MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXIX, Mathematics No. 1, 1955.

On the stability of solutions of a system of differential equations

By

Taro YOSHIZAWA

(Received March 31, 1954)

Introduction. About the non-linear differential equation as well as about the linear one, various authors have discussed the boundedness, the convergence(1) of solutions and the existence of a periodic solution(2) which we have also discussed recently. Moreover we have obtained necessary and sufficient conditions for the boundedness of solutions of a certain type.(3) In these researches we have utilized the existence of a function of points which is characterised by the fact that it is non-increasing or non-decreasing along any solution of the given differential equation. Now by this idea, we will discuss the stability of solutions about which Liapounoff(4) and various mathematicians and physicists have discussed actively and they have obtained many remarkable results in connection with the above mentioned problems.

Now we consider a system of differential equations,

$$\frac{dy}{dx}=F(x, y), \tag{1}$$

where y denotes an n-dimensional vector and $F(x, y)$ is a given vector field which is defined and *continuous* in the domain

$$\Delta : 0\leqq x<\infty, \quad |y|\leqq R \quad (|y|=\sqrt{y_1^2+\cdots+y_n^2}).$$

Before the research of a solution, we assume at first that $y(x)\equiv 0$ is a solution of (1), so that $F(x, 0)$ is *identically* zero. And let $y=y(x\,;y_0, x_0)$ be any solution of (1) satisfying the initial condition $y=y_0$ when $x=x_0$.

Here we state the following definitions.(5)

a) The solution $y\equiv 0$ is said to be *stable* if, given any $\varepsilon>0$, there exists a $\delta>0$ such that if $|y_0|\leqq\delta$, then $|y(x\,;y_0, 0)|\leqq\varepsilon$ for all $x\geqq 0$.

b) The solution $y \equiv 0$ is said to be *asymptotically stable* if it is stable and there exists a (fixed) $\delta_0 > 0$ such that if $|y_0| \leqq \delta_0$, then $\lim_{x \to +\infty} y(x; y_0, 0) = 0$.

In this paper we will see the relations between these notions and the *strong stability* introduced by Okamura[6] and one more, say *equistability* for the sake of convenience as compared with the above defined stability.

1. Stability. At first, we will obtain a necessary and sufficient condition in order that $y \equiv 0$ is stable.

Theorem 1. *In order that the solution $y \equiv 0$ of* (1) *is stable, it is necessary and sufficient that there exists a function $\Phi(x, y)$ defined in Δ satisfying the following conditions; namely*

1° $\Phi(x, y) > 0$, *provided* $|y| \neq 0$,

2° $\Phi(x, 0) = 0$ *for all* x,

3° *there exists a* $\kappa(\varepsilon)$ *for any* $(R \geqq) \varepsilon > 0$ *such that*

$$\Phi(x, y) \geqq \kappa(\varepsilon) \quad (> 0),$$

whenever $0 \leqq x < \infty$ *and* $|y| = \varepsilon$,

4° $\Phi(0, y)$ *is continuous at* $y = 0$,

5° *for any solution of* (1), $y = y(x)$, *the function* $\Phi(x, y(x))$ *is a non-increasing function of* x.

Proof. At first, we show that the condition is sufficient. Now we assume that there exists such a function $\Phi(x, y)$. By 3° there is a $\kappa(\varepsilon)$ for any $\varepsilon > 0$ and by 4°, we can choose a suitable positive number $\delta(< \varepsilon)$ for this $\kappa(\varepsilon)$ such that

$$\Phi(0, y) < \kappa(\varepsilon), \quad \text{if} \quad |y| \leqq \delta, \tag{2}$$

since $\Phi(0, y)$ is continuous at $y = 0$ and $\Phi(x, 0) \equiv 0$. Now for a solution of (1) $y = y(x; y_0, 0)$ such as $|y_0| \leqq \delta$, we suppose that we have at some x, say x_1, $|y(x_1; y_0, 0)| = \varepsilon$. Thus we have by 3°

$$\Phi(x_1, y(x_1; y_0, 0)) \geqq \kappa(\varepsilon)$$

and hence we have

$$\Phi(0, y(0; y_0, 0)) \geqq \kappa(\varepsilon),$$

since $\Phi(x, y(x))$ is non-increasing for x by 5°. On the other hand we have

$$\Phi(0, y(0; y_0, 0)) < \kappa(\varepsilon)$$

by (2) and therefore there arises a contradiction. Hence any solution of (1) $y = y(x; y_0, 0)$ such as $|y_0| \leqq \delta$ satisfies for $0 \leqq x < \infty$

$$|y(x\,;\,y_0,\,0)|<\varepsilon\,,$$

that is to say, the solution $y\equiv 0$ is stable.

