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On the arithmetic normality of hyperplane sections of algebraic varieties

By

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It is well known that the general hyperplane sections of the everywhere locally normal varieties are also locally normal (A. Seidenberg (3)). But when the variety is arithmetically normal, we cannot expect, in general, that the hyperplane section posseses also this property. We shall give here a necessary and sufficient condition for a variety such that whose general hyperplane section is arithmetically normal. This is one of the simple applications of the theory of stacks on algebraic varieties, developed recently by J. P. Serre (4).

1. Let V be a variety imbedded in a projective space, defined over an algebraically closed field k.[1] We shall say that V is arithmetically normal, if the homogeneous coordinate ring of V over k is integrally closed in its quotient field. Let us denote by $L_n(V)$ the linear system on V, which are composed of all the sections of V with the hypersurfaces of order n in the ambient projective space. Then we know the following:[2] Suppose that V has no singular subvarieties of dimension (dim $V-1$), then V is arithmetically normal if and only if the linear system $L_n(V)$ is complete for any integer n. Let $\mathfrak{P}$ be the homogeneous prime ideal of V, H a hyperplane in the projective space corresponding to the linean form l. and put $C=V\cdot H$. We shall say that C is a general hyperplane section of V if C has the following properties:

(1) C is an irreducible variety defined over k.

(2) C has no divisoriel singularity.

1) The terminologies used in this papar are due to A. Weil (5).

2) Cf. Muhly (2) or Zariski (6). The results of Muhly are more general than the one stated here, but this form is sufficient for our purposes.

(3) $\mathfrak{P}:(l)=\mathfrak{P}$

We shall denote as usual by $\chi(V, m)$ the number of linearly independent forms of degree m mod $\mathfrak{P}$. Then we have for a general hyperplane section C, the equality

$$\chi(V, m)=\chi(V, m-1)+\chi(C, m)$$

known as the modular property of $\chi(V, m)$.[3]

Let o be the stack of local rings of V, and n a non-negative integer. Then we have $o(n)\cong\mathfrak{L}(C_n)$, where $\mathfrak{L}(C_n)$ denotes the stack of germs of rational functions on V such that $(f)+C_n>0$, and the q-th cohomology group of $o(n)$ vanishes if n is sufficiently large. We shall denote by $h^q(o(n))$ the dimention of the vector space $H^q(V, o(n))\cong H^q(V, \mathfrak{L}(C_n))$ over k. Under these preparations we can state and prove the theorem.

2. THOREM 1. *Let V^r be a projective variety defined over an algebraically closed field k, having no singular subvriety of dimension $r-1$. Then the general hyperplane section C of V is arithmetically normal if and only if* i) *V is arithmetically normal;* ii) $h^1(o(n))=0$, *for $n\geqq 0$. Moreover in this case, the general hypersurface section C_m of V is also arithmetically normal for any integer $m>0$.*

PROOF. Let C_n be an arbitrary member of $L_n(V)$ which has not C as a component. Then we have the exact sequence of stacks

$$0\to\mathfrak{L}(C_n-C)\xrightarrow{i}\mathfrak{L}(C_n)\xrightarrow{r}\underset{C}{\mathfrak{L}}(C\cdot C_n)\to 0$$

where i is the injection and r the restriction map on C. Then we have the associated exact sexuence of cohomology groups

$$\text{(a)}\quad \begin{aligned}&0\to H^0(V, \mathfrak{L}(C_n-C))\to H^0(V, \mathfrak{L}(C_n))\xrightarrow{r_0^*}H^0(C, \underset{C}{\mathfrak{L}}(C\cdot C_n))\\ &\to H^1(V, \mathfrak{L}(C_n-C))\xrightarrow{i_1^*}H^1(V, \mathfrak{L}(C_n))\to\cdots\end{aligned}$$

As is easily be seen $H^0(C, \underset{C}{\mathfrak{L}}(C\cdot C_n))$ is isomorphic to the defining module of the linear system $|L_n(C)|$ on C.

Suppose that C is arithmetically normal, then $L_n(C)$ is complete for $n\geqq 1$. Hence the homomorphism r_0^* is an onto mapping. From this we see that i_1^* is an into isomorphism and $h^1(o(n-1))\leqq h^1(o(n))$ for $n\geqq 1$. On the other hand we know that $h^1(o(n))=0$

3) Cf. Krull (1)

4) Concerning the definitions and the results in the theory of stacks we refer to J. P. Lerre (4).

for sufficiently large n, and $h^1(o) \geqq 0$. It follows immediately that $h^1(o(n))=0$ for $n \geqq 1$. This proves the second assertion.

Returning back to the exact sequence (a), we have

$$\dim_k H^0(V, \mathfrak{L}(C_n)) = \dim_k H^0(V, \mathfrak{L}(C_n - C)) + \dim_k H^0(C, \underset{C}{\mathfrak{L}}(C \cdot C_n)),$$

Using the induction on the number n, suppose that $L_{n-1}(V)$ is complete. Then we have

$$\dim_k H^0(V, \mathfrak{L}(C_n - C)) = \dim|L_{n-1}(V)| + 1 = \chi(V, n-1)$$

and by definition

$$\dim_k H^0(V, \mathfrak{L}(C_n)) = \dim|L_n(V)| + 1 \geqq \chi(V, n)$$

The arithmetical normality of C implies

$$\dim_k H^0(C, \underset{C}{\mathfrak{L}}(C \cdot C_n)) = \dim|L_n(C)| + 1 = \chi(C, n-1)$$

Combining these with the modular property of $\chi(V, m)$ we have immediately

$$\dim_k H^0(V, \mathfrak{L}(C_n)) = \chi(V, n)$$

This proves that the linear system $L_n(V)$ is complete.

