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On the normality of the Chow variety of positive 0-cycles of degree m in an algebraic variety

By

Masayoshi NAGATA

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When V is a projective variety defined over a field k,*) there corresponds a projective variety $V(m)$ whose points are in a one-to-one correspondence with positive 0-cycles of degree m in V, by virtue of results on associated forms due to W. L. Chow and B. L. van der Waerden [3]. When V is a non-singular curve, then $V(m)$ is also non-singular, as was proved by Chow [2] (cf. van der Waerden [8]). From this, a question arises: *If V is normal,*) then is $V(m)$ normal?*

In the present paper, we will prove the following results (which answer the question):

1) *When k is of characteristic zero, then the question is affirmative.*

2) *Though in the other case the question is not affirmative, there exists a biregularly equivalent*) variety V' to V such that the variety $V'(m)$ is normal.*

We will prove further the following result:

3) *If k is of characteristic zero and if V' is a variety which is biregularly equivalent to V, then $V(m)$ and $V'(m)$ are biregularly equivalent to each other, without assumption that V is normal.*

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§ 1. Definitions and notations.

(1) Let k be a field, let $(x^{(i)}) = (x_1^{(i)}, \cdots, x_n^{(i)})$ $(i=1, \cdots, m)$ be vectors with indeterminates $x_j^{(i)}$'s and let $\mathfrak{S}$ be the symmetric group

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*) For the definitions, see § 1.

of degree m. For any element σ of $\mathfrak{S}$, there corresponds a uniquely determined automorphism, which will be denoted also by σ, such that $\sigma(x_j^{(i)})=x_j^{(\sigma(i))}$. An element f of the ring $k[x]=k[x_1^{(1)}, \cdots, x_n^{(1)}, x_1^{(2)}, \cdots, x_n^{(m)}]$ is called a *symmetric form* on vectors $(x^{(i)})$'s if $\sigma(f)=f$ for any element σ of $\mathfrak{S}$. Let g be any element of $k[x]$ and let H be the set of elements σ of $\mathfrak{S}$ such that $\sigma(g)=g$; H is obviously a subgroup of $\mathfrak{S}$. Then $f=\sum_{\sigma_i \in \mathfrak{S},\ \sigma_i H \neq \sigma_j H} \sigma_i(g)$ is a symmetric form; this symmetric form f is called the *symmetric form generated by g* and will be denoted by $(g)_s$.

On the other hand, let $u_1, \cdots, u_n$ be indeterminates. Then the coefficients of the polynomial $\Pi_i(1+\sum_j x_j^{(i)}u)$, regarded as a polynomial in u_j's, are called the *fundamental symmetric forms* on the vectors $(x^{(i)})$'s. As is easily seen, a fundamental symmetric form is of the form $(x_{j_1}^{(i_1)}\cdots x_{j_r}^{(i_r)})_s$ $(i_k \neq i_l$ if $k \neq l)$ and conversely. The form $(x_{j_1}^{(i_1)} \cdots x_{j_r}^{(i_r)})_s$ will be denoted by $s_{j_1,\ldots,j_r}^{i_1,\ldots,i_r}$.

(2) A ring will mean a commutative ring. A ring is called a *normal ring* if it is an integrity domain which is integrally closed in its field of quotients. When $\mathfrak{o}$ is an integrity domain, the integral closure of $\mathfrak{o}$ in its field of quotients is called the *derived normal ring* of $\mathfrak{o}$.

An integrity domain $\mathfrak{o}$ is said to be a *regular extension* of its subring I if the field of quotients of $\mathfrak{o}$ is a regular extension of that of I in the sense of Weil [9].

(3) An *affine variety* V defined over a field k is a variety in the sense of Weil [9] defined over k. When $(x)=(x_1, \cdots, x_n)$ is a generic point of V over k, the ring $k[x]=k[x_1, \cdots, x_n]$ is uniquely determined within isomorphisms over k; this ring is called the (affine) *co-ordinate ring* of V and the field of quotients of this ring is called the *function field* of V.

