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On the imbedding of a non-singular variety in an irreducible complete intersection

By

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In the present paper, we shall discuss the following question: Let V^r be a non-singular variety in an ambient projective space L^N, then does there exist a non-singular irreducible complete intersection[1)] $U^n(r+1\leqq n\leqq N-1)$, containing V, in L^N? When the dimension n of U is not less than $2r$, the above question can be solved affirmatively. But in general, this is not true, and a counter example will be given at § 2. It must be noticed that this example also shows the fact that there does not necessarily exist a non-singular variety which contains the given non-singular varety, excepting the ambient space itself.

§ 1. The imbedding theorem.

The following lemma is not new and is essencially the same as Lemma 3 of T. Matsusaka [2].[2)]

LEMMA 1. *Let X, X' be two cycles in a projective space L^N defined over k and P, P' two points lying on X, X' respectively such that (X', P') is a specialization of (X, P) over k. Then, if P' is contained in only one component of X' such that its coefficient is* 1 *and P' is simple on it, the same is true for X and P.*

LEMMA 2. *Let V^r $(r\geqq 2)$ be a projective model in L^N and k a field of definition for both L^N and V. Then there exist a positive integer $M(V)$ and a rational number $R(V)$, both depending only on V, with the following property; if H_m is a hypersurface of degree m in L^N such that $m\geqq M(V)$ and that dim $_k(C(H_m))\geqq l(N, m)-R(V)$*

1) In what follows, we mean by an irreducible complete intersection such a variety U^n that is represented as a complete intersection of $(N-n)$ hypersurfaces in L^N.

2) Numbers in brackets refer to the bibliography at the end of this paper.

m^{r-1}, the intersection-product $V\cdot H_m$ is defined and irreducible, where $l(N, m)=\binom{N+m}{N}-1$.

This lemma is a precise formulation of Theorem 1 of M. Nishi and Y. Nakai [3], and the proof will be stated at § 3.

THEOREM 1. *Let V^r be a projective model in L^N. Then there exists an irreducible complete intersection $U^n (r+1\leqq n\leqq N)$ such that V is contained in U and that the singular locus of U lies on V.*

PROOF. Suppose that there exists an irreducible complete intersection $U'^s (r+2\leqq s\leqq N)$ satisfying the required conditions of our theorem, that is to say, U' contains V and the singular locus of U' lies on V. One should notice that such a variety surely exists for some s; in fact the ambient space L^N itself satisfies these conditions.

It can easily be seen that the totality of hypersurfaces of degree m on which the given variety V lies constitutes a projective space L' of dimension $\varphi(V, m)$[3]-1, and it is defined over any field of definition of V.

Let K be a common field of definition for V, U' and L, then L' is also defined over K, and let $\bar{H}_m$ be the hypersurface of degree m corresponding to the generic point of L' over K. Clearly we have $\dim_K C(\bar{H}_m)=\varphi(V, m)-1$.

If m is sufficiently large, $\chi(V, m)$ shows the Hilbert's characteristic function and therefore is a polynomial of m whose degree is r. Since $s\geqq r+2$, the conditions of Lemma 2 are fulfiled by U' and $\bar{H}_m$ for sufficiently large m. Hence the intersection-product $U'\cdot\bar{H}_m$ is defined and irreducible. Let us put $U=U'\cdot\bar{H}_m$.

Now we shall show that the singular locus of U is contained in V. Let P be any point of U not belonging to V. Let $P=(x_0, x_1, \cdots, x_N)$, then we may assume, without loss of generality, that $x_0=1$.

Let Y be the locus of $C(\bar{H}_m)$ over the algebraic closure $\overline{K(x)}$ of $K(x)$, and Z the locus of $C(\bar{H}_1+H_{m-1})$ over the algebraic closure $\bar{K}$ of K, where $\bar{H}_1$ is a generic hyperplane in L^N over K and H_{m-1}

