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Kelvin principle and some inequalities in the theory of functions. III

By

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In the previous papers I[4] and II[5] the author established, by means of the Kelvin minimum energy principle and a variant of this principle, various inequalities which may be reduced to the statements regarding the properties of harmonic functions with a vanishing normal derivative on some of boundary components of a given domain.

It is the aim of this paper to derive several inequalities which may be reduced to statements regarding the properties of harmonic functions considered as the solutions of some mixed boundary value problems. Here we mean by the mixed boundary value problem to find a function $u(z)$ satisfying the following conditions:

(i) $u(z)$ is bounded and harmonic in a domain D,

(ii) $u(\zeta)=f_1(\zeta)$ on the open boundary arcs α of D,

(iii) $\dfrac{\partial u(\zeta)}{\partial n}=f_2(\zeta)$ on the open boundary arcs β of D,

where $f_1(\zeta)$ and $f_2(\zeta)$ are the functions defined on arcs α and β on the boundary C of D, respectively, and $C=\alpha+\beta+$ (end-points of α or β), and n denotes the direction of the outer normal at each point on β.

1. Preliminary considerations. Let H_r be a half-circle: $|z| < r$, Im $z>0$, in the z-plane and let $u(z)$ be a function satisfying the following conditions:

(i) $u(z)$ is bounded and harmonic in H_r,

(ii) $u(z)=0$ on the segment $0<z<r$,

(iii) $\tilde{u}(z)=0$ on the segment $-r<z<0$, $\tilde{u}(z)$ denoting a conjugate harmonic function of $u(z)$.

Moreover let $f(z)=u(z)+i\tilde{u}(z)$. Then it is easily confirmed that the function $\{f(z)\}^2=u^2-\tilde{u}^2+2iu\tilde{u}$ is regular in H_r and takes real values on the diameter $-r<z<r$ of H_r except at $z=0$. Therefore, by the principle of reflection, $\{f(z)\}^2$ must be regular in $|z|<r$, except at $z=0$. Since the real part of $\{f(z)\}^2$ is bounded from above by the condition (i), the point $z=0$ must be a removable singular point of $\{f(z)\}^2$ and $\{f(0)\}^2=0$. Accordingly, it holds that in the vicinity of $z=0$

$$\{f(z)\}^2=a_m z^m(1+\alpha_1 z+\ldots\ldots) \qquad (a_m\neq 0,\ m(\geqq 1) \text{ is an integer}),$$

$$\therefore f(z)=Az^{\frac{m}{2}}(1+\beta_1 z+\ldots\ldots) \qquad A\neq 0.$$

Therefore $|f'(z)|=O(r^{\frac{m}{2}-1})$, $(z=re^{i\theta},\ r\to 0)$. Hence, for the Dirichlet integral of the function $u(z)$ over a half circular ring $H(r_1, r_2)$: $r_1<|z|<r_2$, Im $z>0$, we obtain the following inequality,

$$(u,\ u)_{H(r_1,r_2)}=\iint_{H(r_1,r_2)}|f'(z)|^2 d\tau\leqq A'\int_{r_1}^{r_2} r^{m-1}dr$$
$$(d\tau=dxdy,\ z=re^{i\theta}),$$

for sufficiently small r_1 and r_2, A' being a positive constant. Hence we can assert that the harmonic function $u(z)$ surely *possesses a finite Dirichlet integral* over a half circle $H_{r'}$ $(0<r'<r)$ in the sense of improper integral.

Since a Dirichlet integral is invariant under any conformal mapping of the domain of reference, the above result remains to hold also in the case of any domain D bounded by a finite number of analytic curves and for harmonic function satisfying the same boundary condition of mixed type with that of $u(z)$ on the boundary of $H_{r'}$.