Next we shall show the last assertion. As we can see easily, we have, in general, the inequalities $h^1(o(-n-1)) \leqq h^1(o(-n))$ for $n \geqq 0$. In our case we have $h^1(o)=0$, hence $h^1(o(-n))=0$ for any $n \geqq 0$. Now the arithmetical normality of general hypersurface sections of V can be proved easily by the similar argument as above, using the exact sequence of stacks

$$0 \to \mathfrak{L}(C_n - C_m) \to \mathfrak{L}(C_n) \to \underset{C_m}{\mathfrak{L}}(C_m \cdot C_n) \to 0$$

Thus the theorem is proved completely. q. e. d.

Corollary. *Let V be an arithmetically normal variety such that $h^1(o)=0$. Then there exists a biregulary birationally corresponding variety V' such that the general hyperplane section of V' is also arithmentically normal.*

Proof. Let n be an integer such that $h^1(o(m))=0$ for $m \geqq n$. Let V' be the transform of V by the Veronese transformation of order n. Then V' satisfy the condition of the Corollary.

3. We shall now study the special class of varieties which satisfy the condition $h^1(o(n))=0$ for any integer. The result is the following.

Proposition. *Let V^r be a variety such that V is the complete* *intersection in the projective space and has no singular subvariety of dimension* $r-1$. *Then* V *has the following properties:*

(b) $h^q(o(n))=0$ for $1\leqq q<r$,

where n *is an arbitrary integer.*

This is already proved in J. P. Serre (4). But the following proof is very elementary and we shall write down. To prove this we shall need two lemmas.

LEMMA 1. *Let* V^r *be a variety without singular subvarieties of dimension* $r-1$ *satisfying the condition* (b). *Then the general hyperplane section* C_m *of* V *satisfies also the condition* (b).

PROOF. Let C_n be any member of $L_n(V)$, which has not C_m as a component. Let

$$0\to\mathfrak{L}(\pm C_n-C_m)\to\mathfrak{L}(\pm C_n)\to\mathfrak{L}(\pm C_n\cdot C_m)\to 0$$

be the exact sequence of stacks. Then we have

$$\cdots\to H^q(V, \mathfrak{L}(\pm C_n)\to H^q(C, \mathfrak{L}(\pm C_n\cdot C_m))$$
$$\to H^{q+1}(V, \mathfrak{L}(\pm C_n-C_m))\to\cdots$$

By hypothesis $H^q(V, o(s))\cong 0$ for any s, if $1\leqq q<r$. Then we have $H^q(C_m, \mathfrak{L}(\pm C_n\cdot C_m))\cong 0$ for $1\leqq q<r-1$ and for any $n\geqq 0$. q. e. d.

LEMMA 2. *The projective space* P^N *has the property* (b).

PROOF. We shall use the induction on the dimesion N. It is trivially true for $N=1$. Let us denote by C the hyperplane in P. Then as before we have the exact sequence of cohomology groups

$$\cdots\to H^{q-1}(C, o'(n))\to H^q(P, o(n-1))\to H^q(P, o(n))\to\cdots$$

Since C can be considered as a projective space of dimension $N-1$, we can apply the induction assumption on C. Then $H^{q-1}(C, o'(n))\cong 0$ if $1\leqq q-1< N-1$. Hence we have $h^q(o(n-1))\leqq h^q(o(n))$ for $2\leqq q<N$. But since $h^q(o(n))=0$ for sufficiently large n we have immediately $h^q(o(n))=0$, for $2\leqq q<N$. Concernining $h^1(o(n))=0$, it is an immediate consequence of Theorem 1. q. e. d.

Proposition is the direct consequence of the above lemmas. Now we have the

THEOREM 2.[5] *Let* V^r *be a complete intersection variety in a projective space, without singular subvarieties of dimension* $r-1$. *Then* V *is arithmetically normal.*

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5) This result is stated implicitely in Serre (4).

Bibliography

(1) Krull, W. Idealtheorie, Ergeb. der Math., Vol. 4, No. 3 (1935).

(2) Muhly, T. H. A remark on normal varieties. Ann. of Math. 42 (1941).

(3) Seidenberg, A. The hyperplane sections of normal varieties. Trans. of Amer. Math. Soc. Vol. 69 (1950).

(4) Serre, J. P. Faisceaux cohérént algébriques. will appear in Ann. of Math.

(5) Weil, A. Foundations of algebraic geometry. Amer. Math. Colloq. pub. Vol. XXIX (1946).

(6) Zariski, O. Complete linear systems on normal varieties and a generalization of a lemma of Enriques-Severi. Ann. of Math. Vol. 55 (1952).