A Variety in the sense of [9] is called an *abstract variety*. A complete abstract variety in a projective space is called a *projective variety*. Let V be a projective variety defined over a field k and let $(z)=(z_0, \cdots, z_n)$ be the homogeneous co-ordinate of a generic point of V over k. Then the function field of V is $k(z_0/z_i, \cdots, z_n/z_i)$ (with $z_i \neq 0$). When z_i is transcendental over the function field of V, we call the ring $k[z]=k[z_0, \cdots, z_n]$ the *homogeneous co-ordinate ring* of V.

Two abstract varieties V and V' defined over the same field k are said to be *biregularly equivalent* to each other if they are in everywhere biregular correspondence to each other; in other words, if V and V' correspond to the same model over k in the sense of Nagata [5, I].

We will notice here that the function field of a variety over a field k is a regular extension of k, by the definition in Weil [9]. Therefore the co-ordinate ring of an affine variety over its field k of definition and the homogeneous co-ordinate ring of a projective variety over its field k of definition are regular extensions of k.

In our treatment on projective varieties, we need to consider fields of definition which contain infinitely many elements. Therefore for the simplicity of statements, we will assume always that the fields of consideration contains infinitely many elements.

(4) An abstract variety V defined over a field k is called a *normal variety* (over k) if the model over k, in the sense of Nagata [5, I], which corresponds to V, is a normal model, namely, if the specialization ring of any point of V in the function field of V over k is normal. Then an affine variety is a normal variety if the co-ordinate ring of the variety (over the field of definition of consideration) is a normal ring. Even when a projective variety V defined over a field k is normal, the homogeneous co-ordinate ring $\mathfrak{h}$ of V defined over k may not normal. When $\mathfrak{h}$ is normal, we call V an *arithmetically normal variety*. As is easily seen, any arithmetically normal variety is a normal variety.

(5) Let V be an affine variety defined over a field k and let $(x^{(1)}), \cdots, (x^{(m)})$ be independent generic point of V over k. Let $s_1, \cdots, s_t$ be all of fundamental symmetric forms on $(x^{(i)})$'s. Then the affine variety with the generic point $(s_1, \cdots, s_t)$ defined over k is called the *Chow variety* of positive 0-cycles of degree m in V and will be denoted by $V(m)$.

Let V be a projective variety defined over a field k and let $(z^{(i)})=(z_0^{(i)}, \cdots, z_n^{(i)})$ $(i=1, \cdots, m)$ be independent generic points of V over k. Choose one j such that $z_j^{(i)}\neq 0$ and set $(x^{(i)})=(z_0^{(i)}/z_j^{(i)}, \cdots, z_n^{(i)}/z_j^{(i)})$. Let $s_0, \cdots, s_t$ be all of fundamental symmetric forms on $(x^{(i)})$'s. Then the projective variety with generic point $(s)=(s_0, \cdots, s_t)$ is called the *Chow variety* of positive 0-cycles of degree m in V and will be denoted by $V(m)$. (Observe that there exists one l such that $s_l=1$, because $x_j^{(i)}=1$ for any i.) The affine variety with the generic point (s) defined over k is called the affine representative of $V(m)$ defined by $z_j=1$. (Observe that this affine variety is really an affine representative of $V(m)$.)

As is well known (see [3] or [7]), the definition of $V(m)$ does not depend on the choice of the subscript j (up to linear transformations). Further the following must be noted:

Let A be a linear transformation of the vector space S^n of dimension n into a vector space S^r. Set $(y^{(i)})=(x^{(i)})A$. Then the fundamental symmetric forms on $(y^{(i)})$'s are linear combinations of these on $(x^{(i)})$'s. From this we see immediately.

When a given affine or projective variety V is transformed by a linear transformation (in an affine or projective space which contains the ambient space of V), then the variety $V(m)$ is transformed by a linear transformation.

§ 2. Normality of $V(m)$ (1).