3) Let $\mathfrak{A}$ be an homogeneous ideal of the polynomial ring $k[X_0, X_1, \cdots, X_N]$, then we denote by $\chi(\mathfrak{A}, m)$ the maximal number of linearly independent forms of degree m modulo $\mathfrak{A}$, and by $\varphi(\mathfrak{A}, m)$ that of linealy independent forms of degree m in $\mathfrak{A}$. Then clealy we have $\varphi(\mathfrak{A}, m)+\chi(\mathfrak{A}, m)=\binom{N+m}{N}$. When W^r is a projective model in L^N defined over k, we put $\chi(W, m)=\chi(\mathfrak{A}(W), m)$ and $\varphi(W, m)=\varphi(\mathfrak{A}(W), m)$, where $\mathfrak{A}(W)$ is the defining homogeneous ideal of W in the polynomial ring $k[X_0, X_1, \cdots, X_N]$. The numbers $\chi(W, m)$ and $\varphi(W, m)$ are independent of the choice of the defining field k.

a hypersurface of degree $m-1$ defined over $\bar{K}$ such that it contains V but not P and the intersection-product $U'\cdot H_{m-1}$ is defined and irreducible. Then, since $\dim_K P\leqq s$, we have

$$\dim Y\geqq\varphi(V, m)-1-s,$$

and

$$\dim Z=N.$$

Now the varieties Y and Z are embedded in a projective space L'. Then the fact that $\dim Y+\dim Z\geqq\varphi(V, m)-1+N-s$ leads us to the conclusion that there exists a point ξ in $Y\cap Z$ such that $\dim_{K(x)}\xi\geqq N-s$. There corresponds to ξ a hypersurface H_m of the form H_1+H_{m-1}, where H_1 is a hyperplane in L^N, and $\dim_{K(x)}C(H_m)\geqq N-s$. Since $C(H_m)\in Y$, we have the specialization $\bar{H}_m\to H_m$ with reference to $K(x)$. Then the point $P=(x)$ must lie on H_1, since H_{m-1} does not contain P.

Suppose that H_1 and U' are transversal to each other at P on L^N. Then the intersection-product $U'\cdot H_m$ is defined and therefore $U'\cdot H_m$ is the uniquely determined specialization of $U'\cdot\bar{H}_m$ over $K(x)$. Moreover the transversality shows that there exists only one component of the cycle $U'\cdot H_m$ to which the point P belongs (the coefficient of this component is 1) and P is simple on this component. Hence, by Lemma 1, P must be simple on $U=U'\cdot\bar{H}_m$.

We shall now show that U' and H_1 are transversal to each other at P. Since $\dim_{K(x)}C(H_m)\geqq N-s$ and H_{m-1} is defined over $\bar{K}$, we have $\dim_{K(x)}C(H_1)\geqq N-s$. Let T^s be the tangential linear variety of U' at P, then all the hyperplanes in L^N passing through T^s build up the $(N-s-1)$-dimensional linear subspace in the dual space of L^N. Hence H_1 cannot contain T, and the intersection-product $T\cdot H_1$ is defined. Thus the proof of our lemma is completed. q. e. d.

Now we are in position to prove the imbedding theorem.

THEOREM 2. *Let V^r be a non-singular projective model in L^N. Then, if n is a positive integer such that $2r\leqq n\leqq N$, there exists a non-singular complete intersection U^n, containing V as a subvariety.*

PROOF. Suppose that there exists a non-singular complete intersection U'^s $(2r+1\leqq s\leqq N)$, containing V as a subvariety. As in Theorem 1, such a variety surely exists for some s.

Let $\mathfrak{A}(V)$, $\mathfrak{A}(U')$ be the defining homogeneous ideal of V, U' respectively and $(f^{(1)}, \cdots, f^{(N-r+t)})$, $(f'^{(1)}, \cdots, f'^{(N-s)})$ be homogeneous ideal bases of $\mathfrak{A}(V)$, $\mathfrak{A}(U')$ respectively. Let K be a common field of definition for both V and U', and $\bar{H}_m$ the hypersurface of degree m introduced in the proof of Theorem 1, namely the most general one over K containing V. Then the defining equation $\bar{H}_m(X)=0$ of $\bar{H}_m$ is as follows;

$$\bar{H}_m(X)=\sum_{l=1}^{N-r+t}\bar{H}_{m_l}^{(l)}(X)f^{(l)}(X)=0,$$

where $\bar{H}_{m_l}(X)$ $(l=1, \cdots, N-r+t)$ are independent generic forms of degree $m_l=m-\deg(f^{(l)})$. Let $\{u_i^{(l)}\,;\, i=0, \cdots, i_l=\binom{N+m_l}{N}-1\}$ be the coefficients of the form $\bar{H}_{m_l}^{(l)}(X)$, then we may assume that $\{u_j^{(l)}\,;\, l=1, \cdots, N-r+t, j=0, 1, \cdots, i_l\}$ are $\sum_l (i_l+1)$ independent variables over K.

