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Dimensional differentiation of harmonic tensors for variations of Riemannian metric

By

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The purpose of this paper is to find some properties of harmonic tensors defined in a domain with boundary, when the Riemannian metric undergoes an infinitesimal change. The variations of characteristic roots and Green's tensor are obtained. The notion of abstract dimension is introduced to preserve the duality between differential and codifferential under the change of metric. An application of the abstract dimension to a physical problem is in the last paragraph.

§ 1. Notations and formulas.

Let M be an orientable Riemannian space of dimension n and of class C^∞ for simplicity, the positive definite metric tensor be g_{ij}. Let D be a bounded connected open set with regular boundary B.

If A^i_{jk} and A_{ijk}, for example, are associated tensors, we shall denote them by one and the same symbol A. A tensor is called skew-symmetric, if its associated covariant tensor is skew-symmetric.

We shall adopt the following notations for skew-symmetric tensors A and B:

$$(A^*)_{i_1 i_2 \cdots i_{n-p}} = \frac{1}{p!} A_{j_1 \cdots j_p} \varepsilon^{j_1 \cdots j_p}{}_{i_1 \cdots i_{n-p}},$$

$$(\varDelta A)_{i_1 i_2 \cdots i_{p+1}} = \frac{1}{p!} \delta^{j_1 j_2 \cdots j_{p+1}}_{i_1 i_2 \cdots i_{p+1}} D_{j_1} A_{j_2 \cdots j_{p+1}}, \quad \text{in } D,$$

$$(\nabla A)_{i_2 \cdots i_p} = D^{i_1} A_{i_1 i_2 \cdots i_p}, \quad \text{in } D,$$

$$(1)\qquad (\perp A)_{i_1 i_2\cdots i_{p+1}}=\frac{1}{p!}\delta^{j_1 j_2\cdots j_{p+1}}_{i_1 i_2\cdots i_{p+1}} N_{j_1} A_{j_2\cdots j_{p+1}}\quad \text{on } B,$$

$$(\top A)_{i_2\cdots i_p}=N^{i_1}A_{i_1 i_2\cdots i_p},\qquad \text{on } B,$$

$$(A\cdot B)=\frac{1}{p!}A_{i_1\cdots i_p}B^{i_1\cdots i_p},$$

where D_i denotes the covariant differentiation, N the outwards unit normal vector to the boundary B, and $\varepsilon_{i_1\cdots i_n}=\sqrt{g}\,\delta_{i_1\cdots i_n}$.

We get easily

$$(2)\qquad \begin{aligned} &\varDelta\varDelta A=0, \qquad \varGamma\varGamma A=0\\ &A=\perp\top A+\top\perp A\\ &\top\top A=0, \qquad \perp\perp A=0\\ &\top\perp\top A=\top A, \qquad \perp\top\perp A=\perp A. \end{aligned}$$

Thus $\perp\top A$ is the normal part of A and $\top\perp A$ the tangential part.

Putting $dV=\varepsilon_{i_1\cdots i_n}dx^{i_1}\cdots dx^{i_n}$, $d\sigma_{i_1}=\varepsilon_{i_1 i_2\cdots i_n}dx^{i_2}\cdots dx^{i_n}$, $dS=N^i d\sigma_i$, we have the following well-known formulas for skew-symmetric tensors A and B:

