}^{1}\Gamma_{11}{}^{2}+\Gamma_{21}{}^{0}-H_{12}\Gamma_{31}{}^{1}), \\ H_{2221}=\dfrac{\partial K_{222}}{\partial u}+3\Gamma_{11}{}^{1}K_{222}, \\ H_{2222}=\dfrac{\partial K_{222}}{\partial v}-3\Gamma_{22}{}^{2}K_{222}, \end{cases}$$

and H_{ijhk} are symmetric relative to i, j, h, k.

For order 6, by (3·7), we have

$$\begin{aligned}(3\cdot10)\qquad 0&\equiv D_k D_j D_i R^{3}_{0rs} \\ &=H_{ijka}R^{a}_{0rs} \\ &\quad+(H_{ija}R^{a}_{krs}+H_{kia}R^{a}_{jrs}+H_{jka}R^{a}_{irs}) \\ &\quad+(H_{ij}H_{ka}+H_{ki}H_{ja}+H_{jk}H_{ia})R^{a}_{3rs} \\ &\quad-H_{ijk}R^{0}_{0rs}-(H_{ij}R^{0}_{krs}+H_{ki}R^{0}_{jrs}+H_{jk}R^{0}_{irs}).\end{aligned}$$

And if we put $\lambda=5$ in (2·3),

$$
(3\cdot 11)\quad
\begin{cases}
H_{11111} = \dfrac{\partial^2 K_{111}}{(\partial u)^2} - 7\Gamma_{11}^{\ 1}\dfrac{\partial K_{111}}{\partial u} - 4\Gamma_{11}^{\ 2}\dfrac{\partial K_{111}}{\partial v} \\
\qquad - K_{111}\left\{3\dfrac{\partial \Gamma_{11}^{\ 1}}{\partial u} - 8\Gamma_{11}^{\ 0} - 12(\Gamma_{11}^{\ 1})^2 + 12\Gamma_{11}^{\ 2}\Gamma_{22}^{\ 2} + 4H_{12}\Gamma_{31}^{\ 2}\right\}, \\
H_{11112} = \dfrac{\partial^2 K_{111}}{\partial u\,\partial v} - 3\Gamma_{11}^{\ 1}\dfrac{\partial K_{111}}{\partial v} + 4\Gamma_{22}^{\ 2}\dfrac{\partial K_{111}}{\partial u} \\
\qquad - K_{111}\left\{3\dfrac{\partial \Gamma_{11}^{\ 1}}{\partial v} - 8\Gamma_{12}^{\ 0} + 12\Gamma_{11}^{\ 1}\Gamma_{22}^{\ 2} + 4H_{12}\Gamma_{32}^{\ 2}\right\}, \\
H_{11122} = \dfrac{\partial^2 K_{111}}{(\partial v)^2} + 5\Gamma_{22}^{\ 2}\dfrac{\partial K_{111}}{\partial v} - \Gamma_{22}^{\ 1}\dfrac{\partial K_{111}}{\partial u} \\
\qquad + K_{111}\left\{3\dfrac{\partial \Gamma_{22}^{\ 2}}{\partial v} + 2\Gamma_{22}^{\ 0} + 3\Gamma_{11}^{\ 1}\Gamma_{22}^{\ 1} + 6(\Gamma_{22}^{\ 2})^2 - 4H_{12}\Gamma_{32}^{\ 1}\right\}, \\
H_{22211} = \dfrac{\partial^2 K_{222}}{(\partial u)^2} + 5\Gamma_{11}^{\ 1}\dfrac{\partial K_{222}}{\partial u} - \Gamma_{11}^{\ 2}\dfrac{\partial K_{222}}{\partial v} \\
\qquad + K_{222}\left\{3\dfrac{\partial \Gamma_{11}^{\ 1}}{\partial u} + 2\Gamma_{11}^{\ 0} + 3\Gamma_{22}^{\ 2}\Gamma_{11}^{\ 2} + 6(\Gamma_{11}^{\ 1})^2 - 4H_{12}\Gamma_{31}^{\ 2}\right\}, \\
H_{22221} = \dfrac{\partial^2 K_{222}}{\partial u\,\partial v} - 3\Gamma_{22}^{\ 2}\dfrac{\partial K_{222}}{\partial u} + 4\Gamma_{11}^{\ 1}\dfrac{\partial K_{222}}{\partial v} \\
\qquad - K_{222}\left\{3\dfrac{\partial \Gamma_{22}^{\ 2}}{\partial u} - 8\Gamma_{21}^{\ 0} + 12\Gamma_{11}^{\ 1}\Gamma_{22}^{\ 2} + 4H_{12}\Gamma_{31}^{\ 1}\right\}, \\
H_{22222} = \dfrac{\partial^2 K_{222}}{(\partial v)^2} - 7\Gamma_{22}^{\ 2}\dfrac{\partial K_{222}}{\partial v} - 4\Gamma_{22}^{\ 1}\dfrac{\partial K_{222}}{\partial u} \\
\qquad - K_{222}\left\{3\dfrac{\partial \Gamma_{22}^{\ 2}}{\partial v} - 8\Gamma_{22}^{\ 0} - 12(\Gamma_{22}^{\ 2})^2 + 12\Gamma_{11}^{\ 1}\Gamma_{22}^{\ 1} + 4H_{12}\Gamma_{32}^{\ 1}\right\}.
\end{cases}
$$