By Theorem 1, if m is sufficiently large, $U^{s-1}=U'\cdot\bar{H}_m$ is defined and is an irreducible variety such that the singular locus of U lies on V. Let now $P=(x)$ be any point of V. We may again assume, without loss of generality, that $x_0=1$.

Let us define the matrix

$$A=\begin{pmatrix} \dfrac{\partial f'^{(1)}}{\partial X_1} & \cdots & \dfrac{\partial f'^{(1)}}{\partial X_N} \\ & \cdots & \\ \dfrac{\partial f'^{(N-S)}}{\partial X_1} & \cdots & \dfrac{\partial f'^{(N-S)}}{\partial X_N} \\ \dfrac{\partial \bar{H}_m}{\partial X_1} & \cdots & \dfrac{\partial \bar{H}_m}{\partial X_N} \end{pmatrix}$$

Now we shall show that U is a non-singular variety. For this purpose, it is sufficient to prove that the matrix A is of rank $N-s+1$ at P.

Suppose that the rank of matrix A is not greater than $N-s$ at P, then we have the following N equations

$$\frac{\partial \bar{H}_m}{\partial x_i}=\sum_{j=1}^{N-s}\lambda_j\frac{\partial f'^{(j)}}{\partial x_i} \quad (i=1, \cdots, N)$$

for some quantities $(\lambda_1, \cdots, \lambda_{N-s})$.

Since

$$\frac{\partial \bar{H}_m}{\partial x_i}=\sum_{l=1}^{N-r+t}\bar{H}_{m_l}^{(l)}\frac{\partial f^{(l)}}{\partial x_i},$$

we have

$$(*) \qquad \sum_{l=1}^{N-r+t} \bar{H}^{(l)}_{m_l} \frac{\partial f^{(l)}}{\partial x_i} = \sum_{j=1}^{N-s} \lambda_j \frac{\partial f'^{(j)}}{\partial x_i} \quad (i=1, \cdots, N).$$

Now P is simple on V, hence we can assume that

$$\det \begin{vmatrix} \frac{\partial f^{(1)}}{\partial x_1} & \cdots & \frac{\partial f^{(N-s)}}{\partial x_1} \\ & \cdots & \\ \frac{\partial f^{(1)}}{\partial x_{N-r}} & \cdots & \frac{\partial f^{(N-r)}}{\partial x_{N-r}} \end{vmatrix} \neq 0.$$

Then from (*), we have

$$\bar{H}^{(l)}_{m_l}(x) \in K(x, \lambda_1, \cdots, \lambda_{N-s}) \quad (l=1, 2, \cdots, N-r)$$

This shows that

$$\dim_{K(x,\lambda)} \{u_j^{(l)}\,;\ 1 \leqq l \leqq N-r+t,\ 0 \leqq j \leqq i_l\} \leqq$$
$$\dim_K \{u_j^{(l)}\,;\ 1 \leqq l \leqq N-r+t,\ 0 \leqq j \leqq i_l\} - (N-r).$$

Hence we have $\dim_K K(x, \lambda) \geqq N-r$. Therefore $\dim_{K(x)} K(x, \lambda) \geqq N-2r$, since $P = (x)$ lies on V defined over K. This is a contradiction.

q. e. d.

COROLLARY. *If Γ is a non-singular projective curve in L^N, then, for any integer $n (2 \leqq n \leqq N)$, there exists a non-singular complete intersection U^n, containing Γ as a subvariety.*

§ 2. A counter example.

In Theorem 2 it is desirable to eliminate the additional condition on the dimension n of the variety U, but it is not true in general as will be shown in the following counter example.

Let k be the field of rational numbers and t_1, t_2 two independent variables over k. In an affin 4-space S^4, we consider a variety V'^2, which is the locus of the point $(t_1t_2, t_1, t_2, t_1^2)$ over k. Then it is easy to see that the defining ideal of V' in the polynomial ring $k[X_1, X_2, X_3, X_4]$ are generated by $f'^{(1)} = X_2X_3 - X_1$ and $f'^{(2)} = X_2^2 - X_4$. Now if we immerse V' in a projective 4-space L^4, we can get the projective model V^2 and V is the locus of the point $(\lambda, \lambda t_1 t_2, \lambda t_1, \lambda t_2, \lambda t_1^2)$ of L^4 over k, where λ is a variable over $k(t_1, t_2)$.