Let $u(z)$ be a harmonic function in D satisfying the boundary condition of mixed type above described and let $w(z)$ be any harmonic function in the closure of D. From the above assertion and the triangle inequality for Dirichlet integrals

$$(u+w,\ u+w)^{\frac{1}{2}}\leqq(u,\ u)^{\frac{1}{2}}+(w,\ w)^{\frac{1}{2}},$$

it is easily confirmed that the function $u+w$ also possesses a finite Dirichlet integral.

2. Decreasing domain functional. In the following, for the simplicity, we restrict ourselves to the case of finite plane domains bounded by a finite number of analytic curves. Let D be a domain (of multiple-connectivity, in general,) in the z-plane, α and β be two sets of finite number of open arcs on the outer boundary component of D such that the outer boundary component $=\alpha+\beta+$ end-points of α or β, and γ be a totality of inner boundary components. By extending D outward beyond α (or some parts of α) and remaining fixed β and γ, we obtain a new domain D_1 whose outer boundary component consists of open arcs α', β, and end-points of α' or β. For the shorter formulation we denote by C and C_1 the total boundaries of D and D_1, respectively. Then we obtain the following

THEOREM V. *Let D and D_1 ($D\subset D_1$) be two domains as described above and $S(z)$ be a singularity function having the same meaning as in Theorem* I. *Let $p(z)$ denote the function satisfying the following conditions*: (i) $p(z)=0$ *on* α *and* $p(z)=$const. *on each component belonging to* γ, (ii) $\partial p(z)/\partial n=0$ *on* β, (iii) $p(z)+S(z)$ *is bounded and harmonic in* D, (iv) $\int_\delta [\partial(p+S)/\partial n]\, ds=0$ *for every closed path* δ *in* D, ds *denoting line element of* δ. *If* $p_1(z)$ *is the corresponding function associated with* D_1, *then there holds the following inequality*

$$\int_{\alpha+\gamma} S\frac{\partial p}{\partial n}ds-\int_\beta p\frac{\partial S}{\partial n}ds \geqq \int_{\alpha'+\gamma} S\frac{\partial p_1}{\partial n}ds-\int_\beta p_1\frac{\partial S}{\partial n}ds, \tag{1}$$

i. e., the value of the left-hand side of (1) *is a monotone decreasing domain functional, when the domain D increases, remaining fixed β and γ.*

Proof. We define two functions $u(z)$ and $v(z)$ as follows:

$$u(z)=p_1(z)+S(z) \qquad \text{in } D_1,$$

and

$$v(z)=\begin{cases} p(z)+S(z) & \text{in } D \\ S(z) & \text{in } D_1-D. \end{cases}$$

Both functions $u(z)$ and $v(z)$ are, of course, continuously differentiable (or stepwisely so) in the domain D_1 and further, by the preliminary considerations of Sec. 1, have finite Dirichlet integrals. Hence, applying the Green's theorem, we have the following relations,

$$(u, u)_{D_1}=\int_{C_1}(p_1+S)\frac{\partial(p_1+S)}{\partial n}ds$$

$$(2) \qquad =\int_{\alpha'}S\frac{\partial(p_1+S)}{\partial n}ds+\int_{\beta}(p_1+S)\frac{\partial S}{\partial n}ds+\int_{\gamma}S\frac{\partial(p_1+S)}{\partial n}ds$$

(by (i), (ii) and (iv))

$$=\int_{\alpha'}S\frac{\partial p_1}{\partial n}ds+\int_{\beta}p_1\frac{\partial S}{\partial n}ds+\int_{\gamma}S\frac{\partial p_1}{\partial n}ds+\oint_{D_1}S\frac{\partial S}{\partial n}ds$$

and

$$(v, v)_{D_1}=(v, v)_D+(v, v)_{D_1-\bar{D}}$$

$$(3) \qquad =\int_{\alpha}S\frac{\partial p}{\partial n}ds+\int_{\beta}p\frac{\partial S}{\partial n}ds+\int_{\gamma}S\frac{\partial p}{\partial n}ds$$

$$+\oint_{D}S\frac{\partial S}{\partial n}ds+\oint_{D_1-\bar{D}}S\frac{\partial S}{\partial n}ds.$$