$$(3)\qquad \begin{aligned} &\oint(\top A\cdot B)dS=\int\!\!\int(\varGamma A\cdot B)dV+\int\!\!\int(A\cdot\varDelta B)dV=\oint(A\cdot\perp B)dS,\\ &\oint(\top\varDelta A\cdot B)dS=\int\!\!\int(\varGamma\varDelta A\cdot B)dV+\int\!\!\int(\varDelta A\cdot\varDelta B)dV\\ &\qquad\qquad=\oint(\varDelta A\cdot\perp B)dS,\\ &\oint(\top A\cdot\varGamma B)dS=\int\!\!\int(\varGamma A\cdot\varGamma B)dV+\int\!\!\int(A\cdot\varDelta\varGamma B)dV\\ &\qquad\qquad=\oint(A\cdot\perp\varGamma B)dS,\\ &\int\!\!\int(\Box A\cdot B)dV+\int\!\!\int(\varDelta A\cdot\varDelta B)dV+\int\!\!\int(\varGamma A\cdot\varGamma B)dV\\ &=\oint(\top\varDelta A\cdot B)dS+\oint(\perp\varGamma A\cdot B)dS\\ &=\oint(\varDelta A\cdot\perp B)dS+\oint(\varGamma A\cdot\top B)dS, \end{aligned}$$

where $\oint$ denotes the $(n-1)$-ple integration on B, $\int\!\!\int$ the n-ple integration on D and $\Box=\varDelta\varGamma+\varGamma\varDelta$. (See [1] and [2] in references. The notations are some what different.)

§ 2. Dimensional differentiation of tensors for a infinitesimal variation of the metric.

Definition (2·1). *An abstract dimension of a tensor* A, $[A]$, *is a real number corresponding to each A and satisfying the following conditions*:

(I) $[\delta_{i_1\cdots i_n}]=[\delta^{i_1\cdots i_n}]=0$,

(II) if A' is any contraction of A, then $[A']=[A]$,

(III) if $[A]=[B]$, then $[A+B]=[A]+[B]$,

(IV) $[A\times B]=[A]+[B]$,

(V) if A is a scalar, then $[A^k]=k[A]$,

(VI) $[g_{ij}dx^i dx^j]=2$,

(VII) if DA is covariant derivative of A,
then $[DA]=[A]-[dx^i]$.

There can be two kinds of dimensions, [ ] and [ ]', satisfying the above conditions.

Definition (2·2). *If $[A]=0$ implies $[A]'=0$ and conversely for all scalars A, then we shall call the dimensions equivalent.*

Theorem (2·1). *If two dimensions are equivalent, then they coincide for all scalars.*

Proof. Putting $dV=\sqrt{g}\ \delta_{i_1\cdots i_n}dx^{i_1}\cdots dx^{i_n}$ we have $[dV]=[dV]'=n$ by the conditions in definition (2·1). If $[A]=a$, $[A]'=a'$ for a scalar A, then $[AdV^{-\frac{a}{n}}]=0$ by the above conditions. Hence $[AdV^{-\frac{a}{n}}]'=0$ by definition (2·2). It follows that $[A]'+[dV^{-\frac{a}{n}}]'=0$, $a'=a$.

Theorem (2·2). *If two dimensions are equivalent and $[dx^i]=[dx^i]'$, then they coincide for all relative or absolute tensors.*

Proof. If $[dx^i]=[dx^i]'=\xi$, then $[g_{ij}]=[g_{ij}]'=2-2\xi$ by conditions in definition (2·1), hence $[dx_i]=[g_{ij}dx^j]=2-\xi$. If $A^{i_1\cdots i_p}{}_{j_1\cdots j_q}$ is a tensor of weight w,

$$B=\sqrt{g}^{\,-w}A^{i_1\cdots i_p}{}_{j_1\cdots j_q}dx^{j_1}\cdots dx^{j_q}dx_{i_1}\cdots dx_{i_p}$$

is a scalar, and $[B]=[A]-wn(1-\xi)+q\xi+p(2-\xi)$. Similarly $[B]'=[A]'-wn(1-\xi)+q\xi+p(2-\xi)$. By theorem (2·1), we get $[B]=[B]'$, hence $[A]=[A]'$.

Definition (2·3). *If $[dx^i]=0$ or $[g_{ij}]=2$, the dimension may be called absolute dimension, and if $[dx^i]=1$ or $[g_{ij}]=0$, relative dimension.* [3].