Nextly we show that the condition is necessary. Now we suppose that $y\equiv 0$ is stable. Here we consider the solutions of (1) going to the right from a point P in $\varDelta$ and we represent this family by $\mathfrak{Y}_P$. We will define $\delta(P)$ as follows; namely if there exists a solution of $\mathfrak{Y}_P$ which arrives at $|y|=R$, then we take

$$\delta(P)=R$$

and if otherwise, i.e. if all the solutions of (1) passing through P lie in the interior of $\varDelta$,

$$\delta(P)=\sup_{\substack{x_P\leqq x<\infty\\ y(x)\in\mathfrak{Y}_P}}|y(x)|$$

and when P lies on $|y|=R$, $\delta(P)=R$. Then if P lies on the x-axis, we have $\delta(P)=0$, since for any point P on the x-axis the solution through P is unique to the right and it is $y\equiv 0$ by the condition that $y\equiv 0$ is stable. When P does not lie on the x-axis, we have $\delta(P)>0$. Thus for any ε $(\leqq R)$ and P on $|y|=\varepsilon$, we have $\delta(P)\geqq\varepsilon$ whenever $0\leqq x<\infty$. This ε is $\kappa(\varepsilon)$ in the condition 3°. Moreover if for any $\varepsilon>0$ we choose a suitable $\delta>0$, we have for y_0 such as $|y_0|\leqq\delta$ and $x\geqq 0$

$$|y(x\,;\,y_0,0)|\leqq\varepsilon\,,$$

since the solution $y\equiv 0$ is stable. Therefore we have

$$0\leqq\delta(P)\leqq\varepsilon$$

for P lying in $x=0$, $|y|\leqq\delta$. Clearly if P and Q lie on the same solution of (1) and $x_P\leqq x_Q$, we have

$$\delta(P)\geqq\delta(Q).$$

Here we put with $P=(x,y)$

$$\varPhi(x,y)=\delta(P),$$

and then it is clear from the above mentioned that this $\varPhi(x,y)$ satisfies the conditions 1°, 2°, 3°, 4° and 5°. Therefore this $\varPhi(x,y)$ is the desired.

2. Equistability. In § 1 we have considered the case where $y\equiv 0$ is stable for solutions starting from a neighborhood of $y=0$ at $x=0$. Compared with this stability, here we consider the following case.

Definition. The solution $y \equiv 0$ is said to be *equistable* if, given any $\varepsilon > 0$, there exists a $\delta(x_0) > 0$ for every x_0 such that if $|y_0| \leqq \delta(x_0)$, then $|y(x; y_0, x_0)| \leqq \varepsilon$ for all $x \geqq x_0$.

Namely for every x_0, $y \equiv 0$ is stable when we consider $y \equiv 0$ as a solution going to the right starting from $x = x_0$. Often the equistability is said merely stable,[7] specially in physical problems. As Theorem 3 below shows, the stability and the equistability are equivalent if $y \equiv 0$ is the unique solution starting from any point on the x-axis. If we can choose δ independent of x_0, we shall say it *uniformly equistable*.

Then we have in the same way as in the preceding theorem of the stability.

Theorem 2. *In order that the solution $y \equiv 0$ of* (1) *is equistable, it is necessary and sufficient that there exists a similar function $\Phi(x, y)$ as one in Theorem 1, but only the condition $4°$ being replaced by*

$4°'$ *$\Phi(x_0, y)$ is continuous at $y = 0$ for every x_0.*

For the uniform equistability, $\Phi(x, y)$ is continuous at $y = 0$ with regard to y uniformly for x.