Finally, for order 7, we have

$$
\begin{aligned}
(3\cdot 12)\qquad 0 \equiv{} & H_{ijhk}R^{a}_{0rs} \\
& + (H_{ijha}R^{a}_{krs} + H_{kija}R^{a}_{hrs} + H_{hkia}R^{a}_{jrs} + H_{jhka}R^{a}_{irs}) \\
& + (H_{ij}H_{hka} + H_{ih}H_{jka} + H_{ik}H_{jha} + H_{hk}H_{ija} + H_{jk}H_{iha} \\
& \quad + H_{jh}H_{ika} + H_{ijh}H_{ka} + H_{kij}H_{ha} + H_{hki}H_{ja} + H_{jhk}H_{ia})R^{a}_{3rs} \\
& - 2H_{ijhk}R^{0}_{0rs} \\
& - 2(H_{ijh}R^{0}_{krs} + H_{kij}R^{0}_{hrs} + H_{hki}R^{0}_{jrs} + H_{jhk}R^{0}_{irs}) \\
& - 2(H_{ij}H_{hk} + H_{ih}H_{jk} + H_{ik}H_{jh})R^{0}_{3rs}\,.
\end{aligned}
$$

§ 4. Conveniently, we use the following notations:

$$(4\cdot1)\qquad \begin{cases} \varepsilon=H_{12}=H_{21}=\pm1\,, \\ K_1=K_{111}, \quad K_2=K_{222}, \\ a_1=H_{1111}, \quad b_1=H_{1112}, \quad c=H_{1122}, \\ a_2=H_{2222}, \quad b_2=H_{2221}, \\ L_1=H_{11111}, \quad M_1=H_{11112}, \quad N_1=H_{11122}, \\ L_2=H_{22222}, \quad M_2=H_{22221}, \quad N_2=H_{22211}. \end{cases}$$

At this time, (3·1), (3·3), (3·7), (3·8), (3·10) and (3·12) are expressed as follows:

$$(4\cdot2)\qquad \begin{cases} 0=R^{3}_{0rs}\,, \\ 0=\varepsilon R^{2}_{0rs}-R^{3}_{1rs}\,, \\ 0=\varepsilon R^{1}_{0rs}-R^{3}_{2rs}\,, \\ 0=R^{0}_{0rs}+R^{3}_{3rs}\,, \\ 0=R^{1}_{1rs}+R^{2}_{2rs}\,, \\ 0=K_1R^{1}_{0rs}+2\varepsilon R^{2}_{1rs}\,, \\ 0=K_2R^{2}_{0rs}+2\varepsilon R^{1}_{2rs}\,, \\ 0=a_1R^{1}_{0rs}+b_1R^{2}_{0rs}+3K_1R^{1}_{1rs}-K_1R^{0}_{0rs}\,, \\ 0=b_1R^{1}_{0rs}+cR^{2}_{0rs}+K_1R^{1}_{2rs}+2R^{2}_{3rs}-2\varepsilon R^{0}_{1rs}\,, \\ 0=cR^{1}_{0rs}+b_2R^{2}_{0rs}+K_2R^{2}_{1rs}+2R^{1}_{3rs}-2\varepsilon R^{0}_{2rs}\,, \\ 0=b_2R^{1}_{0rs}+a_2R^{2}_{0rs}+3K_2R^{2}_{2rs}-K_2R^{0}_{0rs}\,, \\ 0=L_1R^{1}_{0rs}+M_1R^{2}_{0rs}+4a_1R^{1}_{1rs}+4b_1R^{2}_{1rs}+4\varepsilon K_1R^{2}_{3rs} \\ \qquad -2a_1R^{0}_{0rs}-8K_1R^{0}_{1rs}\,, \\ 0=M_1R^{1}_{0rs}+N_1R^{2}_{0rs}+a_1R^{1}_{2rs}+2b_1R^{1}_{1rs}+3cR^{2}_{1rs}+4\varepsilon K_1R^{1}_{3rs} \\ \qquad -2b_1R^{0}_{0rs}-2K_1R^{0}_{2rs}\,, \\ 0=N_1R^{1}_{0rs}+N_2R^{2}_{0rs}+2b_1R^{1}_{2rs}+2b_2R^{2}_{1rs}-2cR^{0}_{0rs}-4R^{0}_{3rs}\,, \\ 0=N_2R^{1}_{0rs}+M_2R^{2}_{0rs}+a_2R^{2}_{1rs}+2b_2R^{2}_{2rs}+3cR^{1}_{2rs}+4\varepsilon K_2R^{2}_{3rs} \\ \qquad -2b_2R^{0}_{0rs}-2K_2R^{0}_{1rs}\,, \\ 0=M_2R^{1}_{0rs}+L_2R^{2}_{0rs}+4b_2R^{1}_{2rs}+4a_2R^{2}_{2rs}+4\varepsilon K_2R^{1}_{3rs} \\ \qquad -2a_2R^{0}_{0rs}-8K_2R^{0}_{2rs}\,. \end{cases}$$

Now, we shall compute the determinant of the matrix Δ consisting of the coefficients of R's in above equations. For the sake, if we put $R^{1}_{0\,rs}=0$, then we have $R^{3}_{2\,rs}=0$ and $R^{2}_{1\,rs}=0$ by (3·3) and (3·8). In this situation, the above matrix is essentially equivalent to the following matrix:

$$\Delta' = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & K_2 & 0 & 0 & 0 & 2\varepsilon & 0 & 0 & 0 \\ K_1 & b_1 & 0 & 3K_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & \varepsilon & 0 & 0 & K_1 & 0 & 0 & 1 \\ 0 & b_2 & 0 & 0 & \varepsilon & 0 & 0 & 1 & 0 \\ K_2 & a_2 & 0 & 0 & 0 & 0 & 3K_2 & 0 & 0 \\ 2a_1 & M_1 & 4K_1 & 4a_1 & 0 & 0 & 0 & 0 & 2\varepsilon K_1 \\ 2b_1 & N_1 & 0 & 2b_1 & K_1 & a_1 & 0 & 2\varepsilon K_1 & 0 \\ 2b_2 & M_2 & K_2 & 0 & 0 & 3c & 2b_2 & 0 & 2\varepsilon K_2 \\ 2a_2 & L_2 & 0 & 0 & 4K_2 & 4b_2 & 4a_2 & 2\varepsilon K_2 & 0 \end{pmatrix}.$$

Then, we shall denote by Δ_i the square matrix omitted the i-th column from Δ'. Let us compute the determinant $|\Delta|$ by the aid of those matrices.