It follows immediately for equivalent absolute and relative dimensions that:

$$[A]_{\text{abs}}=[A]_{\text{rel}}+q-p+nw,$$
$$[dV]_{\text{rel}}=n,\quad [N]_{\text{rel}}=0,\quad [dS]_{\text{rel}}=n-1,$$
$$[DA]_{\text{rel}}=[A]_{\text{rel}}-1,\quad [A]_{\text{rel}}=[|A|]_{\text{rel}},$$

where $|A|$ is the absolute value of A, that is,

$$|A|^2=g^{-w}A^{i_1\cdots i_p}{}_{j_1\cdots j_q}A_{i_1\cdots i_p}{}^{j_1\cdots j_q}.$$

Since $[g_{ij}]=0$ and $[\sqrt{g}]=0$ for relative dimension, all associated absolute or relative tensors have the same dimension. It is adequate to use relative dimension for our simplified notation A.

When components of a tensor $A(x, t)$ are differentiable functions of independent variables $x_1, \cdots, x_n$ and a parameter t, vA denotes the variation of A, that is, $\dfrac{\partial A}{\partial t}dt$. vA is a tensor of the same kind as A. We assume that A is of C^∞ for simplicity.

Let $A^{i_1\cdots i_p}{}_{j_1\cdots j_q}$ be components of a tensar A of weight w, and $vg_{ij}(x, t)=2\omega_{ij}(x, t)$.

Definition (2·4). *Dimensional derivatives of A are defined as follows*:

$$\begin{aligned}\delta A^{i_1\cdots i_p}{}_{j_1\cdots j_q}&=vA^{i_1\cdots i_p}{}_{j_1\cdots j_q}+w^{i_1}_{s}A^{s i_2\cdots i_p}{}_{j_1\cdots j_q}+\cdots+w^{i_p}_{s}A^{i_1\cdots i_{p-1}s}{}_{j_1\cdots j_q}\\ &\quad-\omega^{s}_{j_1}A^{i_1\cdots i_p}{}_{s j_2\cdots j_q}-\cdots-\omega^{s}_{j_q}A^{i_1\cdots i_p}{}_{j_1\cdots j_{q-1}s}\\ &\quad-\left(w+\frac{\alpha}{n}\right)\omega A^{i_1\cdots i_p}{}_{j_1\cdots j_q},\end{aligned}\tag{4}$$

where $\alpha=[A]_{\text{rel}}$ and $\omega=\omega_{ij}g^{ij}$.

δA is a tensor of the same kind as A and $[\delta A]=[A]$.

We get

$$\begin{aligned}&\delta g_{ij}=0.\qquad \delta g=0,\qquad \delta\delta_{i_1\cdots i_n}=0,\\ &\delta dx^i=\omega^i_j dx^j-\frac{\omega}{n}dx^i,\quad \delta d\sigma_i=-\omega^j_i d\sigma_j+\frac{\omega}{n}d\sigma_i,\quad \delta dV=0,\end{aligned}\tag{5}$$

since $vdx^i=0$.

Let Ω and Ω' are linear operators defined by:

$$\Omega A^{i_1\cdots i_p}{}_{j_1\cdots j_q}=\omega^{i_1}_{s}A^{s\cdots i_p}{}_{j_1\cdots j_q}+\cdots+\omega^{i_p}_{s}A^{i_1\cdots s}{}_{j_1\cdots j_q}$$

$$+\omega^{s}_{j_1}A^{i_1\cdots i_p}{}_{s\cdots j_q}+\cdots+\omega^{s}_{j_q}A^{i_1\cdots i_p}{}_{j_1\cdots s}$$