When the solution $y \equiv 0$ is equistable, then it is of course stable; but there is a case where it is stable and yet it is not equistable. For example, let us consider such an equation,

$$\frac{dy}{dx} = f(x, y), \tag{3}$$

where

$$f(x, y) = \begin{cases} 2(x-1)y & (x \geqq 1) \\ 2(x-1) & (0 \leqq x \leqq 1,\ (x-1)^2 \leqq y) \\ -2y^{1/2} & (0 \leqq x \leqq 1,\ 0 \leqq y \leqq (x-1)^2) \\ 2(-y)^{1/2} & (0 \leqq x \leqq 1, -(x-1)^2 \leqq y \leqq 0) \\ -2(x-1) & (0 \leqq x \leqq 1,\ y \leqq -(x-1)^2). \end{cases}$$

Clearly $y \equiv 0$ is stable, being considered as the solution starting from the point x such as $0 \leqq x < 1$, but it is not stable for x such as $x \geqq 1$. The solution going to the left from x such as $x \geqq 1$ on the x-axis is *not unique* in this case. This fact characterises the non-equistability. About this matter we have

Theorem 3. *When the solution $y \equiv 0$ of* (1) *is unique to the* *left for all points on the x-axis in Cauchy-problem,*[8] *if $y\equiv 0$ is stable, it is also equistable. Namely in this case the stability and the equistability are the same.*

Proof. Suppose that $y\equiv 0$ is not equistable. Then we assume that, even if we take any neighborhood of $y=0$ at $x=x_0$ for some $x_0\,(>0)$ and some ε, there exists a solution going to the right from there such as $|y|\geqq\varepsilon$ at an $x\ (>x_0)$. For this ε and a suitable $\delta>0$ we have $|y(x\,;y_0,0)|<\varepsilon$ if $|y_0|\leqq\delta$, since $y\equiv 0$ is stable. By the uniqueness to the left of $y\equiv 0$, if we choose a suitable neighborhood $U(x_0, 0)$ of $y=0$ at $x=x_0$, all the solutions of (1) going to the left from U arrive at the plane $x=0$ and moreover we have $|y|\leqq\delta$ at $x=0$. Then there exists a solution starting from $|y|\leqq\delta$ at $x=0$ such as $|y(x\,;y_0,0)|\geqq\varepsilon$ and there arises a contradiction. Therefore $y\equiv 0$ is equistable.

In the case where the right-hand side of the system (1) does *not depend* on x, i.e. where

$$\text{(4)} \qquad \frac{dy}{dx}=F(y),$$

we can prove easily the following theorem, where we assume that $F(0)=0$.

Theorem 4. *If the solution $y\equiv 0$ is stable for the system* (4), *it is also equistable* (*uniformly*).

The proof is omitted.

3. Strong stability. Okamura has introduced a notion, i.e. the strong stability.

Definition. Let $Y(x)$ be an arbitrary function such that

$$Y(0)=0,\ |Y(x)|<R\ \ (0\leqq x\leqq X)$$

which is of bounded variation in $0\leqq x<X$ (X being arbitrary, $0\leqq X<\infty$). The solution $y\equiv 0$ of (1) is said to be *strongly stable* when for any given $\eta>0$ and for a suitable positive number ε, we have $|Y(X)|<\eta$, provided

$$\underset{0\leqq x\leqq X}{V}\left[Y(x)-\int F(x,\,Y(x))\,dx\right]<\varepsilon \text{ [9]}$$

holds, where V denotes the total variation.

Okamura has obtained the following theorems for the strong stability.

Theorem 5. *A function $\varphi(y)$ being defined in a neighborhood of $y=0$, if $\varphi(y)$ is positive or zero according as $|y|$ is positive or zero* *and $\varphi(y)$ satisfies the Lipschitz condition*

$$|\varphi(y_1)-\varphi(y_2)|\leqq K|y_1-y_2|$$

and finally it is a non-increasing function of x along any solution $y(x)$ of (1), *then the solution $y\equiv 0$ of* (1) *is strongly stable.*

Theorem 6. *In order that the solution $y\equiv 0$ of the system* (4) *is strongly stable, it is necessary and sufficient that there exists such a function $\varphi(y)$ as in Theorem 5 for F of the right-hand side of* (4).

