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Note on the continuation of harmonic and analytic functions

By

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1. In the present paper we shall state some notes concerned with the following problem for which P. J. Myrberg has found a wonderful result[1]:

Let the notations $HB(AB)$, $HD(AD)$ denote respectively the classes of bounded harmonic (analytic) functions and of harmonic (analytic) functions with bounded Dirichlet integral. Let R be an arbitrary Riemann surface and E be a closed subset of R. Then what conditions are necessary and sufficient, in order that E should be *removable* for each family defined on $R-E$ (i.e., it would be possible to continuate without singularities all the functions belonged to the class harmonically or analytically onto E)?

2. The case of HB and HD

Lemma.[2] *Let R be an arbitrary Riemann surface of hyperbolic type. Let $g(P, P_0)$ be the Green's function on R with a pole P_0 and let U be an arbitrary neighbourhood which contains the pole P_0. Then the Dirichlet integral $D_{R-U}[g]$ of g taken over $R-U$ is finite, especially*

$$D_{R-U}[g]=\int_{\partial U} g d\bar{g}$$

if the boundary[3] *∂U of U is analytic, where, in general, the barred letter stands for the conjugate harmonic function.*

Proof. It suffices to assume that ∂U is analytic. Consider the exhaustion of R

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1) P. J. Myrberg: Über die analytische Fortsetzung von beschränkten Funktionen. Ann. Acad. Sci. Fenn. Ser. A, I. 58 (1949).

2) Cf. Nevanlinna: Uniformisierung. 1953.

3) In the following ∂A denotes the boundary of A.

$$U=R_0\subset R_1\subset R_1\subset\cdots\subset R_n\subset\cdots\to R,$$

where $\partial R_n=\Gamma_n$ consists of a finite number of closed analytic curves. Let $g_n(P, P_0)$ be Green's function on R_n with the pole P_0, then the sequence $\{g_n\}$ is convergent uniformly on every compact subset of $R-P_0$ to the Green's function $g(P, P_0)$ on R. Now we have for $n>m$

$$0<\left|\int_{\Gamma_m} g_n d\bar{g}\right|^2=D_{R_n-R_m}[g_n, g]^2\leqq D_{R_n-R_m}[g_n]\,D_{R_n-R_m}[g]$$

$$=\int_{\Gamma_m} g_n d\bar{g}_n\cdot D_{R_n-R_m}[g],$$

where the integrations are taken in negative direction with respect to R_n $(n=1, 2, \cdots)$. For $n\to\infty$ (m: fixed) we have

$$0\leqq\left|\int_{\Gamma_m} g d\bar{g}\right|^2\leqq\int_{\Gamma_m} g d\bar{g}\cdot D_{R-R_m}[g]. \tag{1}$$

Hence we have for any m

$$\int_{\Gamma_m} g d\bar{g}\geqq 0,$$

therefore

$$D_{R_m-R_0}[g]=\int_{\Gamma_0} g d\bar{g}-\int_{\Gamma_m} g d\bar{g}\leqq\int_{\Gamma_0} g d\bar{g}<\infty. \tag{2}$$

Since this implies $D_{R-R_0}[g]<\infty$, it follows that for $m\to\infty$ $D_{R-R_m}[g]\to 0$. We obtain therefore from (1) and (2) the desired result.

Theorem.[4] *Let E be a closed set of inner points on an arbitrary Riemann surface R. Then in order that E should be removable for the classes HB, HD defined on $R'=R-E$, it is necessary and sufficient that the logarithmic capacity of E vanishes.*

Proof. Sufficiency: We consider a domain F (on R), whose boundaries are composed of E and the closed analytic curves Γ_0. Let $\{F_n\}$ be an usual exhaustion of F and ω_n be the harmonic measure of F_n, then $\omega_n\to 0$ $(n\to\infty)$ implies $D_{F_n}[\omega_n]\to 0$, vice versa. Since the theorem[5] of Nevanlinna remains true for this component F, it follows that $u\in HD$ is also bounded. $(|u|\leqq M)$. Let u_0 be

4) For the case HB, see P. J. Myrberg loc. cit. 1).

5) R. Nevanlinna: Über Mittelwerte von Potentialfunktionen. Ann. Acad. Sci. Fenn. Ser. A, I 57 (1949).

R. Nevanlinna: Über das Anwachsen des Dirichletintegrals einer analytischen Funktion auf einer Riemannschen Flächen. Ibid. 45 (1948).

a harmonic function on $F \cup E$ which takes on Γ_0 the value u, then by the inequalities $-2M\omega_n \leqq u - u_0 \leqq 2M\omega_n$ (in F_n) we have for $n \to \infty$ $u \equiv u_0$ i.e., u is harmonic on E.