Case I. $K_1 K_2 \equiv 0$.
If $K_1 \equiv 0$, we have

$$a_1 = b_1 = L_1 = M_1 = 0,$$

therefore

$$|\Delta_i| = 0 \quad \text{for every } i.$$

Hence it follows that

$$|\Delta| = 0.$$

For $K_2 \equiv 0$, we have

$$a_2 = b_2 = L_2 = M_2 = 0$$

therefore, similarly,

$$|\Delta_i| = 0 \quad \text{for every, } i,$$

and so,

$$|\Delta| = 0.$$

Case II. $K_1 \not\equiv 0$ and $K_2 \not\equiv 0$.
We shall omit the points where $K_1 = 0$ or $K_2 = 0$. Then, by the elementary computation, we get

$$
(4\cdot3)\quad \begin{cases}
|\varDelta_{10}|=4\varepsilon K_1\{3K_1(K_2)^2M_1+6(K_1)^2K_2M_2-8(K_1)^2a_2b_2 \\
\qquad -K_1K_2a_1a_2-4K_1K_2b_1b_2-5(K_2)^2a_1b_1 \\
\qquad -27\varepsilon(K_1K_2)^2c+9(K_1K_2)^3\}, \\
|\varDelta_9|=2\dfrac{K_1}{K_2}|\varDelta_7|, \\
|\varDelta_8|=2\dfrac{K_2}{K_1}|\varDelta_{10}|, \\
|\varDelta_7|=4\varepsilon K_2\{3(K_1)^2K_2L_2+6K_1(K_2)^2N_1-8(K_2)^2(b_1)^2 \\
\qquad -5K_1K_2a_2b_1-5(K_1)^2(a_2)^2-24\varepsilon(K_1K_2)^2b_2 \\
\qquad -3\varepsilon K_1(K_2)^3a_1\}, \\
|\varDelta_6|=\dfrac{8K_1b_2+K_2a_1}{3(K_2)^2}|\varDelta_7|-\dfrac{5K_1a_2+4K_2b_1}{3K_1K_2}|\varDelta_{10}|, \\
|\varDelta_5|=6\varepsilon K_2|\varDelta_{10}|, \\
|\varDelta_4|=6\varepsilon K_1|\varDelta_7|, \\
|\varDelta_3|=\dfrac{8K_2b_1+K_1a_2}{3(K_1)^2}|\varDelta_{10}|-\dfrac{5K_2a_1+4K_1b_2}{3K_1K_2}|\varDelta_7|, \\
|\varDelta_2|=3K_1\dfrac{\varepsilon c-K_1K_2}{K_2}|\varDelta_7|-\varepsilon\dfrac{K_2a_1+2K_1b_2}{K_1}|\varDelta_{10}|, \\
|\varDelta_1|=\dfrac{4K_1b_2+K_2a_1}{K_2}|\varDelta_7|-\dfrac{K_1a_2+4K_2b_1}{K_1}|\varDelta_{10}|.
\end{cases}
$$

Consequently,

$$
\begin{aligned}
\rho|\varDelta|=&-2\varepsilon a_1|\varDelta_3|+2\varepsilon b_1|\varDelta_4|+(K_1K_2-2\varepsilon c)|\varDelta_5| \\
&+2\varepsilon b_2|\varDelta_6|+(4K_1b_1-2\varepsilon L_1)|\varDelta_7| \\
&+(2\varepsilon M_1-3K_1c)|\varDelta_8|+(K_1a_2-2\varepsilon N_2)|\varDelta_9|+2\varepsilon M_2|\varDelta_{10}|,
\end{aligned}
$$

where ρ is a certain constant.
Then if we substitute the above expressions for $|\varDelta_i|$,