Moreover for an equation of the *first order*

$$\frac{dy}{dx}=f(y) \qquad (f(0)=0),$$

in order that $y\equiv 0$ is strongly stable, it is necessary and sufficient that, given any $\varepsilon>0$, both the measure of the set of y such as $[0<y<\varepsilon, f(y)\leqq 0]$ and that of the set of y such as $[-\varepsilon<y<0, f(y)\geqq 0]$ are *positive*. Hence for the equation,

$$(5) \qquad \frac{dy}{dx}=\begin{cases} y\sin^2\dfrac{1}{y} & (y\neq 0) \\ 0 & (y=0), \end{cases}$$

the solution $y\equiv 0$ is stable, but it is not strongly stable. Besides in (5) $y\equiv 0$ is uniformly equistable by Theorem 4. Hence there is a case where $y\equiv 0$ is uniformly equistable, but it is not strongly stable. By considering $\varphi(y)$ in Theorems 5 and 6 as $\Phi(x, y)$ in Theorem 2, we can see in these cases that if $y\equiv 0$ is strongly stable, then it is also uniformly equistable. And yet we have always

Theorem 7. *If the solution $y\equiv 0$ of* (1) *is strongly stable, it is also uniformly equistable.*

Proof. For every x_0 $(\geqq 0)$, given any $\eta>0$ and for a suitable $\varepsilon>0$, let $Y(x)$ be a function such that $Y(x)\equiv 0$ for $0\leqq x\leqq x_0$ and it is a solution of (1) satisfying $|y|<\varepsilon$ as $x\to x_0+0$ for $x_0<x\leqq X$. Then by the strong stability of $y\equiv 0$, we have $|Y(X)|<\eta$, since for this $Y(x)$ we have

$$\underset{0\leqq x\leqq X}{V}[Y(x)-\int F(x, Y(x))dx]<\varepsilon .$$

Therefore $y\equiv 0$ is uniformly equistable.

Thus the equation (5) shows that the strong stability is *stronger* than the uniform equistability, *a fortiori*, than the stability. Therefore in the case where $y\equiv 0$ is equistable simply (not uniformly),

$y \equiv 0$ can not be strongly stable. For example, in

$$\frac{dy}{dx} = \begin{cases} y - 2e^{-x} & (y \geqq e^{-x}) \\ -y & (y \leqq e^{-x}), \end{cases}$$

$y \equiv 0$ is equistable, but it is not uniformly; hence it shall not be strongly stable.

Finally we add a theorem[10] due to Okamura showing a relation between the strong stability and the asymptotical stability.

Theorem 8. *If the solution $y \equiv 0$ of the system* (4) *is asymptotically stable, $y \equiv 0$ is strongly stable.*

REFERENCES

(1) Yoshizawa; "*On the convergence of solutions of the non-linear differential equation,*" Mem. Coll. Sci. Kyoto Univ., A Vol. 28, Mathematics, No. 2 (1953), pp. 143-151.

(2) Yoshizawa; "*On the non-linear differential equation,*" *ibid.*, pp. 133-141 and "*Note on the existence theorem of a periodic solution of the non-linear differential equation,*" *ibid.*, pp. 153-159.

(3) Yoshizawa; "*Note on the boundedness of solutions of a system of differential equations,*" these Memoirs, Vol. 28, No. 3 (1953), pp. 293-298.

(4) Liapounoff; "*Problème général de la stabilité du mouvement,*" Annals of Math. Studies, No. 17.

(5) Massera; "*On Liapounoff's conditions of stability,*" Ann. of Math., Vol. 50 (1949), pp. 705-721.

(6) Okamura; "*On the strong stability of the stationary point in the current,*" Functional Equations (in Japanese), Vol. 40 (1943), pp. 6-19.

(7) Lefschetz; "*Lectures on differential equations,*" Annals of Math. Studies, No. 14.

(8) Hayashi and Yoshizawa; "*New treatise of solutions of a system of ordinary differential equations and its application to the uniqueness theorems,*" these Memoirs, Vol. 26 (1951), pp. 225-233.

(9) Okamura; "*Sur une sorte de distance relative à système differentiel,*" Proc. of Physico-Math. Soc. of Japan, 3rd Ser., Vol. 25 (1943), pp. 514-523.

(10) Okamura; pp. 16-17 in (6).