And further, there holds the following equality for the Dirichlet product:

$$(v, u-v)_{D_1}=(v, u-v)_D+(v, u-v)_{D_1-\bar{D}}$$

$$=\oint_{D}v\frac{\partial(u-v)}{\partial n}ds+\oint_{D_1-\bar{D}}v\frac{\partial(u-v)}{\partial n}ds$$

$$=\int_{\alpha+\beta+\gamma}(p+S)\frac{\partial(p_1-p)}{\partial n}ds+\int_{\alpha'-\alpha}S\frac{\partial p_1}{\partial n}ds$$

$$(4) \qquad =\int_{\alpha}S\frac{\partial(p_1-p)}{\partial n}ds+\int_{\gamma}S\frac{\partial(p_1-p)}{\partial n}ds$$

$$+\int_{\alpha'}S\frac{\partial p_1}{\partial n}ds-\int_{\alpha}S\frac{\partial p_1}{\partial n}ds \qquad \text{(by (i)}\sim\text{(iv))}$$

$$=\int_{\alpha'}S\frac{\partial p_1}{\partial n}ds-\int_{\alpha}S\frac{\partial p}{\partial n}ds+\int_{\gamma}S\frac{\partial p_1}{\partial n}ds-\int_{\gamma}S\frac{\partial p}{\partial n}ds.$$

From the positive definiteness of the Dirichlet integral $(u-v, u-v)_{D_1}$, we easily obtain the inequality

$$(5) \qquad (u, u)_{D_1}-(v, v)_{D_1}-2(v, u-v)_{D_1}\geqq 0.$$

Inserting (2), (3) and (4) into (5), we have

$$\int_{\alpha'}S\frac{\partial p_1}{\partial n}ds-\int_{\alpha}S\frac{\partial p}{\partial n}ds+\int_{\beta}p_1\frac{\partial S}{\partial n}ds-\int_{\beta}p\frac{\partial S}{\partial n}ds$$

$$+\int_{\gamma} S\frac{\partial p_1}{\partial n}ds-\int_{\gamma} S\frac{\partial p}{\partial n}ds-2\int_{\alpha'} S\frac{\partial p_1}{\partial n}ds+2\int_{\alpha} S\frac{\partial p}{\partial n}ds$$

$$-2\int_{\gamma} S\frac{\partial p_1}{\partial n}ds+2\int_{\gamma} S\frac{\partial p}{\partial n}ds\geqq 0.$$

Thus we obtain the required inequality (1). Q. E. D.

Remark. It is easily confirmed that, if $p(z)$ takes the constant values zero on each inner boundary component belonging to γ, the conclusion of Theorem V will hold regardless of the condition (iv).

3. Increasing domain functional. Under the same assumptions with those of Theorem V concerning $p(z)$, $S(z)$, α, β and γ, we consider a new domain D_1 by extending the domain D outward beyond β instead of α and remaining fixed α and γ, and denote the total boundary C_1 of D_1 by $\alpha+\beta'+\gamma+$end-points of α or β'. Thus we obtain the following.

THEOREM VI. *Let D and D_1 $(D\subset D_1)$ be domains described above and let $p(z)$ and $S(z)$ be functions satisfying the same assumptions with those of Theorem V. If $p_1(z)$ is the corresponding function associated with D_1, then there holds the following inequality*

$$\int_{\alpha+\gamma} S\frac{\partial p}{\partial n}ds-\int_{\beta} p\frac{\partial S}{\partial n}ds \leqq\int_{\alpha+\gamma} S\frac{\partial p_1}{\partial n}ds-\int_{\beta'} p_1\frac{\partial S}{\partial n}ds, \tag{6}$$

i.e., the value of the left-hand side of (6) *is a monotone increasing domain functional, when the domain D increases, remaining fixed α and γ.*