Now we are going to prove that the defining homogeneous ideal $\mathfrak{A}(V)$ of V is generated by three forms $f^{(1)} = X_2X_3 - X_0X_1$, $f^{(2)} = X_2^2 - X_0X_4$, and $f^{(3)} = X_1X_2 - X_3X_4$. Let the ideal generated by $f^{(1)}$,

$f^{(2)}$ and $f^{(3)}$ be $\tilde{\mathfrak{A}}$; and we shall show that $\mathfrak{A}(V)=\tilde{\mathfrak{A}}$. Let g be any homogeneous form in $\mathfrak{A}(V)$, then for suitable choice of an positive integer m, we can find that, by using the fact that $X_0X_1\equiv X_2X_3(\tilde{\mathfrak{A}})$ and $X_0X_4\equiv X_2^2(\tilde{\mathfrak{A}})$, $X_0^m\cdot g\equiv h\ (X_0, X_2, X_3)\ (\tilde{\mathfrak{A}})$, where h is a form in the polynonial ring $k[X_0, X_2, X_3]$. Since t_1, t_2 are variables over k, the right hand side must be identically zero. Hence $X_0^m g\equiv 0\ (\tilde{\mathfrak{A}})$. Now, for the proof that $\mathfrak{A}(V)=\tilde{\mathfrak{A}}$, it is sufficient to show that, if $X_0 f\equiv 0(\tilde{\mathfrak{A}})$ for any form f in $\mathfrak{A}(V)$, we have $f\equiv 0\ (\tilde{\mathfrak{A}})$. Since $X_0 f$ belongs to the ideal $\tilde{\mathfrak{A}}$, we can find the following expression $X_0 f=g_1f^{(1)}+g_2f^{(2)}+g_3f^{(3)}$. Let us put $g_j=g_{j1}+g_{j2}$ $(j=1, 2, 3)$, where $g_{j1}\equiv 0(X_0)$ and g_{j2} is free from X_0 (g_{j2} may be zero for some j). Then we have

$$g_{12}X_2X_3+g_{22}X_2^2+g_{32}(X_1X_2-X_3X_4)=0,$$

and hence $g_{32}\equiv 0(X_2)$. Put $g_{32}=g'_{32}X_2$. Again we have

$$g_{12}X_3+g_{22}X_2+g'_{32}(X_1X_2-X_3X_4)=0,$$

and $(g_{12}-g'_{32}X_4)X_3+(g_{22}+g'_{32}X_1)X_2=0$.

Therefore we can get the follewing expression

$$g_{12}=g'_{32}X_4+X_2q,\ g_{22}=-g'_{32}X_1-X_3q,$$

where q is a form, and it follows that

$$\begin{aligned} g_{12}X_0X_1+g_{22}X_0X_4&=X_0(g_{12}X_1+g_{22}X_4)\\ &=X_0q(X_1X_2-X_3X_4)\\ &=X_0qf^{(3)}. \end{aligned}$$

Hence $f\equiv 0(\tilde{\mathfrak{A}})$.

The Jacobian matrix J of V is as follows;

$$J=\begin{pmatrix} -X_1 & -X_0 & X_3 & X_2 & 0\\ -X_4 & 0 & X_2 & 0 & -X_0\\ 0 & X_2 & X_1 & -X_1 & -X_3 \end{pmatrix}.$$

And it can easily be shown that the rank of this matrix is of 2 at each point of V. Hence V is a non-singular variety.

Now we shall prove that the singular locus of any 3-dimensional varity passing through V is not empty. For this purpose, by Lemma 1, we have only to prove it for the most general one which passes through V. Let V_m^3 be the most general hypersurface of degree m which passes through V. We shall examine two caces separately:

Case 1. $m=2.$[4]

In this case, $\bar{V}_2$ is defined by the equation

$$\bar{H}_2(X)=uf^{(1)}+vf^{(2)}+wf^{(3)}=0,$$

where u, v, w are three variables over k. The point $(w^2, -uv, uw, -vw, u^2)$ of V is surely a multiple point of $\bar{V}_2$.

