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On transformations of differential equations whose second members are discontinuous

By

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In the previous paper[1] the transformations of the ordinary differential equations, whose second members are continuous, have been investigated. In the present paper, however, we will treat the equations whose second members are discontinuous.

Now consider a system

$$\frac{dy}{dx}=f(x, y) \tag{1}$$

where $f(x, y)$ is defined in a region

$$E_{n+1}\ :\ 0\leqq x\leqq a, \quad |y|<+\infty^{2)},$$

measurable with respect to x and continuous with respect to y and, for any $t(\geqq 0)$, $\max_{|y|\leqq t}|f(x, y)|$ is a summable function of x in the interval

$$I\ :\ 0\leqq x\leqq a.$$

(The case, where $f(x, y)$ is continuous, is a special case of the above.)

Since $f(x, y)$ is measurable with respect to x and continuous with respect to y, we can choose such a sequence of closed subsets $e_1, e_2, \dots$ of I, that $\lim_{m\to+\infty} m(e_m)=a$ and $f(x, y)$ is continuous in every region $[x\epsilon e_m, |y|<+\infty]$[3]. It is clear that we can suppose that

1) K. Hayashi, "*On transformations of differential equations*", Mem. Coll. Sci. Univ. Kyoto, A 28 (1953), pp. 315-325.

2) $f=(f_1, f_2, \dots, f_n)$, $y=(y_1, y_2, \dots, y_n)$, $|y|=\sqrt{y_1^2+y_2^2+\dots+y_n^2}$ etc..

3) G. Scorza Dragoni, "*Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile*", Rend. Padova. Vol. 17 (1948), pp. 102-106; "*Una applicazione della quasicontinuità semiregolare della funzioni misurabili rispetto ad una e continue rispetto ad un'altra variabili*", Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. (8) 12 (1952), p. 57.

$$e_1 \subset e_2 \subset \ldots.$$

and

$$\int_{I-e_m} \max_{|y|\leqq m} |f(x, y)| dx \leqq \frac{1}{2^m} \qquad (m=1, 2, \ldots.).$$

(When $f(y, y)$ is continuous we may take I for every e_m.)

Now put

$$f_1(s) = \max_{\substack{x\in e_m \\ |y|\leqq m}} |f(x, y)| \text{ for } m-1 \leqq s < m \qquad (m=1, 2, \ldots.)$$

then we have

$$f_1(|y|) = \max_{\substack{x\in e_m \\ |y'|\leqq m}} |f(x, y')| \geqq |f(x, y)|$$

when $x \in e_m$ and $m-1 \leqq |y| < m$. In general, if $m-1 \leqq |y|$, there exists a positive integer j such that $m+j-1 \leqq |y| < m+j$ and therefore we have

$$|f(x, y)| \leqq f_1(|y|) \text{ for } x \in e_{m+j}.$$

Since $e_m \subset e_{m+j}$, whenever $x \in e_m$ and $m-1 \leqq |y|$ we have

$$|f(x, y)| \leqq f_1(|y|).$$

1. Transformations of (1)

Let $f(t)$ be the greatest value of 1, t and $\int_t^{t+1} f_1(s) ds$, then $f(t)$ is a positive continuous function of t, not less than unity, in $0 \leqq t < +\infty$. Since $f_1(t) \leqq f(t)$, whenever $x \in e_m$ and $m-1 \leqq |y|$, we obtain

$$(2) \qquad |f(x, y)| \leqq f(|y|).$$

Now for a given positive constant σ, consider as in the previous paper[4] the function defined by the relation

$$\frac{1}{\{\lambda(r)\}^\sigma} = \int_r^{r+1} \frac{dt}{\{f(t)\}^2},$$

then $\lambda(r)$ and its derivative $\lambda'(r)$ are continuous functions of r in $0 \leqq r < +\infty$ where $\lambda(r) \geqq 1$, $\lambda'(r) \geqq 0$ and $\lim_{r\to+\infty} \lambda(r) = +\infty$. Next put

$$\rho(r) = r\lambda(r),$$

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4) K. Hayashi, *loc. cit.*.