LEMMA 1. Let $\mathfrak{o}$ and $\mathfrak{o}'$ be normal rings which contains a field k. If $\mathfrak{o}$ and $\mathfrak{o}'$ are regular extensions of k, then $\mathfrak{o}\otimes_k\mathfrak{o}'$ is a normal ring. (Nakai)

For the proof see [5, II].

Now, we first consider the affine case. Let V be an affine variety defined over a field k and let $(x^{(i)})$ $(i=1, \cdots, m)$ be independent generic points of V over k. Then

LEMMA 2. Let f be an element of the ring $\mathfrak{o}=k[x^{(1)}, \cdots, x^{(m)}]$. If f is invariant under any automorphisms induced by the permutations of superscripts of $(x^{(i)})$'s, then f is expressible as a symmetric form on $(x^{(i)})$'s.[1) 2)]

Proof. We denote by $\mathfrak{o}^{(i)}$ the ring $k[x^{(i)}]$ for each i. Then there exists an isomorphism σ_i from $\mathfrak{o}^{(1)}$ and $\mathfrak{o}^{(i)}$ which mapps $(x^{(1)})$ to $(x^{(i)})$ for each i. Let $\{u_\lambda^{(1)}\}$ be a linearly independent base of $\mathfrak{o}^{(1)}$ over k and set $u_\lambda^{(i)}=\sigma_i(u_\lambda^{(1)})$. Since $\mathfrak{o}$ may be regarded as the m-ple tensor product of $\mathfrak{o}^{(1)}$, f is expressible uniquely in the form $\sum a_{\lambda_1,\dots,\lambda_m}u_{\lambda_1}^{(1)}\cdots u_{\lambda_m}^{(m)}$ $(a_{\lambda_1,\dots,\lambda_m}\in k)$. Then the uniqueness of this representation and the invariance of f under permutations of $(x^{(i)})$'s shows that the above expression is a symmetric form and we see our assertion.

Proposition 1. *Let V be a normal affine variety defined over a field k. Then there exists an affine variety V' defined over k*

1) In general, there are relations among $x_j^{(i)}$'s. Therefore that f is invariant does not mean that the expression of f is a symmetric form (formally).

2) When k is of characteristic zero, the proof is easy: Let $\mathfrak{S}$ be the symmetric group of degree m which operates to the superscripts of $(x^{(i)})$'s. Since $f=\sigma(f)$ for any $\sigma\in\mathfrak{S}$, $f=\sum_{\sigma\in\mathfrak{S}}\sigma(f)/(m!)$ and the right hand of this equality is a symmetric form.

which is biregularly equivalent to V such that $V'(m)$ is normal.

Proof. We will use the same notations as in Lamma 2. By Lemma 1, the ring $\mathfrak{o}$ is a normal ring in this case. Let $\mathfrak{S}$ be the co-ordinate ring of $V(m)$. If $\mathfrak{S}$ is not normal, then there exists an element f of the derived normal ring $\mathfrak{S}'$ of $\mathfrak{S}$ which is not $\mathfrak{S}$. Since $\mathfrak{o}$ is noraml, f is in $\mathfrak{o}$. Since f is in $\mathfrak{S}'$, f is invariant under any automorphisms induced by permutations of $(x^{(i)})$'s. Therefore by Lemma 2 there exist elements $u_j^{(i)}$'s of $\mathfrak{o}^{(i)}$'s $(u_j^{(i)}=\sigma_i(u_j^{(1)}))$ such that f is the sum of fundamental symmetric forms on $(u^{(i)})$'s. Let V'' be the affine variety defined over k with the generic point $(x^{(1)}, u^{(1)})$. Then V'' is biregularly equivalent to V and the co-ordinate ring of $V''(m)$ contains f and $\mathfrak{S}$. Repeating the same procedure, we see the existence of the required variety V' by the finiteness of $\mathfrak{S}'$ over $\mathfrak{S}$.