Case 2. $m\geqq 3$.

The defining equation of $\bar{V}_m$ is as follows;

$$\bar{H}_m(X)=\bar{H}^{(1)}_{m-2}(X)f^{(1)}(X)+\bar{H}^{(2)}_{m-2}(X)f^{(2)}(X)+\bar{H}^{(3)}_{m-2}(X)f^{(3)}(X)=0,$$

where $\bar{H}^{(i)}_{m-2}(X)$ $(i=1, 2, 3)$ are independent generic forms of degree $m-2$ over k. We first consider the following equations:

$$-\bar{H}^{(1)}_{m-2}(X)X_0+\bar{H}^{(3)}_{m-2}(X)X_2=0$$

$$-\bar{H}^{(2)}_{m-2}(X)X_0-\bar{H}^{(3)}_{m-2}(X)X_3=0,$$

and let $\boldsymbol{X}$, $\boldsymbol{Y}$ be the cycles on $\boldsymbol{L}^4$ defined by the above equations ($\boldsymbol{X}$ by the former and $\boldsymbol{Y}$ the latter) respectively. Further let $\boldsymbol{X}'$, $\boldsymbol{Y}'$ be the cycles defined by the following equations respectively:

$$-\bar{H}^{(1)}_{m-2}(X)X_0=0, \quad -\bar{H}^{(2)}_{m-2}(X)X_0=0.$$

Then we can see immediately that $\boldsymbol{X}'$ and $\boldsymbol{Y}'$ are the specializations of $\boldsymbol{X}$ and $\boldsymbol{Y}$ over k respectively. Therefore we have the following specialization $(\boldsymbol{V}\cdot\boldsymbol{X}, \boldsymbol{Y})\rightarrow(\boldsymbol{V}\cdot\boldsymbol{X}', \boldsymbol{Y}')$ with reference to k.

The intersection-product $(\boldsymbol{V}\cdot\boldsymbol{X}')\cdot\boldsymbol{Y}'$ is not defined, but if we denote by $\bar{\boldsymbol{H}}^{(i)}_{m-2}(i=1, 2)$ the hypersurfaces of degree $m-2$ defined by the equations $\bar{H}^{(i)}_{m-2}(X)=0$ $(i=1, 2)$, each component of the cycle $\boldsymbol{V}\cdot\bar{\boldsymbol{H}}^{(1)}_{m-2}\cdot\bar{\boldsymbol{H}}^{(2)}_{m-2}$ is a proper component of the intersection $(\boldsymbol{V}\cdot\boldsymbol{X}')\cap\boldsymbol{Y}'$. Let $\boldsymbol{M}'$ be such a component, then $\boldsymbol{M}'$ is clearly a generic point of $\boldsymbol{V}$ over k. As is well known, there exists a proper component $\boldsymbol{M}$ of the intersection $(\boldsymbol{V}\cdot\boldsymbol{X})\cap\boldsymbol{Y}$ such that $\boldsymbol{M}'$ is a speciallzation of $\boldsymbol{M}$ over the specialization $(\boldsymbol{V}\cdot\boldsymbol{X}, \boldsymbol{Y})\rightarrow(\boldsymbol{V}\cdot\boldsymbol{X}', \boldsymbol{Y}')$ with reference of k. The point $\boldsymbol{M}$ must be a generic point of $\boldsymbol{V}$ over k, since $\boldsymbol{M}'$ is so.

Let us put $\boldsymbol{M}=(x_0, x_1, x_2, x_3, x_4)$, then $x_0\neq 0$. At this point $\boldsymbol{M}=(x)$, we have

$$\frac{\partial\bar{H}_m}{\partial x_1}=-\bar{H}^{(1)}_{m-2}(x)\cdot x_0+\bar{H}^{(3)}_{m-2}(x)\cdot x_2=0$$

4) It is easy to see that V is not contained in a hyperplane in L^4.

$$\frac{\partial \bar{H}_m}{\partial x_4} = -\bar{H}^{(2)}_{m-2}(x)x_0 + \bar{H}^{(3)}_{m-2}(x)\cdot x_3 = 0,$$

and since $\boldsymbol{M}$ is a generic point of $\boldsymbol{V}$ over k, it follows naturally that

$$\frac{\partial \bar{H}_m}{\partial x_j} = 0 \quad (j=0, 2, 3).$$

This yields the conclusion that $\boldsymbol{M}$ is a multiple point of $\bar{\boldsymbol{V}}_m$.