(6) $$+\frac{\alpha}{n}\,\omega A^{i_1\cdots i_p}{}_{j_1\cdots j_q},$$

$$\Omega' A^{i_1\cdots i_p}{}_{j_1\cdots j_q}=-\omega^{i_1}_{s}A^{s\cdots i_p}{}_{j_1\cdots j_q}-\cdots-\omega^{i_p}_{s}A^{i_1\cdots s}{}_{j_1\cdots j_q}$$

$$-\omega^{s}_{j_1}A^{i_1\cdots i_p}{}_{s\cdots j_q}-\cdots-\omega^{s}_{j_q}A^{i_1\cdots i_p}{}_{j_1\cdots s}$$

$$+\left(\frac{\alpha}{n}+1\right)\omega A^{i_1\cdots i_p}{}_{j_1\cdots j_q}.$$

If A and ΔA are covariant skew-symmetric tensors of weight w, we get easily $(v-w\omega)\Delta A=\Delta(v-w\omega)A$ and $v-w\omega=\delta+\Omega$ by (4) and (6), it follows that

(7) $$(\delta+\Omega)\Delta A=\Delta(\delta+\Omega)A.$$

This identity holds not only for convariant skew-symmetric tensor A, but for any skew-symmetric tensor A, since the operations, Ω, Ω' and δ, are commutable with the operations, uppering or lowering of the indices of A and weighting A by $\sqrt{g}$. Similary we get

(8) $$(\delta+\Omega')\nabla A=\nabla(\delta+\Omega')A.$$

We also have the identity

(9) $$(\Omega A\cdot B)+(A\cdot\Omega' B)=0,$$

provided that $[A]_{\rm rel}+[B]_{\rm rel}=-n$

If $(\top A\cdot B)dS=(A\cdot\perp B)dS=\frac{1}{(p-1)!}A^{i_1\cdots i_p}d\sigma_{i_1}B_{i_2\cdots i_p}$ is a scalar of relative dimension 0, we have

$$(T\Omega' A\cdot B)dS+(\top A\cdot\Omega B)dS$$

(10) $$=(\Omega' A\cdot\perp B)dS+(A\cdot\perp\Omega B)dS$$

$$=\frac{1}{(p-1)!}A^{i_1i_2\cdots i_p}B_{i_2\cdots i_p}\delta d\sigma_{i_1}$$

by (5) and (6).

If A is a scalar of relative dimension $(-n)$, that is, if AdV is a scalar of dimension 0, then $v\int\int AdV=\int\int v(AdV)=\int\int\delta(AdV)$, and if A^i is a tensor of relative dimension $(1-n)$, that is, if $A^i d\sigma_i$ is a scalar of dimension 0, $v\oint A^i d\sigma_i=\oint v(A^i d\sigma_i)=\oint\delta(A^i d\sigma_i)$. We shall denote them $\delta\int\int A dV$ and $\delta\oint A' d\sigma_t$ for simplicity, that is, *the operation* δ *upon a integral is applicable if and only if the relative dimension of the integral is* 0.

We get from (10) and (3)

$$
\begin{aligned}
&\delta(\top A\cdot B)dS\\
&=(\top\delta A\cdot B)dS+(\top A\cdot\delta B)dS+(\top\Omega' A\cdot B)dS+(\top A\cdot\Omega B)dS\\
&=(\delta A\cdot\bot B)dS+(A\cdot\bot\delta B)dS+(\Omega' A\cdot\bot B)dS+(A\cdot\bot\Omega B)dS\\
&=\delta(A\cdot\bot B)dS,
\end{aligned}
$$

(11)

$$
\begin{aligned}
&\delta\oint(\top A\cdot B)dS\\
&=\int\int(\nabla\delta A\cdot B)dV+\int\int(\delta A\cdot\Delta B)dV+\int\int(\nabla A\cdot\delta B)dV\\
&+\int\int(A\cdot\Delta\delta B)dV+\int\int(\nabla\Omega' A\cdot B)dV+\int\int(\Omega' A\cdot\Delta B)dV\\
&+\int\int(\nabla A\cdot\Omega B)dV+\int\int(A\cdot\Delta\Omega B)dV=\delta\oint(A\cdot\bot B)dS,
\end{aligned}
$$

for skew-symmetric tensors A and B, sum of whose relative dimension is $(1-n)$.