Next, we will consider projective varieties. Let V be a projective variety defined over a field k and contained in a projective space P^n of dimension n. Then

LEMMA 3. There exist hyperplanes $H_1, \cdots, H_t$ in P^n defined over k such that for any m points $Q_1, \cdots, Q_m$ of V (m being a given integer), there exists one i such that H_i does not contain any of Q_j's.[3]

Proof.[4] Let t be an integer greater than mn and let $H_1, \cdots, H_t$ be hyperplanes of P^n such that any of $n+1$ of them have no common point; since k contains infinitely many elements, we can choose them so that they are defined over k. For any point Q_k, there exist at most n of H_j's which contain Q_k. Since $t>mn$, we see that there exists one H_i which does not contain any of Q_j's and we complete the proof.

Now we will prove

Theorem 1. *If V is a normal projective variety defined over a field k, then there exists a projective variety V' defined over k which is biregularly equivalent to V such that $V'(m)$ is normal.*

Proof. Let $\mathfrak{h}$ be the homogeneous co-ordinate ring of V. By Lemma 3, there exist linear forms $\varphi_1, \cdots, \varphi_t$ in $\mathfrak{h}$ such that for any m points $Q_1, \cdots, Q_m$ there exists one i such that $\varphi_i \neq 0$ at any of Q_j's. Let (z) be the homogeneous co-ordinate of the generic point

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3) In this lemma, the assumtion that k contains infinitely many elements cannot be omitted.

4) The writer owes the present proof to Mr. H. Hironaka.

of V such that $\mathfrak{h}=k[z]$. Then the projective variety with the generic point (z, ψ) is a linear transform of V. Therefore we may assume that the affine representatives $A_0, \cdots, A_n$ of V defined by $z_0=1, \cdots, z_n=1$ $((z)=(z_0, \cdots, z_n))$ satisfies the condition that for any m points $Q_1, \cdots, Q_m$ of V, there exists one i such that A_i contains all of Q_j's. For each A_i, there exists an affine variety A'_i which is biregularly equivalent to A_i such that $A'_i(m)$ is normal by Proposition 1. Let $(y^{(i)})$ be a generic point of A'_i; we may regard that $y^{(i)}$'s are rational functions of $z_0/z_i, \cdots, z_n/z_i$. Then there exists a natural number N such that $z_i^N y_j^{(i)} \in \mathfrak{h}$ for all i and j. Let $m_0, \cdots, m_a$ $(m_i=z_i^N$ for $i \leq n)$ be a set of generators of the forms of degree N in $\mathfrak{h}$ and let V' be the projective variety with the generic point $(m)=(m_0, \cdots, m_a)$ (defined over k). We shall show that $V'(m)$ is normal. By the invariance of $V'(m)$ under linear transformations, we may assume that $z_i^N y_j^{(i)}$'s are contained among m_k's. Let V'_i be the affine representative of $V'(m)$ defined by $m_i=1$ and let $\mathfrak{o}_i$ be the co-ordinate ring of V'_i for each $i=0, \cdots, n$. Then $\mathfrak{o}_i$ contains the coordinate ring $\mathfrak{o}'_i$ of $A'_i(m)$ and therefore $\mathfrak{o}_i=\mathfrak{o}'_i$ because $\mathfrak{o}'_i$ is normal. Thus we see that V'_i is normal. By our construction, for any points $Q_1, \cdots, Q_m$ of V', there exists one $i(\leq n)$ such that the affine representative of V' defined by $m_i=1$ contains all of Q_j's. Therefore any point in $V'(m)$ has a representative in at least one V'_i. Thus we see that $V'(m)$ is normal.

§ 3. A preliminary on symmetric forms.