Proof. We define two functions $u(z)$ and $v(z)$ in the same way as in the proof of Theorem V and then analogously obtain the following relations:

$$(u,u)_{D_1}=\int_{\alpha+\gamma} S\frac{\partial p_1}{\partial n}ds+\int_{\beta'} p_1\frac{\partial S}{\partial n}ds+\oint_{D_1} S\frac{\partial S}{\partial n}ds, \tag{7}$$

$$\begin{aligned}(v,v)_{D_1}&=(v,v)_D+(v,v)_{D_1-\bar D}\\ &=\int_{\alpha+\gamma} S\frac{\partial p}{\partial n}ds+\int_{\beta} p\frac{\partial S}{\partial n}ds+\oint_{D} S\frac{\partial S}{\partial n}ds+\oint_{D_1-\bar D} S\frac{\partial S}{\partial n}ds.\end{aligned} \tag{8}$$

And further we have

$$
(9) \quad
\begin{aligned}
(v, u-v)_{D_1} &= (v, u-v)_{D} + (v, u-v)_{D_1-\bar{D}} \\
&= \oint_D (u-v)\frac{\partial v}{\partial n} ds + \oint_{D_1-\bar{D}} (u-v)\frac{\partial v}{\partial n} ds \\
&= \int_{\alpha+\beta+\gamma} (p_1 - p)\frac{\partial (p+S)}{\partial n} ds + \int_{\beta'-\beta} p_1 \frac{\partial S}{\partial n} ds \\
&= \int_{\beta} (p_1 - p)\frac{\partial S}{\partial n} ds + \int_{\beta'} p_1 \frac{\partial S}{\partial n} ds - \int_{\beta} p_1 \frac{\partial S}{\partial n} ds
\end{aligned}
$$

(because of $p_1 - p = 0$ on α, $p_1 - p =$ const. on γ, and (iv))

$$
= \int_{\beta'} p_1 \frac{\partial S}{\partial n} ds - \int_{\beta} p \frac{\partial S}{\partial n} ds.
$$

Inserting (7), (8) and (9) into the inequality (5), we obtain

$$
\int_{\alpha+\gamma} S\frac{\partial p_1}{\partial n} ds + \int_{\beta'} p_1 \frac{\partial S}{\partial n} ds - \int_{\alpha+\gamma} S\frac{\partial p}{\partial n} ds - \int_{\beta} p \frac{\partial S}{\partial n} ds
$$

$$
-2\int_{\beta'} p_1 \frac{\partial S}{\partial n} ds + 2\int_{\beta} p \frac{\partial S}{\partial n} ds \geqq 0.
$$

Thus we obtain the required Theorem. Q.E.D.

Remark. It is easily confirmed that, if $p(z)$ takes the constant values zero on each inner boundary component belonging to γ, the conclusion of Theorem VI will also hold regardless of the condition (iv).

4. Green's functions for mixed boundary value problems. To illustrate the applications of Theorems V and VI, we concern with the Green's function $G(z, \xi)$ for the mixed boundary value problem, i.e. the function satisfying the following conditions:

(i) $G(z, \xi)$ is harmonic in D, except at $z = \xi \ (\xi \in D)$,

(ii) $G(z, \xi) + \log|z-\xi|$ is harmonic at $z = \xi$,

(iii) $G(z, \xi) = 0$ on α and γ,

(iv) $\partial G(z, \xi)/\partial n = 0$ on β.