It follows from (7) and (8) that

$$
\begin{aligned}
&\delta\Delta=\Delta\delta+\Delta\Omega-\Omega\Delta,\\
&\delta\nabla=\nabla\delta+\nabla\Omega'-\Omega'\nabla,\\
&\delta\Delta\nabla=\Delta\nabla\delta+\Delta\nabla\Omega'-\Delta\Omega'\nabla+\Delta\Omega\nabla-\Omega\Delta\nabla,\\
&\delta\nabla\Delta=\nabla\Delta\delta+\nabla\Delta\Omega-\nabla\Omega\Delta+\nabla\Omega'\Delta-\Omega'\nabla\Delta.
\end{aligned}
\tag{12}
$$

Let A be a skew-symmetric tensor of relative dimension $\left(1-\frac{n}{2}\right)$, ρ a positive scalar of relative dimension (-2).

Consider the elliptic differential equation

$$
\begin{aligned}
&\Box A+\lambda\rho A=0 &&\text{in } D,\\
&A=0 &&\text{on } B,
\end{aligned}
\tag{13}
$$

where λ is a characteristic root. We shall assume that λ and A are differentiable with respect to the parameter t and A is normalized by $\int\int\rho(A\cdot A)dV=1$.

From (13) we get $\int\int(A\cdot\Box A)dV=-\lambda$;

Operating δ, we have

$$
\begin{aligned}
-\delta\lambda=2\int\int[&(\Omega' A\cdot\Delta\nabla A)+(\Omega A\cdot\nabla\Delta A)+(\Omega'\nabla A\cdot\nabla A)\\
&+(\Omega\Delta A\cdot\Delta A)]dV+\lambda\int\int\delta\rho(A\cdot A)dV,
\end{aligned}
$$

by (3), (9), (12) and (13).

If A is a scalar, then $\nabla A=0$, $\Box A=\nabla \Delta A$ by definition, and

$$-\delta\lambda=-2\lambda\int\int\rho(\Omega A\cdot A)dV+2\int\int(\Omega\Delta A\cdot\Delta A)dV+\lambda\int\int\delta\rho(A\cdot A)dV.$$

Let $\mu_1, \cdots, \mu_n$ be the characteristic roots of $|\omega^i_j-\mu\delta^i_j|=0$, $\theta=\frac{v\rho}{\rho}$, and we shall assume $\mu_m\leqq\mu_1, \cdots, \mu_n\leqq\mu_M$, $\theta_m\leqq\theta\leqq\theta_M$, where μ_m, μ_M, θ_m and θ_M are constants. Since $\int\int(\nabla\Delta A\cdot A)dV=-\int\int(\Delta A\cdot\Delta A)dV$, *we get an inequality for* $\delta\lambda$:

$$[n(\mu_M-\mu_m)-(2\mu_m+\theta_m)]>\frac{\delta\lambda}{\lambda}>[n(\mu_m-\mu_M)-(2\mu_M+\theta_M)],\quad (n\geqq 3).$$

Putting $K=\int\int\rho^{\frac{n}{2}}dV$, we also get an inequality:

$$2\mu_M+\theta_M>\frac{2}{n}\frac{\delta K}{K}>2\mu_m+\theta_m.$$

§ 3. Conformal change of metric.

