MEMOIRS OF THE COLLEGE OF SCIECE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXIX, Mathemamatics No. 2, 1955.

Tangential vector bundle and Todd canonical systems of an algebraic variety

By

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(Received Dec. 20, 1954)

In a previous paper [2], the author considered complex analytic vector bundles over a non-singular algebraic variety immersed in a projective space, and proved that the Chern (homology) classes of these bundles contain algebraic cycles. Now we can get in general the

THEOREM *For a non-singular algebraic variety in a projective space, the Chern classes of the tangential vector bundle coincide with Todd canonical systems.*

This theorem was proved by W. V. D. Hodge [1] for a non-singular variety which is a complete intersection of hypersurfaces, but the proof for arbitrary non-singular varieties seems not to have been published.[1)]

We shall make use of notations in [2].

1. Let V^r be a non-singular algebraic variety in a projective space L^N of complex dimension N. To every point P of V, we associate the tangential linear variety $T(P)$, considered as a point of the Grassman variety $H(r+1, N+1)$. Then we have an everywhere regular rational mapping Φ from V into $H=H(r+1, N+1)$.

In H, there are $r+1$ subvarieties $\Omega_{(p)}$ $(p=1, \cdots, r+1)$ which generate, together with $\Omega_{(0)}=H$, the homology ring of H. $\Omega_{(p)}$ represent the Chern classes of the universal bundle over H. We denote by X a generic hyperplane section of V, and by X^h the intersection product of h independent X's.

By Todd canonical systems, we understand the cycles

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1) I learned from J. Igusa that K. Kodaira and J. P. Serre have this result already, but I would like to complete my paper [2] by this note.

$$(1)\qquad t_p(V)=\sum_{h=0}^{p}(-1)^h\binom{r-p+1+h}{r-p+1}\Phi^{-1}(\Omega_{(p-h)})\cdot X^h \qquad (p=0,\cdots,r),$$

where $\cdot$ denotes the intersection product on V. $t_p(V)$ are not defined uniquely, but the arbitrariness lies in that we may replace $\Omega_{(p)}$ and X^h by linearly equivalent ones. (Here we say that two cycles are linearly equivalent on a variety, if they belong to an algebraic system on that variety, and if the parameter variety of the system is a rational variety.) Hence $t_p(V)$ are well defined as homology classes. We shall also define $t_{r+1}(V)$ to be equal to 0.

2. Consider $\Phi^{-1}(\mathfrak{R})$, the bundle induced on V, by the universal bundle $\mathfrak{R}$ over H. Then $c_p'=\Phi^{-1}(\Omega_{(p)})$ is the p-th Chern class of $\Phi^{-1}(\mathfrak{R})$, and therefore

$$F(\lambda)=\lambda^{r+1}+c_1'\lambda^r+\cdots+c'_{r+1}$$

is the characteristic polynomial of $\Phi^{-1}(\mathfrak{R})$.

It is easy to see that

$$(2)\qquad F(\lambda-X)=\lambda^{r+1}+t_1\lambda^r+\cdots+t_{r+1}$$

where $t_p=t_p(V)$. This suggests that $\Phi^{-1}(\mathfrak{R})$ will be a $\otimes$-product of a complex line bundle $\mathfrak{B}=\{-X\}$ and a vector bundle whose characteristic classes are t_p's. (See [2], the end of § 1.)

3. We shall now seek for a system of transition functions of $\Phi^{-1}(\mathfrak{R})$. Let $(\xi_0,\cdots,\xi_N)$ be homogeneous coordinate functions on V, and let P be a point on V such that $\xi_{i_0}(P)\neq 0$ and $(x_{i_1},\cdots,x_{i_r})$ form a system of local parameters at P. (Here we set $x_\lambda=\xi_\lambda/\xi_{i_0}$.) Then for a generic point $(z_0,\cdots,z_N)$ of the tangent linear variety $T(P)$, we have

$$z_\lambda/z_{i_0}-x_\lambda(P)=\sum_{\alpha=1}^{r}(\partial x_\lambda/\partial x_{i_\alpha})_P(z_{i_\alpha}/z_{i_0}-x_{i_\alpha}(P)),$$

or

$$z_\lambda\xi_{i_0}-z_{i_0}\xi_\lambda=\sum_{\alpha=1}^{r}(\partial x_\lambda/\partial x_{i_\alpha})_P(z_{i_\alpha}\xi_{i_0}-z_{i_0}\xi_{i_\alpha}).$$

Hence the homogeneous conrdinates $(z_0,\cdots,z_N)$, which are now considered as a vector in the fiber over P, in the fiber bundle $\Phi^{-1}(\mathfrak{R})$, are determined by $r+1$ components $z_{i_0},\cdots,z_{i_r}$ among them.