there exists the inverse function of $\rho(r)$, written $r(\rho)$. $\rho(r)$ and $\rho'(r)$ are continuous functions of r in $0 \leqq r < +\infty$ such that $\rho(0) = 0$, $\rho(r) > 0$ for $r > 0$, $\lim_{r \to +\infty} \rho(r) = +\infty$ and $\rho'(r) \geqq 1$. $r(\rho)$ and $r'(\rho)$ are also continuous functions of ρ in $0 \leqq \rho < +\infty$ such that $r(0) = 0$, $r(\rho) > 0$ for $\rho > 0$, $\lim_{\rho \to +\infty} r(\rho) = +\infty$ and $r'(\rho) > 0$.

Now consider a mapping from y-space onto η-space, effected by

$$\eta = \frac{y}{|y|} \rho(|y|) \equiv y\lambda(|y|) \tag{3}$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$. Since $|\eta| = \rho(|y|)$, (3) gives immediately the inverse

$$y = \frac{\eta}{|\eta|} r(|\eta|). \tag{4}$$

Thus (3) maps topologically the whole y-space onto the whole η-space.

By (3), x being unchanged, (1) will be transformed to the system

$$\frac{d\eta}{dx} = \lambda(|y|) f(x, y) + \frac{(yf)\lambda'(|y|)}{|y|} y,$$

where $(yf) = y_1 f_1 + y_2 f_2 + \cdots + y_n f_n$. Consider its second member as a function of (x, η), written $g(x, \eta) = (g_1, g_2, \ldots, g_n)$, then we obtain a system, where the unknown is η:

$$\frac{d\eta}{dx} = g(x, \eta) \tag{5}$$

where $g(x, \eta)$ is measurable with respect to x and continuous with respect to η in the region $[0 \leqq x \leqq a,\ |\eta| < +\infty]$ and, for any $t (\geqq 0)$, $\max_{|\eta| \leqq t} |g(x, \eta)|$ is a summable function of x in I. Since

$$|g(x, \eta)| \leqq |f(x, y)| \{\lambda(|y|) + |y|\lambda'(|y|)\},$$

$$f(r) \leqq \{\lambda(r)\}^{\frac{\sigma}{2}} \leqq \{\lambda(r)\}^{\sigma}$$

and

$$\lambda'(r) < \frac{1}{\sigma} \frac{\{\lambda(r)\}^{1+\sigma}}{\{f(r)\}^2},$$

whenever $x \epsilon e_m$ and $m - 1 \leqq |y|$ we have by (2)

$$|g(x,\eta)|\leqq f(|y|)\{\lambda(|y|)+f(|y|)\lambda'(|y|)\}$$
$$<(1+\frac{1}{\sigma})\{\lambda(|y|)\}^{1+\sigma}$$

and finally

$$\frac{|g(x,\eta)|}{|\eta|^{1+\sigma}}<\frac{1+\frac{1}{\sigma}}{|y|^{1+\sigma}}.$$

Therefore we have for $x\epsilon e_m$

$$\lim_{|\eta|\to+\infty}\frac{|g(x,\eta)|}{|\eta|^{1+\sigma}}=0.$$

Now let e denote the union of all $e_m(m=1,2,\ldots.)$, then e is a measurable subset of I of measure equal to that of I. If $x\epsilon e$ there exists such a positive integer m_0 that $x\epsilon e_{m_0}$. Consequently we have for any $x\epsilon e$

$$\lim_{|\eta|\to+\infty}\frac{|g(x,\eta)|}{|\eta|^{1+\sigma}}=0. \tag{6}$$

In the following suppose σ as $0<\sigma\leqq 1$. Then, whenever $x\epsilon e_m$ and $m-1\leqq|y|$ we have

$$\frac{|g(x,\eta)|}{1+|\eta|^2}<\left(1+\frac{1}{\sigma}\right)\frac{\{\lambda(|y|)\}^{1+\sigma}}{1+|y|^2\{\lambda(|y|)\}^2}$$
$$\leqq\left(1+\frac{1}{\sigma}\right)\min\left[\frac{1}{|y|^2},\ \{\lambda(|y|)\}^{1+\sigma}\right]$$
$$\leqq\left(1+\frac{1}{\sigma}\right)\{\lambda(1)\}^{1+\sigma}=\text{const. }(=K).$$