Let x and y be two points in D. Set of functions $A^{i|j}(x,y)$ of variables (x) and (y) may be called a double tensor, [1], if

$$A^{i|j}(x,y)=\frac{\partial x^i}{\partial\bar{x}^k}\frac{\partial y^j}{\partial\bar{y}^l}\bar{A}^{k|l}(\bar{x}.\bar{y})$$

for any coordinates transformations: $(x)\to(\bar{x})$ and $(y)\to(\bar{y})$. The covariant differentiation with respect to (x) or (y) and the dimensional differentiation can be defined as follows:

$$D_{k(x)}A^{i|j}=\frac{\partial A^{i|j}}{\partial x^k}+\left\{{i \atop pk}\right\}_{(x)}A^{p|j},$$

$$D_{l(y)}A^{i|j}=\frac{\partial A^{i|j}}{\partial y^l}+\left\{{i \atop pl}\right\}_{(y)}A^{i|p},$$

$$\delta A^{i|j}=vA^{i|j}+\omega^i_s(x)A^{s|j}+\omega^j_s(y)A^{i|s}$$
$$-\frac{\alpha}{n}\omega(x)A^{i|j}-\frac{\alpha'}{n}\omega(y)A^{i|j},$$

where α and α' are relative dimensions of $A^{i|j}$ with respect to (x) and (y), which we denote $[A]=(\alpha,\alpha')$ for simplicity.

The notions above may be extended for any tensor, and relations analogous to (12) hold for δ, $D_{(x)}$ and $D_{(y)}$.

If $A^{i|j}(x,y)$ and $B_{i|j}(x,y)$ are double tensors for example, and $A^{i|j}$

$(x,y)B_{|j}(x,y)$ is of dimension $(-n)$ with respect to (y), then its integration over D with respect to (y), $\int\int A^{i|j}(x,y)B_{|j}(x,y)dV(y)$, is a contravariant tensor with respect to (x), and moreover $\delta\int\int ABdV(y)=\int\int\delta(ABdV)$, since $ABdV(y)$ is of 0 dimension with respect to y.

For conformal change of metric, $\omega_{ij}=\tau g_{ij}$, from (4), (5) and (6) we get

$$
(15)\quad
\begin{aligned}
&\delta A^{i_1\cdots i_p}{}_{j_1\cdots j_q}=vA^{i_1\cdots i_p}{}_{j_1\cdots j_q}+(p-q-nw-\alpha)\tau A^{i_1\cdots i_p}{}_{j_1\cdots j_q},\\
&\delta dx^i=0.\qquad \delta d\sigma_i=0,\qquad \delta dV=0,\\
&\Omega A^{i_1\cdots i_p}{}_{j_1\cdots j_q}=(m+\alpha)\tau A^{i_1\cdots i_p}{}_{j_1\cdots j_q},\\
&\Omega' A^{i_1\cdots i_p}{}_{j_1\cdots j_q}=(m'+\alpha)\tau A^{i_1\cdots i_p}{}_{j_1\cdots j_q},
\end{aligned}
$$

where $m=p+q=$ the degree of A, $m'=n-m$, we shall call m' the dual degree of A.

Since degree of A is smaller than degree of ΔA by one, and greater than degree of ∇A by one, relative dimension of A is greater than relative dimension of ΔA or ∇A by one, we also have:

$$
(16)\quad
\begin{aligned}
\delta\Delta A&=\Delta\delta A+(m+\alpha)(\Delta\tau A-\tau\Delta A),\\
\delta\nabla A&=\nabla\delta A+(m'+\alpha)(\nabla\tau A-\tau\nabla A),\\
\delta\nabla\Delta A&=\nabla\Delta\delta A+(m'+\alpha)(\Delta\nabla\tau A-\Delta\tau\nabla A)\\
&\qquad+(m+\alpha-2)(\Delta\tau\nabla A-\tau\Delta\nabla A),\\
\delta\nabla\Delta A&=\Delta\nabla\delta A+(m+\alpha)(\nabla\Delta\tau A-\nabla\tau\Delta A\\
&\qquad+(m'+\alpha-2)(\nabla\tau\Delta A-\tau\nabla\Delta A),
\end{aligned}
$$

by (12).

Let the Riemannian space be euclidean, (x) and (y) the cartesian coordinates of two points, r^2 the square of distance of (x) and (y). We regard r^2 as a double scalar of relative dimension (1,1).