If another set of indices $j_0,\cdots,j_r$ are such that $\xi_{j_0}(P)\neq 0$ and $(y_{j_1},\cdots,y_{j_r})$ with $y_\lambda=\xi_\lambda/\xi_{j_0}$ form a system of local parameters at P, then the vector $(z_0,\cdots,z_N)$ can also be determined by $(z_{j_0},\cdots,z_{j_r})$

and we have the following relation between $(z_{i_0}, \cdots, z_{i_r})$ and $(z_{j_0}, \cdots, z_{j_r})$;

$$(z_{j_\alpha}\xi_{i_0}-z_{i_0}\xi_{j_\alpha})=\sum_\beta (\partial x_{j_\alpha}/\partial x_{i_\beta})_P(z_{i_\beta}\xi_{i_0}-z_{i_0}\xi_{i_\beta}),$$

or

$$(3) \quad \begin{pmatrix} z_{j_0} \\ \vdots \\ z_{j_r} \end{pmatrix} = \begin{pmatrix} x_{j_0}-\sum_\beta\left(\dfrac{\partial x_{j_0}}{\partial x_{i_\beta}}\right)x_{i_\beta}, & \dfrac{\partial x_{j_0}}{\partial x_{i_1}}, & \cdots, & \dfrac{\partial x_{j_0}}{\partial x_{i_r}} \\ \cdots\cdots & & & \\ x_{j_r}-\sum_\beta\left(\dfrac{\partial x_{j_r}}{\partial x_{i_\beta}}\right)x_{i_\beta}, & \dfrac{\partial x_{j_r}}{\partial x_{i_1}}, & \cdots, & \dfrac{\partial x_{j_r}}{\partial x_{i_r}} \end{pmatrix} \begin{pmatrix} z_{i_0} \\ \vdots \\ z_{i_r} \end{pmatrix}.$$

Hence if we put

$$(4) \quad g_{(j_0,\cdots,j_r)(i_1,\cdots,i_r)}=\text{the matrix in (3)},$$

then $\{g_{(j)(i)}\}$ form a system of transition functions of $\Phi^{-1}(\mathfrak{N})$. Here $\Phi^{-1}(\mathfrak{N})$ is considered to be defined with respect to the open covering $\{U_{(i)}\}$, where $U_{(i)}=U_{i_0,\cdots,i_r}$ is the set of points $P\in V$ such that $\xi_{i_0}(P)\neq 0$ and $x_{i_0},\cdots, x_{i_r}$ form a system of local parameters at P.

Put

$$h_{(i)}=\begin{pmatrix} 1 & 0 \cdots 0 \\ x_{i_1} & \\ \vdots & \boxed{I_r} \\ x_{i_r} & \end{pmatrix},$$

then $h_{(i)}$ is holomorphic and invertible on $U_{(i)}$, and we have

$$h_{(j)}^{-1}g_{(j)(i)}h_{(i)}=\begin{pmatrix} \xi_{j_0}/\xi_{i_0} & \dfrac{\partial x_{j_0}}{\partial x_{i_1}} \cdots\cdots \dfrac{\partial x_{j_0}}{\partial x_{i_r}} \\ 0 & \\ \vdots & \boxed{\dfrac{\partial x_{j_\alpha}}{\partial x_{i_\beta}}-\dfrac{\partial x_{j_0}}{\partial x_{i_\beta}}y_{j_\alpha}} \\ 0 & \end{pmatrix}$$

$$=(\xi_{j_0}/\xi_{i_0})\otimes\begin{pmatrix} 1 & \dfrac{1}{x_{j_0}}\dfrac{\partial x_{j_0}}{\partial x_{i_1}} \cdots\cdots \dfrac{1}{x_{j_0}}\dfrac{\partial x_{j_0}}{\partial x_{i_r}} \\ 0 & \\ \vdots & \boxed{\dfrac{\partial y_{j_\alpha}}{\partial x_{i_\beta}}} \\ 0 & \end{pmatrix}.$$

4. The system $f_{(j)(i)}=\xi_{j_0}/\xi_{i_0}$ defines the complex line bundle $\mathfrak{B}=\{-X\}$, and the system

$$
(5) \qquad g'_{(j)(i)} = \begin{pmatrix} 1 & \frac{1}{x_{j_0}}\frac{\partial x_{j_0}}{\partial x_{i_1}} \cdots\cdots \frac{1}{x_{j_0}}\frac{\partial x_{j_0}}{\partial x_{i_r}} \\ 0 & \\ \vdots & \boxed{\frac{\partial y_{j_\alpha}}{\partial x_{i_\beta}}} \\ 0 & \end{pmatrix}
$$

defines a vector bundle which is topologically equivalent to the Whitney product of a trivial complex line bundle $\mathfrak{N}$ and the tangential vector bundle $\mathfrak{T}$ over V:

$$
\Phi^{-1}(\mathfrak{R}) = \mathfrak{B} \otimes (\mathfrak{N} \dot{+} \mathfrak{T}).
$$