We will use the same notations as in § 1, (1). We assume that the field k is of characteristic zero. Then

LEMMA 4. Any symmetric form f on vectors $(x^{(i)})$'s over the field k (of characteristic zero) is expressible as a polynomial in the fundamental symmetric forms over k. (Weyl [11])

Proof. (1) $(x_1^{(1)} \cdots x_r^{(1)})_s$ is expressible as a polynomial in the fundamental symmetric forms $s_1, \cdots, s_t$ on $(x^{(i)})$'s:

Indeed: When $r=1$, our assertion is obvious because $(x_1^{(1)})_s = s_1^1$. Therefore we will prove our assertion by induction on r. We assume that, for any $i_1, \cdots, i_k$ $(k \geq 1)$, $(x_1^{(1)} \cdots \hat{x}_{i_1}^{(1)} \cdots \hat{x}_{i_k}^{(1)} \cdots x_r^{(1)})_s$[5] is expressible as a polynomial in s_j's. On the other hand, $(x_1^{(1)} \cdots x_r^{(1)})_s = 1/r \cdot (\sum_1^r s_i^1 (x_1^{(1)} \cdots \hat{x}_i^{(1)} \cdots x_r^{(1)})_s - \sum_{i,j,j\neq 1} (x_1^{(1)} \cdots \hat{x}_i^{(1)} \cdots x_r^{(1)} x_i^{(j)})_s$,

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5) The symbol $\wedge$ means that the indicated places are dropped. For example, $x_1 \cdots \hat{x}_i \cdots x_r$ denotes $x_1 \cdots x_{i-1} x_{i+1} \cdots x_r$.

$\sum_{i,j,j\neq 1}(x_1^{(1)}\cdots\hat{x}_i^{(1)}\cdots x_r^{(1)}x_i^{(j)})_s=1/r-1\cdot(\sum_{i,k}(x_1^{(1)}\cdots\hat{x}_i^{(1)}\cdots\hat{x}_k^{(1)}\cdots x_r^{(1)})_s\cdot s_{ik}^{j1}$ $-\sum_{i,k,i',k',i'\neq 1,k'\neq 1,i'\neq k'}(x_1^{(1)}\cdots\hat{x}_i^{(1)}\cdots\hat{x}_k^{(1)}\cdots x_r^{(1)}x_i^{(i')}x_k^{(k')})_s)$ (when $m=2$, this last sum vanishes), and so on. Thus we see that, repeating the similar as above, $(x_1^{(1)}\cdots x_r^{(1)})_s$ is expressible as a polynomial in s_j's.

(2) $(x_{i_1}^{(1)}\cdots x_{i_r}^{(1)})_s$ is expressible as a polynomial in s_j's.

Indeed: $(x_1^{(1)}\cdots x_r^{(1)})_s$ is expressible as a polynomial in s_j's (with respect to vectors $(x^{(i)})=(x_1^{(i)}\cdots,x_r^{(i)})$ $(i=1,\cdots,m)$). The expression is an identical relation and therefore the subscripts $1,\cdots,r$ may be replaced by $i_1,\cdots,i_r$, which proves (2).

(3) Now we prove the lemma. We may assume that f is of the form $f=(M)_s$ with a monomial $M=x_{i_1}^{(1)}\cdots x_{i_r}^{(1)}\cdot M^*$, where M^* is a monomial whose superscripts are greater than 1. We remark here that we can choose M so that $r\geq 1$ because $(M)_s=(\sigma(M))_s$ for any permution σ of superscripts. When $M^*=1$, we have already proved our assertion by (2) and we will prove our assertion by in induction on the degree of M^*. Let f^* be the symmetric form generated by M^*. Then $f^*=(x_{j_1}^{(1)}\cdots x_{j_t}^{(1)}\cdot M^{**})_s$ (with $t\geq 1$ and with a monomial M^{**} whose superscripts are greater than 1). Since M^{**} is of less degree than M^*, we see that f^* is expressible as a polynomial in s_j's by our induction assumption. Therefore $f^{**}=(x_{i_1}^{(1)}\cdots x_{i_r}^{(1)})_s\cdot f^*$ is expressible as a polynomial in s_j's. On the other hand, every term of $f-f^{**}$ is of the form $x_{i_1}^{(l)}\cdots x_{i_u}^{(l)}\cdot M'$ with $u>r$ (for any term of the form $x_{i_1}^{(l)}\cdots x_{i_r}^{(l)}\cdot M'$ (M' being a monomial whose superscripts are different from l), which appears in f or f^{**}, appears in both of them and therefore such term does not appear in $f-f^{**}$). Since M' is of less degree than M^*, we see that $f-f^{**}$ and therefore also f are expressible as a polynomial in s_j's. Thus the proof is completed.