$$
\begin{aligned}
\rho|\varDelta|=&\frac{|\varDelta_{10}|}{K_1}\frac{1}{K_1K_2}\{6K_1(K_2)^2M_1+3(K_1)^2K_2M_2-8(K_2)^2a_1b_1 \\
&-K_1K_2a_1a_2-4K_1K_2b_1b_2-5(K_1)^2a_2b_2-27\varepsilon(K_1K_2)^2c+9(K_1K_2)^3\} \\
&-\frac{|\varDelta_7|}{K_2}\frac{1}{K_1K_2}\{6(K_1)^2K_2N_2+3K_1(K_2)^2L_1-8(K_1)^2(b_2)^2
\end{aligned}
$$

$$-5K_1K_2a_1b_2-5(K_2)^2(a_1)^2-24\varepsilon(K_1K_2)^2b_1-3\varepsilon(K_1)^3K_2a_2\},$$

therefore,

$$\begin{aligned}\rho K_1K_2|\varDelta|=&\{3K_1(K_2)^2M_1+6(K_1)^2K_2M_2-8(K_1)^2a_2b_2-K_1K_2a_1a_2\\&-4K_1K_2b_1b_2-5(K_1)^2a_1b_1-27\varepsilon(K_1K_2)^2c+9(K_1K_2)^3\}\\&\times\{3(K_1)^2K_2M_2+6K_1(K_2)^2M_1-8(K_2)^2a_1b_1-K_1K_2a_1a_2\\&-4K_1K_2b_1b_2-5(K_2)^2a_2b_2-27\varepsilon(K_1K_2)^2c+9(K_1K_2)^3\}\\&-\{3(K_1)^2K_2L_2+6K_1(K_2)^2N_1-8(K_2)^2(b_1)^2-5(K_1)^2(a_2)^2\\&-5K_1K_2a_2b_1-24\varepsilon(K_1K_2)^2b_2-3\varepsilon K_1(K_2)^3a_1\}\\&\times\{3K_1(K_2)^2L_1+6(K_1)^2K_2N_2-8(K_1)^2(b_2)^2-5(K_2)^2(a_1)^2\\&-5K_1K_2a_1b_2-24\varepsilon(K_1K_2)^2b_1-3\varepsilon(K_1)^3K_2a_2\},\end{aligned}$$

where ρ is a certain constant. Using (3·10) and the relation

$$\frac{1}{K}\frac{\partial^2K}{\partial u^r\partial u^s}=\frac{\partial^2\log K}{\partial u^r\partial u^s}+\frac{\partial\log K}{\partial u^r}\frac{\partial\log K}{\partial u^s},$$

we have

$$\begin{aligned}\text{1st term}=(K_1K_2)^2\Big\{&3\frac{\partial^2\log K_1(K_2)^2}{\partial u\partial v}-\frac{\partial\log K_1(K_2)^2}{\partial u}\frac{\partial\log(K_1)^2K_2}{\partial v}\\&+9\frac{\partial\Gamma'^1_{11}}{\partial v}-3\Gamma'^1_{11}\frac{\partial\log(K_1)^2K_2}{\partial u}\Big\},\end{aligned}$$

$$\begin{aligned}\text{2nd term}=(K_1K_2)^2\Big\{&3\frac{\partial^2\log(K_1)^2K_2}{\partial u\partial v}-\frac{\partial\log(K_1)^2K_2}{\partial v}\frac{\partial\log K_1(K_2)^2}{\partial u}\\&+9\frac{\partial\Gamma'^2_{22}}{\partial u}-3\Gamma'^2_{22}\frac{\partial\log K_1(K_2)^2}{\partial u}\Big\},\end{aligned}$$

$$\begin{aligned}\text{3rd term}=(K_1K_2)^2\Big\{&3\frac{\partial^2\log(K_1)^2K_2}{(\partial v)^2}-\frac{\partial\log(K_1)^2K_2}{\partial v}\frac{\partial\log K_1(K_2)^2}{\partial v}\\&+9\frac{\partial\Gamma'^2_{22}}{\partial v}-3\Gamma'^2_{22}\frac{\partial\log K_1(K_2)^2}{\partial v}\Big\},\end{aligned}$$