Necessity: We separate the set E into two disjoint subsets E_1, E_2 such that each harmonic measure is positive. Let $g_i(P, P_0)$ $(i = 1, 2)$ be Green's function on $R - E_i$ whose pole P_0 lies in R'. Then the function $h = g_1 - g_2$ is harmonic everywhere on R'. Let U be a neighbourhood of P_0, then $D_{R'}[h] = D_{R'-U}[h] + D_U[h]$. By Schwarz's inequality

$$D_{R'-U}[h] \leqq [\sqrt{D_{R'-U}[g_1]} + \sqrt{D_{R'-U}[g_2]}]^2.$$

Now by our Lemma we have $D_{R'-U}[g_i] \leqq D_{R-E_i-U}[g_i] < \infty$ $(i=1, 2)$. Since $D_U[h] < \infty$, $h \in HD$. But E is not removable for h, otherwise $g_i = 0$ by the minimum principle, for cap. $E_i > 0$ implies $\inf_{E_i} g_i = 0$.

3. The case of AB and AD

Let R be a Riemann surface which covers exactly m-times $(\infty > m \geqq 2)$ the unit circle K: $|z| < 1$. Then a disc on R would be also removable for AB or AD in the following weak sense[6]. Let $z_1, z_2, \cdots$ denote the projections (arranged with multiplicities) of branch points. Now we take on each i-th sheet the disc K_i $(i = 1, 2, \cdots, m)$ such that

(3) $\quad$ Proj. $K_i \cap$ Proj. $K_j = \phi$ $(i \neq j)$, Proj. $K_i \not\ni \{z_n\}$ for all i and one of the discs K_i is empty ϕ.

Then we have

Proposition 1. *If* $\{z_n\}$ *converge only to the peripherie of* K *and the series*

$$\sum_{n=1}^{\infty} (1 - |z_n|) \tag{4}$$

is divergent, then at least a disc K_j $(\neq \phi)$ *is removable for* AB *or* AD *defined on* $R' = R - \bigcup_{i=1}^{m} K_i$.

Proof. Let $f \in AB$ or AD and $f_i(z)$ be its i-th branch. Consider the function

$$F(z) = \prod_{i<j} [f_i(z) - f_j(z)]^2$$

where the product is taken over all the combinations of the indexes

6) Cf. P. J. Myrberg: loc. cit. 1)

L. Ahlfors: Remarks on the classification of open Riemann surfaces. Ann. Acad. Sci. Fenn. Ser. A. I 87 (1951).

i, j. Then $F(z)$ is single-valued, regular, in the z-plane and moreover $F(z_n)=0$ $(n=1, 2, \cdots)$. Now we assume $F\not\equiv$const. We take the ring domain $K_r: r_0<|z|<r$ such that $K_r\cap(\bigcup_{i=1}^{m}\text{Proj.}\, K_i)=\phi$ and $\partial K_r \not\ni \{z_n\}$. By the argument principle we have usually for $r_0<r<1$

$$\frac{1}{2\pi}\left[\int_{|z|=r_0}+\int_{|z|=r}\frac{\partial}{\partial r}\log|F(re^{i\theta})|\,r\,d\theta\right]=n(r, 0)$$

where $n(r, 0)$ denotes the number of zero points of $F(z)$ lying in K_r. It follows that

$$(5)\qquad \sum_{r_0<|z_n|<r}\log\frac{r}{|z_n|}=\int_{r_0}^{r}\frac{n(r, 0)}{r}dr\leqq O(1)+\int_{|z|=r}\overset{+}{\log}|F|\,d\theta\,.$$

Obviously right hand side is uniformly bounded for $f\in AB$. Next, let F_r be a connected piece of R lying over the circle $|z|<r$. Now we may assume that for r $(\geqq r_0)$ F_r always consists of m-sheets. Then

$$\int_{|z|=r}\overset{+}{\log}|F|\,d\theta\leqq 2(m-1)\sum_{i=1}^{m}\int_{|z|=r}\overset{+}{\log}|f_i|\,d\theta+O(1)$$
$$\leqq 2\pi m(m-1)\,\overset{+}{\log}\int_{\partial F_r}|f|^2 d\theta+O(1).$$

Let

$$\Omega(r)=\int_{\partial F_r}|f|^2 d\theta\,.$$

Since $f=u+iv$ is single-valued on R', we have for $r_0<r<1$

$$\Omega'(r)=\frac{2}{r}\int_{\partial F_r}\left(u\frac{\partial u}{\partial r}+v\frac{\partial v}{\partial r}\right)ds\leqq\frac{4}{r_0}D_{F_r-F_{r_0}}[f]+O(1),\quad ds=rd\theta$$

therefore by integration from r_0 to 1 we find that the right hand side of (5) is also bounded for $f\in AD$. Hence for $r\to 1$ we have $\prod_{n=1}^{\infty}|z_n|>0$. i.e. $\sum(1-|z_n|)<\infty$, which contradicts to (4). Therefore $F(z)\equiv 0$ and it follows at once by the conditions (3) the desired conclusion.