For the shorter formulation we call Robin's function and Robin's constant the function $h(z, \xi) = G(z, \xi) + \log|z-\xi|$ and the constant $h(\xi, \xi)$, respectively, in the same way as in the case of the ordinary Green's function. And take in Theorems V and VI the functions with arbitrary real constants λ_j $(j = 1, \ldots, m)$:

$$
p(z) = \sum_{j=1}^{m} \lambda_j G(z, \xi_j),
$$

$$S(z)=\sum_{j=1}^{m}\lambda_j\log|z-\xi_j|,\quad (\xi_j,\, j=1,\ldots,m, \text{ arbitrary points in } D)$$

$$p(z)+S(z)=\sum_{j=1}^{m}\lambda_j\, h(z,\xi_j).$$

Then we have

$$\begin{aligned}(10)\quad \int_{\alpha+\gamma}S\frac{\partial p}{\partial n}ds-\int_{\beta}p\frac{\partial S}{\partial n}ds&=\int_{\alpha+\gamma}(p+S)\frac{\partial p}{\partial n}ds-\int_{\beta}p\frac{\partial(p+S)}{\partial n}ds\\ &=\int_{\alpha+\gamma}\Big(\sum_j\lambda_j h(z,\xi_j)\Big)\Big(\sum_i\lambda_i\frac{G(z,\xi_i)}{\partial n}\Big)ds\\ &\quad-\int_{\beta}\Big(\sum_i\lambda_i G(z,\xi_i)\Big)\Big(\sum_j\lambda_j\frac{\partial h(z,\xi_j)}{\partial n}\Big)ds\\ &=\sum_{i,j}\lambda_i\lambda_j\Big\{\int_{\alpha+\gamma}h(z,\xi_j)\frac{\partial G(z,\xi_i)}{\partial n}ds-\int_{\beta}G(z,\xi_i)\frac{\partial h(z,\xi_i)}{\partial n}ds\Big\}.\end{aligned}$$

Now, it is well known that, the harmonic function $u(z)$ which is a solution of the mixed boundary value problem introduced in the beginning of the present paper, can be represented by the above described Green's function $G(z,\xi)$ as follows:

$$(11)\quad u(\xi)=-\frac{1}{2\pi}\int_{\alpha+\gamma}f_1(z)\frac{\partial G(z,\xi)}{\partial n}ds+\frac{1}{2\pi}\int_{\beta}f_2(z)\,G(z,\xi)\,ds\quad (\xi\in D).$$ [2]

Applying the formula (11) to (10), there holds

$$\int_{\alpha+\gamma}S\frac{\partial p}{\partial n}ds-\int_{\beta}p\frac{\partial S}{\partial n}ds=-2\pi\sum_{i,j}\lambda_i\lambda_j h(\xi_i,\xi_j).$$

Hence, from Theorems V, VI and Remarks in Sec. 2 and 3, we obtain the following

COROLLARY 11. *The real quadratic form*

$$(12)\qquad \sum_{i,j=1}^{m}\lambda_i\lambda_j h(\xi_i,\xi_j)\qquad (\xi_i\in D,\ i=1,\ldots,m)$$

concerning the Robin's constants and functions of the domain D, monotonously increases or decreases according as D increases beyond α or β.

Remark. We can easily verify that the Robin's function must be symmetric in the same way as in the case of ordinary Green's function, i.e. $h(\xi_i,\xi_j)=h(\xi_j,\xi_i)$. Therefore (12) is a symmetric quadratic form.

Considering the two special cases where $m=1$ $(\lambda_1=1)$ and m

$=2$ ($\lambda_1=1$ and $\lambda_2=-1$), we obtain the following

COROLLARY 12. *The Robin's constant $h(\xi, \xi)$ increases or decreases according as D increases beyond α or β. The same result holds good for the difference of the Robin's constants and the Robin's function, i.e.,*

$$(13) \qquad h(\xi, \xi)+h(\eta, \eta)-2h(\xi, \eta) \qquad (\xi, \eta \in D).$$

It should be noticed that, using the Hadamard's variational formula, Bergman[1] obtained the same result for (13) in the case of ordinary Green's function which may be considered as a special case of Corollary 12, the set of points β being empty.