Put $\Gamma_{i|j}=-\dfrac{1}{2}\dfrac{\partial^2 r^2}{\partial x^i\partial y^j}$, then $\Gamma_{i|j}$ is a double tensor of covariant order (1,1) and relative dimension (0,0). For the special conformal change of metric, $\omega_{ij}=\tau\delta_{ij}$, where τ is a constant, we get

$$\delta r^2=0,\qquad \delta\Gamma_{i|j}=0.$$

It follows that:

Theorem (3·1). *If the space is Riemannian and r is the geodesic distance between two points (x) and (y) sufficiently near, then $\delta r^2 \to 0$, $\delta \Gamma_{i|j} \to 0$ when $(x) \to (y)$, for comformal change of metric.*

Let $G_{i_1 \cdots i_m | j_1 \cdots j_m}(x, y)$ be Green's tensor for the elliptic differential equation:

$$\square A = 0 \quad \text{in } D \text{ with given } \perp\top A \text{ and } \top\perp A \text{ on } B.$$

It is known that Green's tensor is unique under some appropriate topological conditions for domain D and the tensor is characterized by the following properties:

$$\square_x G(x, y) = 0 \qquad \text{in } D,$$

$$G(x, y) = 0, \qquad \text{if } x \text{ is on } B,$$

$$G(x, y) - \gamma(x, y) \text{ is regular at } x = y,$$

where

$$\gamma_{i_1 \cdots i_m | j_1 \cdots j_m}(x, y) = \frac{1}{(n-2) S_n} \frac{1}{r^{n-2}} \begin{vmatrix} \Gamma_{i_1|j_1} \cdots \Gamma_{i_1|j_m} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \Gamma_{i_m|j_1} \cdots \Gamma_{i_m|j_m} \end{vmatrix}, \qquad (n \geqq 3)$$

Sn is the area of the unit $(n-1)$ sphere. [2].

We shall assume $[G]_{\rm rel} = \left(1 - \frac{n}{2},\ 1 - \frac{n}{2}\right)$,

since $[\gamma]_{\rm rel} = \left(1 - \frac{n}{2},\ 1 - \frac{n}{2}\right)$.

By theorem (3·1) $\delta\gamma \to 0$ when $(x) \to (y)$, and we get easily $\delta G = 0$, if (x) is on B. Thus we have:

Theorem (3·2) *δG is regular for any (x) and (y) in D, $\delta G = 0$ if (x) is on B.*

Operating δ on $\square_x G = 0$, it follows

$$\begin{aligned} \square_x \delta G(x, y) = \Big[& \left(\frac{n}{2} - m + 1\right) \{ \Delta_x \tau(x) \nabla_x - \Delta_x \nabla_x \tau(x) \} \\ & + \left(m - 1 - \frac{n}{2}\right) \{ \tau(x) \Delta_x \nabla_x - \Delta_x \tau(x) \nabla_x \} \\ & + \left(m + 1 - \frac{n}{2}\right) \{ \nabla_x \tau(x) \Delta_x - \nabla_x \Delta_x \tau(x) \} \\ & + \left(\frac{n}{2} - m - 1\right) \{ \tau(x) \nabla_x \Delta_x - \nabla_x \tau(x) \Delta_x \} \Big] G(x, y) \end{aligned} \qquad (17)$$

by (16).

Putting the right side in (17) $\tilde{G}(x, y)$, we have

$$\delta G(y, z) = -\iint G(x, z)\tilde{G}(x, y)dV(x)$$

since $\delta G(x, y) = 0$ if (x) is on B.