Remark. When $\{z_n\}$ converge to a inner points z_0 $(|z_0|<1)$, z_0 is removable for F, therefore we have also $F(z)\equiv 0$. i.e. the conclusion of Proposition 1 holds.

Proposition 2. *Let R be a Riemann surface (with a finite or infinite number of sheets) spread over the unit circle K which has only algebraic branch points whose projections are $z_1, z_2, \cdots$. Then* *if the series* (4) *converges, any closed domain E on R is not removable for AB defined on $R-E$.*

Proof.[7] Since the linear transformation which maps K into itself does not change the convergency of (4), we assume at first that E contains the origine $z=0$ and there is no branch point on $z=0$. Let P_0 be a point of R lying on $z=0$. Now we consider the exhaustion R_n $(n=1, 2, \cdots)$ of R where R_n is the connected piece which spreads over the circle $|z|<r_n$ (r_n is determined in later) and contains the point P_0. From the convergency of (4) we find that R_n consists of only a finite number k_n of sheets. Let $g_n(P, P_0)$ be Green's function of R_n. Since $g_n(z, P_0) \leqq \log 1/|z|$, $z \in R_n$, for $n \to \infty$

$$(6) \qquad g(z, P_0) \leqq \log 1/|z|.$$

Every domain $G_\lambda = E\{g>\lambda\}$ becomes compact, therefore by Lemma

$$(7) \qquad D_{R-G_\lambda}[g]=2\pi\lambda.$$

Since for $\lambda=\log 1/r$, Proj. G_λ is contained in $|z|<r$, we have by (7)

$$(8) \qquad \int_r^1 \int_{\Gamma(t)} |p'(te^{i\varphi})|^2 t\,dt\,d\varphi \leqq 2\pi \int_r^1 dt/t,$$

where $p=g+i\bar{g}$ and $\Gamma(t)$ denotes all the circles over $|z|=t$. Hence there exists a sequence $\{r_n\}$ tending to 1 such that

$$(9) \qquad \frac{1}{2\pi}\int_{\partial R_n} |p'|^2 d\varphi \leqq \frac{1}{2\pi}\int_{\Gamma(r_n)} |p'|^2 d\varphi \leqq 1/r_n^2 \quad (n=1, 2, \cdots).$$

As $\overset{+}{\log}\, x \leqq x$ $(x \geqq 0)$, it follows

$$(10) \qquad \frac{1}{2\pi}\int_{\partial R_n} \overset{+}{\log}\, |p'|\, d\varphi \leqq 1/2r_n^2 \quad (n=1, 2, \cdots).$$

Let $B(z)$ denotes the Blaschke product of (4), then the function

$$(11) \qquad \varphi(z) = zB(z)\,p'(z)$$

is regular analytic on R. By means of Poisson integral we construct in the circle $|z|<r_n$ a harmonic function u_n whose boundary value is $\max\limits_{\mu=1,\dots,k_n} \overset{+}{\log}\, |\varphi_\mu|$ where φ_μ denote the branches on R_n of φ. Then the regular function

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7) Cf. H. Selberg: Ein Satz über beschränkte endlich vieldeutige analytische Funktion. Comm. Math. Helv. Vol. 9 (1937).

$$f_n(z)=\varphi(z)e^{-\psi_n(z)}, \quad \psi_n=u_n+i\bar{u}_n$$

is single-valued in $|z|<r_n$ and $|f_n(z)|\leqq 1$, $z\in R_n$ by maximum modulus principle. As $R_n\to R$ we can choose a subsequence, say again $\{f_n(z)\}$, converging uniformly to the bounded regular function $f(z)$ defined on R. Now for all the points P_n (except P_0) over $z=0$, $f(P_n)=0$. While, since

$$b=|B(0)|=\prod_{n=1}^{\infty}|z_n|>0$$

and

$$u_n(0)=\frac{1}{2\pi}\int_0^{2\pi}u_n(r_ne^{i\varphi})d\varphi\leqq\frac{1}{2\pi}\int_{\partial R_n}\overset{+}{\log}|p'|\,d\varphi\,,$$

we have by (10)

$$|f(P_0)|=b\lim_{n\to\infty}e^{-u_n(0)}\geqq be^{-1/2}>0\,.$$

It follows that f is non-constant $\in AB$ on R. Now the function $F(z)=f(z)/z$ is regular on R except a point P_0, where it has a simple pole. By maximum modulus principle $|F(z)|\leqq 1/\rho_0$, $z\in R_n-R_0$ (for any n) where R_0 is a small circle of radius ρ_0 with center P_0 which is contained in E. We see that $F\in AB$ on R' and cannot continuate without singularities onto E. q.e.d.

We have therefore by Propositions 1 and 2 the following

Proposition 3. *Let R' be a Riemann surface defined in Proposition* 1. *Then the necessary and sufficient condition that at least a disc should be removable for the class AB is the divergency of series* (4).

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