$$\begin{aligned}\text{4th term}=(K_1K_2)^2\Big\{&3\frac{\partial^2\log K_1(K_2)^2}{(\partial u)^2}-\frac{\partial\log K_1(K_2)^2}{\partial u}\frac{\partial\log(K_1)^2K_2}{\partial u}\\&+9\frac{\partial\Gamma'^1_{11}}{\partial u}-3\Gamma'^1_{11}\frac{\partial\log(K_1)^2K_2}{\partial u}\Big\}.\end{aligned}$$

Put

$$(4\cdot4)\qquad \begin{cases} X_1=\log(K_1)^2K_2\,, \\ X_2=\log K_1(K_2)^2\,, \\ Y_1=3\Gamma_{11}^{\prime 1}+\dfrac{\partial X_2}{\partial u}\,, \\ Y_2=3\Gamma_{22}^{\prime 2}+\dfrac{\partial X_1}{\partial v}\,, \end{cases}$$

then

$$|\Delta|=\rho(K_1K_2)^3\left\{\left(3\frac{\partial Y_1}{\partial u}-\frac{\partial X_1}{\partial u}Y_1\right)\left(3\frac{\partial Y_2}{\partial v}-\frac{\partial X_2}{\partial v}Y_2\right)\right.$$
$$\left.-\left(3\frac{\partial Y_1}{\partial v}-\frac{\partial X_1}{\partial v}Y_1\right)\left(3\frac{\partial Y_2}{\partial u}-\frac{\partial X_2}{\partial u}Y_2\right)\right\}.$$

Case II-i. $Y_1Y_2\equiv0$.
At this time, clearly,

$$|\Delta|=0.$$

Case II-ii. $Y_1\not\equiv0$ and $Y_2\not\equiv0$.

$$|\Delta|=\rho(K_1K_2)^3Y_1Y_2\left\{\left(3\frac{\partial\log Y_1}{\partial u}-\frac{\partial X_1}{\partial u}\right)\left(3\frac{\partial\log Y_2}{\partial v}-\frac{\partial X_2}{\partial v}\right)\right.$$
$$\left.-\left(3\frac{\partial\log Y_1}{\partial v}-\frac{\partial X_1}{\partial v}\right)\left(3\frac{\partial\log Y_2}{\partial u}-\frac{\partial X_2}{\partial u}\right)\right\}.$$

If we put

$$(4\cdot5)\qquad \begin{cases} U_1=3\log Y_1-X_1, \\ U_2=3\log Y_2-X_2, \end{cases}$$

it follows that

$$|\Delta|=\rho(K_1K_2)^3Y_1Y_2\frac{\partial(U_1U_2)}{\partial(uv)},$$

where ρ is a certain constant.

So, we get the following theorem:

THEOREM. *A two-dimensional space with a dominant projective connexion which cannot be imbeded in a three-dimensional projective space admits an osculating surface of 7-th order if and only if one of the following three canditions is satisfied,*

(A) $$K_1K_2\equiv 0\,,$$

(B) $$Y_1Y_2\equiv 0,$$

(C) $$\frac{\partial(U_1U_2)}{\partial(uv)}\equiv 0.$$

$K_1K_2\equiv 0$ means that *the three Darboux tangents are coincident.*

We next consider a geometrical meaning of the equation $Y_1Y_2 \equiv 0$. If we put

(4·6) $$\begin{cases} E=K_{rst}K^{rst}\,, \\ E_r=\dfrac{\partial E_r}{\partial u^r}\,, \\ \theta_r=H^a_{ast}K^{st}_r\,, \end{cases}$$

those are expressed by

$$E=2\varepsilon K_1K_2\,,$$

$$E_r=2\varepsilon K_1K_2\frac{\partial \log K_1K_2}{\partial u^r}\,,$$

$$\theta_1=2\varepsilon K_1K_2\left(3\Gamma_{11}^{\prime 1}+\frac{\partial \log K_2}{\partial u}\right),$$

$$\theta_2=2\varepsilon K_1K_2\left(3\Gamma_{22}^{\prime 2}+\frac{\partial \log K_1}{\partial v}\right).$$