§ 4. Normality of $V(m)$ (II).

Theorem 2. *Let V be an affine or a projective variety defined over a fied k. If k is of characteristic zero and if V is normal, then $V(m)$ is also a normal variety.*

Proof. (1) When V is an affine variety: We will make use of the same notations as in Lemma 2 and Proposition 1. If f is in the derived normal ring $\mathfrak{S}'$ of the co-ordinate ring $\mathfrak{S}$ of $V(m)$, then f is expressible as a symmetric form on $(x^{(i)})$'s. Since k is of characteristic zero, f is expressible as a polynomial in the fundamental symmentric forms on $(x^{(i)})$'s, hence f is in $\mathfrak{S}$. Thus $\mathfrak{S}'=\mathfrak{S}$

and $\mathfrak{s}$ is normal, which shows that $V(m)$ is a normal variety.

(2) When V is a projective variety: By the invariance of $V(m)$ under linear transformations (see § 1, (5)) and by Lemma 3, we may assume that for any m points of V, there exists one i such that the affine representative A_i of V defined by $z_i=1$ (where $(z_0, \cdots, z_n)$ is the homogeneous co-ordinate of a generic point of V over k) contains all of the m points (see the proof of Theorem 1). Then any point of $V(m)$ has a representative in one of $A_i(m)$. By (1), each $A_i(m)$ is a normal variety and therefore $V(m)$ is also a normal variety. Thus the proof is completed.

§ 5. Biregular invariance of $V(m)$.

Theorem 3. *Let V be an affine or a projective variety defined over a field k. If a variety V' is biregularly equivalent to V and if k is of characteristic zero, then $V(m)$ and $V'(m)$ are biregularly equivalent to each other.*

Proof. By the same way as in the proofs of Theorems 1 and 2, we can reduce the projective case to the affine case. Therefore we will treat the affine case. Let L be the function field of V, let $L(m)$ be the function field of $V(m)$ and let L^* be the function field of the m-ple product V^* of V. Then we see that $L(m)$ is a subfield of L^* and L^* is a normal extension of $L(m)$ with Galois group $\mathfrak{S}$ which is the symmetric group of degree m (which operates as permutations of independent generic points $(x^{(1)}), \cdots, (x^{(m)})$ of V over k).[6] Now we have only to prove the following.

Proposition 2. *The co-ordinate ring $\mathfrak{o}(m)$ of $V(m)$ is the intersection of $L(m)$ with the co-oridinate ring $\mathfrak{o}^*$ of V^*.*

Proof of the proposition. Obviously $\mathfrak{o}(m)$ is a subring of $\mathfrak{o}^*$. An element f of $\mathfrak{o}^*$ is in $L(m)$ if and only if it is invariant under any permutations of $\mathfrak{S}$. Then by Lemma 2, f is in $L(m)$ if and only if f is expressible as a symmetric form on $(x^{(i)})$'s; since k is of characteristic zero, it is equivalent to say that f is expressible as a polynomial in the fundamental symmetric forms on $(x^{(i)})$'s by Lemma 4, that is, f is in $\mathfrak{o}(m)$. Thus the proof is completed.

§ 6. Examples.

We first show an example which shows that 1) Lemma 4 does not hold in general, 2) Theorem 2 (and therefore also Theorem 3)

6) See Weil [10].

does not hold in general and 3) even when V is free from singularities, $V(m)$ may not normal.

Example 1. Let k be a field of characteristic 2 and let $(x^{(i)}) = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$ $(i=1, 2)$ be independent generic points of the affine space S^3 of dimension 3 over k. Then there exist homogeneous symmetric forms of degree 3 which are not expressible as polynomials in the fundamental symmetric forms on $(x^{(1)})$ and $(x^{(2)})$.