Thus we have established that our non-singular surface $\boldsymbol{V}$ cannot be contained in any non-singular 3-dimensional variety.

Remark. After a straightfoward computations, we can see that the degree of the variety $\boldsymbol{V}$ is 3.

§ 3. The proof of Lemma 2.

To prove Lemma 2, we need some lemmas.

Let $\mathfrak{L}(r, d; N)$ be the algebraic system built up by the cycles on $\boldsymbol{L}^N$ whose dimensions are r and degrees d; let $e(r, d; N)$ be the maximal dimension of the components in $\mathfrak{L}(r, d; N)$. Then we have the following lemma:

LEMMA 3. *If d is a sufficiently large positive integer, we have* $e(r, d; N) \leqq (N+1)\cdot d^{r+1}$.

PROOF. We shall use the induction on the dimension N of the ambient projective space $\boldsymbol{L}^N$.

When N is 2, our assertion is trivially valid.

Assume that our lemma is verified for any projective space of dimension $\leqq N-1$. And now we shall proceed to the case $\boldsymbol{L}^N$.

Let $\boldsymbol{\Gamma}$ be a member of $\mathfrak{L}(r, d; N)$ such that $\dim_{k_0}(C(\boldsymbol{\Gamma})) = e(r, d; N)$, where k_0 is a field over which $\boldsymbol{L}^N$ is defined. Let $\boldsymbol{P}$ be a $\bar{k}_0$-rational point of $\boldsymbol{L}^N$, $\bar{k}_0$ being the algebraic closure of k_0, such that $\boldsymbol{P}$ does not belong to $\boldsymbol{\Gamma}$. (Here we assume that $r \leqq N-2$, because our assertion is trivial for $r=N-1$.) Projecting $\boldsymbol{\Gamma}$ from the point $\boldsymbol{P}$, we get a projecting cone $\tilde{\boldsymbol{\Gamma}}^{r+1}$ of $\boldsymbol{\Gamma}$ with the center $\boldsymbol{P}$. Then $\tilde{\boldsymbol{\Gamma}}$ is a $(r+1)$-dimensional irreducible variety, since $\boldsymbol{\Gamma}$ is a r-dimensional irreducible one, and moreover $\deg \tilde{\boldsymbol{\Gamma}} = \deg \boldsymbol{\Gamma} = d$. Let $\boldsymbol{H}$ be a hyperplane, defined over $\bar{k}_0$, such that $\boldsymbol{H} \not\ni \boldsymbol{P}$ and the intersection-product $\tilde{\boldsymbol{\Gamma}}\cdot\boldsymbol{H}$ is defined and irreducible.

Set $\boldsymbol{\Gamma}' = \tilde{\boldsymbol{\Gamma}}\cdot\boldsymbol{H}$. Then clealy we have $\dim_{\bar{k}_0(C(\tilde{\Gamma}))}(C(\boldsymbol{\Gamma}')) = 0$. On the other hand, let $\tilde{\boldsymbol{\Gamma}}'$ be an arbitrary specialization of $\tilde{\boldsymbol{\Gamma}}$ with reference to $\bar{k}_0(C(\boldsymbol{\Gamma}'))$. Then, since the intersection-product $\tilde{\boldsymbol{\Gamma}}'\cdot\boldsymbol{H}$

is defined,[5] $\tilde{\Gamma}\cdot H$ has the uniquely determined specialization $\tilde{\Gamma}'\cdot H$ over the specialization $\tilde{\Gamma}\to\tilde{\Gamma}'$ with reference to $k_0(C(\Gamma'))$. This yields that $\tilde{\Gamma}'\cdot H=\Gamma'=\tilde{\Gamma}\cdot H$, and hence $\tilde{\Gamma}'=\tilde{\Gamma}$, thus we have $\dim_{k_0(C(\Gamma'))}C(\tilde{\Gamma})=0$. Therefore it holds that $\dim_{k_0}(C(\tilde{\Gamma}))=\dim_{k_0}(C(\Gamma'))$. But now, by induction assumption, $\dim_{k_0}(C(\Gamma'))\leqq N\cdot d^{r+1}$ for sufficiently large d. Hence we have $\dim_{k_0}(C(\tilde{\Gamma}))\leqq N\cdot d^{r+1}$ for sufficiently large d.