And for any $(x,y)\epsilon E_{n+1}$ we obtain

$$\frac{|g(x,\eta)|}{1+|\eta|^2}\leqq\frac{|f(x,y)|\{\lambda(|y|)+|y|\lambda'(|y|)\}}{1+|y|^2\{\lambda(|y|)\}^2}$$
$$\leqq\frac{|f(x,y)|[\lambda(|y|)+\frac{1}{\sigma}\{\lambda(|y|)\}^{1+\sigma}]}{1+|y|^2\{\lambda(|y|)\}^2}$$
$$\leqq(1+\frac{1}{\sigma})\frac{\{\lambda(|y|)\}^{1+\sigma}}{1+|y|^2\{\lambda(|y|)\}^2}|f(x,y)|$$
$$\leqq(1+\frac{1}{\sigma})\{\lambda(1)\}^{1+\sigma}|f(x,y)|=K|f(x,y)|.$$

Therefore we have

$$\int_0^a \max_{|y|\leq m} \frac{|g(x,\eta)|}{1+|\eta|^2}dx < \int_0^a K\,dx + \int_{I-e_1} K \max_{|y|\leq 1} |f|dx + \dots$$

$$+\int_{I-e_m} K \max_{|y|\leq m} |f|dx \leqq K(a+\frac{1}{2}+\dots+\frac{1}{2^m}) < K(a+1).$$

If we put

$$\lim_{m\to+\infty} \max_{|y|\leq m} \frac{|g(x,\eta)|}{1+|\eta|^2} = M(x)$$

then $M(x)$ is a summable function of x in I, such that

$$\int_0^a M(x)dx < K(a+1)$$

and

$$\frac{|g(x,\eta)|}{1+|\eta|^2} \leqq M(x).$$

Now consider the second mapping effected by

$$(7)\quad \begin{cases} Y_i = \dfrac{2\eta_i}{1+|\eta|^2} & (i=1, 2, \dots, n) \\[2ex] Y_{n+1} = 1 - \dfrac{2}{1+|\eta|^2} & \end{cases}$$

which maps topologically the whole η-space, the point at infinity $|\eta|=+\infty$ being included, onto the whole unit sphere in $(Y_1, Y_2, \dots, Y_{n+1})$-space:

$$Y_1^2+Y_2^2+\dots+Y_{n+1}^2=1$$

whose pole $(0, 0, \dots, 0, 1)$ is the image of the point at infinity $|\eta|=+\infty$. The inverse of (7) is given by

$$(8)\qquad \eta_i = \frac{Y_i}{1-Y_{n+1}} \qquad (i=1, 2, \dots, n)$$

The system (5) is transformed by (7), x unchanged, to a system of the form

$$(9)\qquad \begin{cases} \dfrac{dY}{dx} = h(x, Y) \\[2ex] |Y| = 1 \end{cases}$$

where $Y=(Y_1, Y_2, \ldots, Y_{n+1})$, $|Y|=\sqrt{Y_1^2+Y_2^2+\ldots+Y_{n+1}^2}$, $h(x, Y)=(h_1, h_2, \ldots, h_{n+1})$, $h_i(x, Y)=\sum_{j=1}^{n}\frac{\partial Y_i}{\partial \eta_j}g_j(x,\eta)$ $(i=1,2,\ldots,n+1)$