For characteristic polynomials, we have

$$
(6) \qquad \begin{cases} G(\lambda + X) = \lambda^{r+1} + c_1'\lambda^r + \cdots + c'_{r+1} \\ G(\lambda) = \lambda(\lambda^r + c_1\lambda^{r-1} + \cdots + c_r), \end{cases}
$$

where c_p is the p-th characteristic class of the tangential bundle. Compared with (2) we have $c_p = t_p$, which proves our theorem announced.

5. If we stand on an analytical point of view instead of topological one, the bundle defined by (5) is not a Whitney product.

Consider the system $g''_{(j)(i)} = {}^t(g'_{(j)(i)})^{-1}$, then just as in [3], we can associate to it an element of $H^1(V;\ \Omega({}^t\mathfrak{T}^{-1}))$, where ${}^t\mathfrak{T}^{-1}$ is the vector bundle defined by the transposed inverses of transition functions of $\mathfrak{T}$, and $\Omega({}^t\mathfrak{T}^{-1})$ denotes the sheaf of germs of holomorphic cross sections of ${}^t\mathfrak{T}^{-1}$.

As we have indicated, $f'_{(i)(j)} = \xi_{j_0}/\xi_{i_0} = x_{j_0}$ may be interpreted as transition functions for $\{X\}$. We shall write $\lambda, \mu, \cdots$ instead of $(i), (j), \cdots$ as indices for neighborhoods, and $x_\lambda^1, \cdots, x_\lambda^r$ instead of $x_{i_1}, \cdots, x_{i_r}$. Then after re-ordering rows and columns of matrices, $g''_{(j)(i)}$ are rewritten as

$$
g''_{\lambda\mu} = \begin{pmatrix} h_{\lambda\mu} & b_{\lambda\mu} \\ 0 & 1 \end{pmatrix},
$$

with

$$
h_{\lambda\mu} = \left(\frac{\partial x^\alpha_\mu}{\partial x^\beta_\lambda}\right), \quad b_{\lambda\mu} = \begin{pmatrix} -\frac{\partial}{\partial x^1_\mu}(\log f'_{\lambda\mu}) \\ \vdots \\ -\frac{\partial}{\partial x_\mu^r}(\log f'_{\lambda\mu}) \end{pmatrix}.
$$

We put $\eta^{(\nu)}_{\lambda\mu} = h_{\nu\lambda} b_{\lambda\mu}$, then $\eta^{(\nu)}_{\lambda\mu} = h_{\nu\rho}\eta^{(\rho)}_{\lambda\mu}$ in $U_\lambda \cap U_\mu \cap U_\nu \cap U_\rho$, and

$\eta_{\lambda\mu}=\{\eta_{\lambda\mu}^{(\nu)}\}$ defines a holomorphic cross section of ${}'\mathfrak{F}^{-1}$ on $U_\lambda \cap U_\mu$.

In $U_\kappa \cap U_\lambda \cap U_\mu$, we have

$$\eta_{\kappa\lambda}+\eta_{\lambda\mu}+\eta_{\mu\kappa}=0$$

and $(\eta_{\lambda\mu})$ defines a 1-cocycle of the nerve of the covering $\{U_\lambda\}$, with coefficients in $\Omega({}'\mathfrak{F}^{-1})$. This cocycle determines the cohomology class in question. (This was indicated by Y. Kawada [4], the note [3] contains this only implicitly.)

Now there is a canonical isomorphism between the sheaves $\Omega({}'\mathfrak{F}^{-1})$ and Ω^1 (the sheaf of the germs of holomorphic 1-forms on V), which is defined by

$$\Omega({}'\mathfrak{F}^{-1}) \ni \eta=\{\eta^{(\nu)}\} \longleftrightarrow \eta_1^{(\nu)}dx_\nu^1+\cdots+\eta_r^{(\nu)}dx_\nu^r=\omega\in\Omega^1.$$

Hence we have

$$H^1(V;\, \Omega({}'\mathfrak{F}^{-1}))\cong H^1(V;\, \Omega^1)\cong H^{1,1}(V,\, C),$$

the second isomorphism being that of Dolbeault.

It is easily seen that our cohomology class corresponds to the homology class of X by this isomorphism.

This describes the deviation of (5) from the Whitney product $\mathfrak{N}\dot{+}\mathfrak{F}$ in analytical sense.

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