Let M^{N-r-2} be a linear variety in L^N defined over $\bar{k}_0$ such that the intersection $\Gamma\cap M$ is empty and that the projecting cone $\tilde{H}^{N-1}$ of Γ with the center M does not contain $\tilde{\Gamma}$. Then it is easy to see that

$$\dim_{k_0}(C(\tilde{H}))\leqq\binom{r+1+d}{r+1}$$

$$\leqq d^{r+1} \text{ for sufficiently large } d,$$

and now we can estimate $e(r, d;N)$ as follows;

$$\begin{aligned} e(r, d;N) &\leqq\dim_{k_0}(C(\tilde{\Gamma}))+\dim_{k_0(C(\tilde{\Gamma}))}(C(\Gamma)) \\ &\leqq\dim_{k_0}(C(\tilde{\Gamma}))+\dim_{k_0(C(\tilde{\Gamma}))}(C(\tilde{\Gamma}\cdot\tilde{H})) \\ &\leqq\dim_{k_0}(C(\tilde{\Gamma}))+\dim_{k_0(C(\tilde{\Gamma}))}(C(\tilde{H})) \\ &\leqq\dim_{k_0}(C(\tilde{\Gamma}))+\dim_{k_0}(C(\tilde{H})) \\ &\leqq N\cdot d^{r+1}+d^{r+1}, \\ &\leqq(N+1)\cdot d^{r+1}, \end{aligned}$$

where we assume that d is sufficiently large. Thus the proof is completed. q. e. d.

LEMMA 4. *For any integer r, $1\leqq r\leqq N-1$, there exists a positive integer $m_0(r)$, depending only on r, with the folloing property; if H_m is a hypersurface of degree m in L^N such that $\dim_k C(H_m)\geqq l(N, m)-m^r/N!$ and that $m\geqq m_0(r)$, then H_m has no subvarieties of dimension r and of degree $d\leqq m^{r/r+1}/N+1$, where k is any field over which L^N is defined.*

PROOF. There exists a positive integer $m_0'(r)$ such that, if $m\geqq m_0'(r)$, then the inequality

$$\binom{r+m}{r}\geqq m^r/N!+(N+1)(m^{r/r+1}/N+1)^{r+1}+1$$

5) We can easily see that each component of $\tilde{\Gamma}'$ is also a cone with the vertex P, and P does not lie on H. Therefore $\tilde{\Gamma}'\cdot H$ can be defined.

holds. Let a positive integer $m_0''(r)$ be such that, by Lemma 3, if $m \geqq m_0''(r)$, then $e(r, [m^{r/r+1}/N+1]; N) \leqq (N+1)[m^{r/r+1}/N+1]^{r+1}$, where [ ] shows the Gauss' symbol. Put $m_0(r) = \max(m_0'(r), m_0''(r))$. The number $m_0(r)$ will satisfy the requirements of our lemma.

In fact, suppose that there exists a hypersurface $\boldsymbol{H}_m$ of degree m in $\boldsymbol{L}^N$ such that $m \geqq m_0(r)$ and $\dim_k(C(\boldsymbol{H}_m)) \geqq l(N, m) - m^r/N!$ and that $\boldsymbol{H}_m$ contains a subvariety $\boldsymbol{\Gamma}$ of dimension r and of degree $d \leqq m^{r/r+1}/N+1$. Then we have

$$\dim_{k(C(\boldsymbol{\Gamma}))}(C(\boldsymbol{H}_m)) \geqq l(N, m) - m^r/N! - e(r, d; N).$$

Now, since $m \geqq m_0(r)$, it follows that

$$\binom{r+m}{r} \geqq m^r/N! + (N+1)\cdot(m^{r/r+1}/N+1)^{r+1} + 1$$
$$\geqq m^r/N! + e(r, [m^{r/r+1}/N+1]; N) + 1$$
$$\geqq m^r/N! + e(r, d; N) + 1.$$

Hence $\dim_{k(C(\boldsymbol{\Gamma}))}(C(\boldsymbol{H}_m)) > l(N, m) - \binom{r+m}{r}$.