Indeed, fundamental symmetric forms are

$$s_i^1 = (x_i^{(1)})_s \quad (i=1, 2, 3)$$

$$s_{ii}^{12} = x_i^{(1)} x_i^{(2)} \quad (i=1, 2, 3)$$

$$s_{ij}^{12} = (x_i^{(1)} x_j^{(2)})_s \quad (i \neq j,\ i=1, 2, 3;\ j=1, 2, 3).$$

Therefore homogeneous symmetric forms of degree 3 which can be expressible as polynomials in s's such that each of their terms has all subscripts 1, 2, 3 are generated by

$$s_1^1 s_2^1 s_3^1,\ s_1^1 s_{23}^{12},\ s_2^1 s_{31}^{12},\ s_3^1 s_{12}^{12}. \tag{1}$$

On the other hand, homogeneous symmetric forms of degree 3, such that each of their terms has all subscripts 1, 2, 3, are generated by

$$\begin{aligned} &\varepsilon_1 = (x_1^{(1)} x_2^{(1)} x_3^{(1)})_s,\ \varepsilon_2 = (x_1^{(1)} x_2^{(1)} x_3^{(2)})_s, \\ &\varepsilon_3 = (x_1^{(1)} x_2^{(2)} x_3^{(1)})_s,\ \varepsilon_4 = (x_1^{(1)} x_2^{(2)} x_3^{(2)})_s. \end{aligned} \tag{2}$$

If we express each of forms in (1) as the linear combination of ε_i's, then the matrix of the coefficients is of the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and its determinant is zero, because k is of characteristic 2. Therefore the forms in (1) cannot generates all of forms in (2). Thus the proof is completed.

REMARK 1. It will be not hard to construct similar examples for any field of non-zero characteristic.

Next we will show an example which shows that even when V is an affine or a projective variety defined over a field of characteristic zero and when V is free from singularities, $V(m)$ may have singularities.

Example 2. Let k be a field of characteristic zero and let $(x^{(i)})=(x_1^{(i)}, x_2^{(i)})$ $(i=1, 2)$ be independent generic points of the affine space S^2 of dimension 2. Then the co-ordinate ring $\mathfrak{o}(2)$ of $S^2(2)$ is the ring $k[s_1^1, s_2^1, s_{11}^{12}, s_{12}^{12}, s_{22}^{12}]$. We denote by $\mathfrak{o}$ the ring $k[x^{(1)}, x^{(2)}]$. Let $\mathfrak{p}$ and $\mathfrak{p}'$ be maximal ideals of $\mathfrak{o}$ and $\mathfrak{o}(2)$ generated by $x^{(i)}$'s and s's respectively (then $\mathfrak{p}'=\mathfrak{p}\cap\mathfrak{o}(2)$). Set $P=\mathfrak{o}_{\mathfrak{p}}$ and $P'=\mathfrak{o}(2)_{\mathfrak{p}'}$. Then $(\mathfrak{p}'P+\mathfrak{p}^2P)/\mathfrak{p}^2P$ is of rank 2 over k, because $\mathfrak{p}'$ contains no linear form in P other than those generated by s_1^1 and s_2^1. Therefore, for any system of parameters (y) of P', the multiplicity[7] of the ideal of P generated by (y) is at least 4. Since $\mathfrak{o}$ is a normal extension of degree 2 over $\mathfrak{o}(2)$, we see that the multiplicity of the local ring P' is at least 2, hence P' is not a regular local ring.

REMARK 2. Let $Q_1, \cdots, Q_m$ be simple points of an affine or a projective variety V defined over a field k of characteristic zero. Let Q^* be the point of $V(m)$ which corresponds to the cycle $Q_1+\cdots+Q_m$. If $Q_i\neq Q_j$ for any $i\neq j$, then Q^* is a simple point.

The proof is easy by Proposition 2, observing that the maximal ideal of the specialization ring $\mathfrak{o}^*$ of Q^* decomposes completely in the integral closure of $\mathfrak{o}^*$ in the function field of the m-ple product of V.

REMARK 3. Similar fact as in Remark 2 holds good even when the ground field is not of characteristic zero, under the assumption that point Q^* is normal.