and

$$(10)\qquad (Yh)\equiv Y_1h_1+Y_2h_2+\ldots+Y_{n+1}h_{n+1}=0.$$

The second member of the former equation of (9) is not defined for $Y=(0, 0, \ldots, 0, 1)$ though it seems clear that it converges to zero as $Y_{n+1}\to 1-0$ $(x\epsilon e)$. And therefore, if we put

$$h(x, (0, 0, \ldots, 0, 1))=0 \text{ for } x\epsilon I,$$

$h(x, Y)$ is a function defined on the whole surface

$$S_{n+1}: 0\leqq x\leqq a, \quad |Y|=1,$$

measurable with respect to x and continuous with respect to Y[5].

Since

$$|h(x, Y)|=2\frac{|g(x, \eta)|}{1+|\eta|^2}\leqq 2M(x),$$

$h(x, Y)$ is dominated by a summable function of x. Finally, consider the product of the mappings (3) and (7),

$$(11)\qquad \begin{cases} Y_i=\dfrac{2y_i\lambda(|y|)}{1+\{\rho(|y|)\}^2} & (i=1, 2, \ldots, n) \\ Y_{n+1}=1-\dfrac{2}{1+\{\rho(|y|)\}^2} & \end{cases}$$

which maps topologically the whole y-space, the point at infinity $|y|=+\infty$ being included, onto the whole unit sphere in Y-space. Then we have the following

Theorem 1. *(1) is transformed by means of (11), x unchanged, to (9) whose second member is defined on the surface S_{n+1} in (x, Y)-space, measurable with respect to x, continuous with respect to Y and dominated by a summable function of x alone. The segment*

$$L: 0\leqq x\leqq a, \quad Y=(0, 0, \ldots, 0, 1)$$

may be regarded as the image of $|y|=+\infty$.

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5) Written more precisely, for any fixed x *merely in* e, $h(x, Y)$ is a continuous function of Y, but since $m(I-e)=0$ $h(x, Y)$ may by regarded as a continuous function of Y for any fixed x in I.

2. Properties of solutions of (1)

Theorem 1 is quite analogous to Theorem 1 of the previous paper[6]. Therefore proceeding in the same way, we may obtain results quite analogous to the previous.

In the following we cite fundamental properties of solutions of (1).

Theorem 2. *For any point P of E_{n+1} there exists a solution of* (1) *passing through P.*

Proof. x unchanged, (1) is transformed by (11) to (9) and P is mapped by (11) to a point Q of $S_{n+1}-L$. Since (9) is a system of Carathéodory's type, there exists a solution passing through Q. Its inverse by (11) is a solution of (1) passing through P.

Theorem 3. *Consider a solution $y=y(x)$ of* (1) *going to the right (or left) from a point P (x_0, y_0) of E_{n+1}. If $y=y(x)$ has no end point whose x-coordinate is equal to a (or 0), there exists a value of x, say x_1 $[x_0<x_1\leqq a$ (or $0\leqq x_1<x_0)]$, such that $y(x)$ is defined in $x_0\leqq x<x_1$ (or $x_1<x\leqq x_0$) and we have $\lim_{x\to x_1-0}|y(x)|=+\infty$ (or $\lim_{x\to x_1+0}|y(x)|=+\infty$).*

Proof. It is not difficult to prove that the image of $y=y(x)$ by (11) reaches at a point of L on S_{n+1}. To prove the theorem it is sufficient to consider the x-coordinate of this point as x_1.

The analogous theorem to Theorem 3 in the previous paper[7] may be stated as follows;

Theorem 4. *A necessary and sufficient condition for every solution of* (1) *to have an end point, whose x-coordinate is equal to a (or o), is that, given any bounded subset of E_{n+1}, say A, the set of all points lying on any of the solutions of* (1), *which issue from A to the right (or left), is also bounded.*

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6) K. Hayashi, *loc. cit.*

7) K. Hayashi, *loc. cit.*