On the other hand, since $\boldsymbol{H}_m$ contains $\boldsymbol{\Gamma}$,

$$\dim_{k(C(\boldsymbol{\Gamma}))}(C(\boldsymbol{H}_m)) \leqq \varphi(\boldsymbol{\Gamma}, m) - 1$$
$$= l(N, m) - \chi(\boldsymbol{\Gamma}, m).$$

Therefore we have

$$\chi(\boldsymbol{\Gamma}, m) < \binom{r+m}{r}.$$

But, if m is sufficiently large, $\chi(\boldsymbol{\Gamma}, m)$ has the following expression[6]; $\chi(\boldsymbol{\Gamma}, m) = (\deg \boldsymbol{\Gamma})\cdot\binom{m}{r} + a_1\binom{m}{r-1} + \cdots + a_{r-1}\binom{m}{1} + a_r$, a_i $(1 \leqq i \leqq r)$ being integers. And this shows that $\chi(\boldsymbol{\Gamma}, m) \geqq \binom{r+m}{r}$. This is a contradiction. q. e. d.

Now we can state the proof of Lemma 2.

Set $\overline{m} = [m^{r-1/r}/d_0(N+1)]$, where d_0 is the degree of $\boldsymbol{V}$. Then there exists a positive integer $m_0''(\boldsymbol{V})$, depending only on $\boldsymbol{V}$, such that if $m \geqq m_0''(\boldsymbol{V})$,

$$l(N, \overline{m}) - \varphi(\boldsymbol{V}, \overline{m}) = \chi(\boldsymbol{V}, \overline{m}) - 1 \geqq m^{r-1}/d_0^r(N+1)^N r!$$

By Lemma 4, there exists a positive integer $m_0'''(r-1)$. Let

6) Cf. W. Krull [1].

us set $M(V)=\max(m_0'', m_0''')$, and $R(V)=1/d_0^r(N+1)^N\cdot r!$ Then these two numbers $M(V)$ and $R(V)$ will satisfy the requirements of Lemma 2. The proof is as follows.

Suppose that there exists a hypersurface $\boldsymbol{H}_m$ of degree m such that $m \geqq M(V)$ and $\dim_k C(\boldsymbol{H}_m) \geqq l(N, m) - R(V)m^{r-1}$ and further that $V\cdot \boldsymbol{H}_m$ is reducible.

Let now Y be the locus of $C(\boldsymbol{H}_m)$ over $\bar{k}$, the algebraic closure of k. There exists a hypersurface $\boldsymbol{H}_{m-\bar{m}}$ of degree $m-\bar{m}$, defined over $\bar{k}$, such that $V\cdot \boldsymbol{H}_{m-\bar{m}}$ is defined and irreducible. Let Z be the locus of $C(\boldsymbol{H}_{m-\bar{m}}+\bar{\boldsymbol{H}}_{\bar{m}})$ over $\bar{k}$, where $\bar{\boldsymbol{H}}_{\bar{m}}$ is a generic hypersurface of degree $\bar{m}$ over $\bar{k}$. Then $\dim Y \geqq l(N, m) - m^{r-1}/d_0^r(N+1)^N r!$ and $\dim Z = l(N, \bar{m})$. Hence there exists a point ξ in $Y\cap Z$ such that $\dim_k \xi \geqq l(N, \bar{m}) - m^{r-1}/d_0^r\cdot(N+1)^N r! \geqq \varphi(V, \bar{m})$. There corresponds to ξ a hypersurface $\boldsymbol{H}_m'$ of degree m such that $\boldsymbol{H}_m' = \boldsymbol{H}_{m-\bar{m}}+\boldsymbol{H}_{\bar{m}}$. Then the fact that $\dim_k \xi \geqq \varphi(V, \bar{m})$ shows that the intersection-product $V\cdot H_m'$ is defined. Since $V\cdot \boldsymbol{H}_m$ is reducible, $\boldsymbol{H}_m$ must contain a subvariety Γ of dimension $r-1$ and of degree $\leqq d_0\bar{m} \leqq m^{r-1/r}/N+1$.

On the other hand,

$$\dim_k(C(\boldsymbol{H}_m)) \geqq l(N, m) - m^{r-1}/d_0^r(N+1)^N\cdot r!$$
$$\geqq l(N, m) - m^{r-1}/N!\,.$$

Hence, by Lemma 4, $\boldsymbol{H}_m$ cannot contain such a variety Γ. This is a contradiction. Thus the proof is completed. q. e. d.

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