The proof is quite similar as that of Remark 2.

§ 7. Arithmetic normality of $V(m)$.

LEMMA 5. Let V and V' be arithmetically normal varieties defined over the same field k. Then the projective product variety V'' of V and V' is also an arithmetically normal variety (over k). (Nakai)

Proof. Let $(z)=(z_0, \cdots, z_m)$ and $(w)=(w_0, \cdots, w_n)$ be independent generic points of V and V' respectively (over k); here we assume that z_0 and w_0 are algebraically independent over the function field of V''. Set $\mathfrak{o}=k[z, w]$. Then $\mathfrak{o}$ is a normal ring by Lemma 1. In $\mathfrak{o}$, we can consider homogeneity and degree of homogeneity with respect to (z) and (w). Then the homogeneous co-ordinate ring $\mathfrak{h}$ of V'' is the subring of $\mathfrak{o}$ generated by all elements of $\mathfrak{o}$ which are homogeneous with respect to both (z) and (w) and

7) For the notion of multiplicity, see [4] or [5, V] (cf. [1] or [6]).

whose degrees of homogeneity with respect to (z) and (w) are the same. Let f be any element of the derived normal ring $\mathfrak{h}'$ of $\mathfrak{h}$. Then f is expressible as the sum of homogeneous elements in $\mathfrak{h}'$. Therefore in order to show that f is in $\mathfrak{h}$, we may assume that f is a homogeneous element in $\mathfrak{h}'$. Since $\mathfrak{o}$ is a normal ring, f is in $\mathfrak{o}$. Since f is in $\mathfrak{h}'$, the degrees of homogeneity of f with respect to (z) and (w) (in $\mathfrak{o}$) are the same. Thus we see that f is in $\mathfrak{h}$ and $\mathfrak{h}$ is normal, which shows that V'' is arithmetically normal.

Proposition 3. *Let k be a field of characteristic zero. Let n and m be given integers and let $\boldsymbol{P}$ be the projective space of dimension n over k. If $\boldsymbol{P}(m)$ is arithmetically normal, then for any arithmetically normal variety V in $\boldsymbol{P}$ defined over k, $V(m)$ is arithmetically normal.*

In order to prove our proposition, we will introduce the notion of homogeneously symmetric forms. Let $(z^{(i)}) = (z_0^{(i)}, \cdots, z_n^{(i)})$ $(i = 1, \cdots, m)$ be vectors with indeterminates $z_j^{(i)}$'s and we denote by $\mathfrak{o}$ the ring $k[z^{(1)}, \cdots, z^{(m)}]$. An element f of $\mathfrak{o}$ is called a homogeneously symmetric form on the vectors $(z^{(i)})$ if 1) f is a symmetric form on the vectors, 2) f is a homogeneous form (in $\mathfrak{o}$) and 3) f is homogeneous on each $(z_j^{(1)}, z_j^{(2)}, \cdots, z_j^{(m)})$ and the degree is independent of the subscript j; this is called the degree of f.

If we regard the vectors $(z^{(i)})$'s as independent generic points of $\boldsymbol{P}$ over k, then obviously the co-ordinate ring $\mathfrak{h}(m)$ of $\boldsymbol{P}(m)$ is generated by homogeneously symmetric forms of degree one. Therefore that $\mathfrak{h}(m)$ is normal means that any homogeneously symmetric form is expressible as a polynomial in those of degree one. This fact can be applied for any arithmetically normal varieties in $\boldsymbol{P}$ defined over k, by the same way as in § 4 by virtue of Lemma 5. Thus the proof is completed.

REMARK. In the above proposition, we need not assume that the field k is of characteristic zero. But, if k is of characteristic $p \neq 0$, then $\boldsymbol{P}(m)$ is not arithmetically normal for any m which is not less than p.

Here we will offer a problem to decide whether $\boldsymbol{P}(m)$, as in the above proposition, is necessarily arithmetically normal or not. (It seems to the writer very likely that the answer is affirmative.)

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