Integrating by part, *we get*

$$\delta G(x, y) = -2\left(m-\frac{n}{2}-1\right)\iint \tau(x)[(\nabla_x G(x, y)\cdot\nabla_x G(x,z)) + (\Delta_x\nabla_x G(x, y)\cdot G(x,z))]dV(x)$$
$$-2\left(\frac{n}{2}-m-1\right)\iint \tau(x)[(\Delta_x G(x, y)\cdot\Delta_x G(x, z)) + (\nabla_x\Delta_x G(x, y)\cdot G(x, z)]dV(x).$$

It follows that:

if $\tau = \text{const.}$ $\delta G = 0$;

if $m=0$, that is, G is a double scalar,

$$\delta G(x, y) = (2-n)\iint \tau[\Delta_x G(x, y)\cdot\Delta_x G(x, z))dV(x),$$

since $\nabla G = 0$, $\Box G = \nabla\Delta G = 0$;

if n is even and $m=\frac{n}{2}$,

$$\delta G(x, z) = 2\iint \tau(x)[(\nabla_x G(x, y)\cdot\nabla_x G(x, z)) + (\Delta_x G(x, y)\cdot\Delta_x G(x, z))]dV(x)$$

since $(\Delta\nabla+\nabla\Delta)G = \Box G = 0$.

4. An application to electrostatic field.

Let $\mu(x)$ be the dielectric constant for an electrostatic field in an n-dimensional Riemannian space with the metric tensor g_{ij}. Physical dimension of μ, $[\mu]_{\text{phy}}$, is $L^{2-n}Q^2E^{-1}$, where L, Q and E denote the dimensions of length, electric charge and energy respectively, and L, Q, E are *independent* since dimensions of mass and time do not appear.

Electric energy contained in a domain D is given by

$$\iint \mu\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j}g^{ij}\varepsilon_{i_1\cdots i_n}dx^{i_1}\cdots dx^{i_n},$$ where φ

is the electrostatic potential, and $[\varphi]_{\text{phy}} = EQ^{-1}$,

$$[g_{ij}]_{\rm phy}=[\varepsilon]_{\rm phy}=Q^0E^0L^0,\ [dx^i]_{\rm phy}=L.$$

Putting $\qquad f_{ij}=\mu^{\frac{2}{n-2}}g_{ij}, \qquad f=\begin{vmatrix} f_{11}\cdots f_{1n} \\ \cdots\cdots\cdots \\ f_{n1}\cdots f_{nn}\end{vmatrix},$

we have

$$\mu\frac{\partial\varphi\partial\varphi}{\partial x^i\partial x^j}g^{ij}\varepsilon_{i_1\cdots i_n}dx^{i_1}\cdots dx^{i_n}=\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j}f^{ij}\sqrt{f}\,\delta_{i_1\cdots i_n}dx^{i_1}\cdots dx^{i_n}.$$

Hence f_{ij} may be used as the fundamental tensor for the electrostatic field instead of g_{ij}.

We have $\quad [f_{ij}]_{\rm phy}=L^{-2}S^2,\ \ [f_{ij}dx_i dx^j]_{\rm phy}=S^2,\ \ [\varphi]_{\rm phy}=S^{\frac{2-n}{2}}E^{\frac{1}{2}},$

where $\quad S=(Q^2E^{-1})^{\frac{1}{n-2}}.$

If physical dimension of the absolute value of a tensor A with respect to the fundamental tensor f_{ij} is $S^{\alpha}E^{\gamma}$, we shall define abstract relative dimension of A, $[A]_{\rm rel}$, as α, since it satisfies the axioms of abstract dimension in § 2.

Thus we have $\quad [f_{ij}]_{\rm rel}=0,\ \ [dx^i]_{\rm rel}=1,\ \ [\varphi]_{\rm rel}=1-\dfrac{n}{2}$

and $\quad \left[\iint\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j}f^{ij}\sqrt{f}\,\delta_{i_1\cdots i_n}dx^{i_1}\cdots dx^{i_n}\right]_{\rm rel}=0.$

Variation of μ is a conformal change of f_{ij}, and *the arguments in previous paragraphs hold for such a kind of problems in physics, providing that f_{ij} is the fundamental tensor.*

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