Accordingly

$$\frac{E_1+\theta_1}{E}=Y_1\,,$$

$$\frac{E_2+\theta_2}{E}=Y_2\,.$$

Since the space $\boldsymbol{R}_2$ admits the surface which has the contact of four-th with it, the canonical pencil can be defined as the same way in the ordinary projective differential geometry: the lines joining the point

$$\frac{p(E_2+\theta_2)-\theta_2}{2\varepsilon E}A_1+\frac{p(E_1+\theta_1)-\theta_1}{2\varepsilon E}A_2+A_3\,,$$

(p: arbitrary constant)

with $\boldsymbol{A}$ are the canonical edges of the first kind. When we make vary p, these edges form a flat pencil having A as vertex. Let z^1, z^2, z^3 be non-homogeneous coordinates of a point refered to the frame $[A A_1 A_2 A_3]$. Now AA_1, AA_2 are asymptotic tangents at A. The plane of said pencil is gived by

$$(E_1+\theta_1)z^1-(E_2+\theta_2)z^2=\frac{E_2\theta_1-E_1\theta_2}{2\varepsilon E}z^3.$$

Therefore, $Y_1Y_2\equiv 0$ means that *the plane of the canonical pencil of the first kind intersects the tangent plane along an asymptotic tangent.*

§ **5.** In the case where torsions are zero, that is,

$$R^{a}_{0rs}=0, \tag{5·1}$$

(3·1), (3·3) and (3·8) become

$$R^{3}_{0rs}=0,\quad R^{3}_{irs}=0, \tag{5·2}$$
$$R^{1}_{2rs}=R^{2}_{1rs}=0.$$

Next, let i, j, k in (3·10) take the values 1, 2, then, if $K_1K_2\neq 0$, we have

$$R^{0}_{0rs}=3R^{1}_{1rs}=3R^{2}_{2rs}, \tag{5·3}$$
$$R^{0}_{krs}=H_{ka}R^{a}_{3rs}.$$

Hence, by (3·7),

$$R^{0}_{0rs}=0,\quad R^{3}_{3rs}=0, \tag{5·4}$$

$$R^{a}_{brs}=0. \tag{5·5}$$

Consequently, *if two-dimensional space with a projective connexion without torsion admits an osculating surface of 6-th order, this connexion must be symmetric.*

In virture of (5·1), (5·2), (5·3), (5·4) and (5·5), (3·12) is turned to

$$\begin{aligned}0=&(H_{ij}H_{hka}+H_{ih}H_{jka}+H_{ik}H_{jha}+H_{hk}H_{ija}+H_{jk}H_{iha}+H_{jh}H_{ika}\\&-H_{ijh}H_{ka}-H_{kij}H_{ha}-H_{hki}H_{ja}-H_{jhk}H_{ia})R^{a}_{3rs}\\&-2(H_{ij}H_{hk}+H_{ih}H_{jk}+H_{ik}H_{jh})R^{0}_{3rs}.\end{aligned}$$

If we put $i=j=k=1$, 2,

$$0=-4\varepsilon K_1R^{2}_{3rs},$$

$$0=-4\varepsilon K_2 R^{1}_{3rs},$$

so,

$$R^{a}_{3rs}=0, \qquad R^{0}_{krs}=0. \tag{5·6}$$

Therefore

$$R^{0}_{3rs}=0. \tag{5·7}$$

Thus, *if a two-dimensional projectively connected space in the wide sense without torsion cannot be imbeded in a three-dimensional projective space, this space admits an osculating surface of 7-th order if and only if the three Darboux tangents do not coincide.*

This result coincides with the general theory obtained by Prof. J. Kanitani.*)

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*) J. Kanitani. (In the press)