Especially consider a simply-connected domain D in the z-plane and let α and β be two open arcs on the boundary C of D which are mutually disjoint such that $C=\alpha+\beta+$end-points of α. And further let the function $w=f(z, \xi)$ $(f(\xi, \xi)=0, \xi \in D)$ map D conformally onto the unit circle $|w|<1$ with a rectilinear slit: $1 \geqq w \geqq \tau (>0)$ on the positive real axis, in such a way that the unit circumference and the slit correspond to α and β, respectively. Such mapping function $w=f(z, \xi)$ and the value of τ are uniquely determined for the domain D and an arc α or β, and there holds the relation

$$(14) \qquad G(z, \xi)=\log\frac{1}{|f(z, \xi)|}\ ^{2)}.$$

Therefore

$$(15) \qquad |f'(\xi, \xi)|=e^{-h(\xi, \xi)}.$$

Then, by Corollary 12, we have the following

COROLLARY 13. *Let $f(z, \xi)$ be a bounded slit mapping function described above of a given simply-connected domain D. Then $|f'(\xi, \xi)|$ decreases or increases according as D increases beyond α or β.*

5. Circular ring with a radial slit. As the second application we consider a doubly-connected domain D in the z-plane. For the brevity let the outer boundary component consist of two open arcs α, β and end-points of α (or β), and let the inner boundary component γ enclose the origin $z=0$. Moreover let the function $w=F(z)$ map D conformally onto a circular ring $q<|w|<1$ with a rectilinear slit: $1 \geqq w \geqq \tau (>q)$ in such a way that the circumference $|w|=1$ and $|w|=q$ correspond to α and γ, respectively, and and the slit to β. Such mapping function and the value of q and τ

are uniquely determined for the domain D and arc α (or β).[5] Thus we may assume in Theorems V and VI as follows:

$$p(z)=\log|F(z)|, \quad S(z)=-\log|z|.$$

It is obvious that all the assumptions of Theorems V and VI are satisfied from the properties of the mapping function $w=F(z)$. Thus we have the following relations:

$$\int_\alpha S\frac{\partial p}{\partial n}ds=\int_\alpha(p+S)\frac{\partial p}{\partial n}ds=\int_\alpha\log\left|\frac{F(z)}{z}\right|d\arg F(z)$$

(16) (by C. R. equations)

$$=\mathrm{Re}\left\{-i\int_\alpha\log\frac{F(z)}{z}d\log F(z)\right\},$$

$$\int_\gamma S\frac{\partial p}{\partial n}ds=\int_\gamma\log\left|\frac{F(z)}{z}\right|d\arg F(z)+\int_\gamma\log\frac{1}{q}d\arg F(z)$$

(17)

$$=\mathrm{Re}\left\{-i\int_\gamma\log\frac{F(z)}{z}d\log F(z)\right\}-2\pi\log\frac{1}{q},$$

$$\int_\beta p\frac{\partial S}{\partial n}ds=\int_\beta p\frac{\partial(p+S)}{\partial n}ds$$

$$=\int_\beta\log|F(z)|\,d\arg\frac{F(z)}{z} \quad \text{(by C.R. equations)}$$

(18)

$$=\left[\log|F(z)|\arg\frac{F(z)}{z}\right]_\beta-\int_\beta\arg\frac{F(z)}{z}d\log|F(z)|$$

$$=-\mathrm{Re}\left\{-i\int_\beta\log\frac{F(z)}{z}d\log F(z)\right\}.$$

From (16), (17) and (18),

$$\int_{\alpha+\gamma}S\frac{\partial p}{\partial n}ds-\int_\beta p\frac{\partial S}{\partial n}ds=\mathrm{Re}\left\{-i\oint_C\log\frac{F(z)}{z}d\log F(z)\right\}-2\pi\log\frac{1}{q}$$

$$=-2\pi\log\frac{1}{q} \quad \text{(by Cauchy theorem).}$$

From Theorems V and VI we obtain the following

COROLLARY 14. *Let $w=F(z)$ be a slit-mapping function described above of a given doubly-connected domain D. Then the quantity $\log\frac{1}{q}$ increases or decreases according as D increases beyond α or β.*

It should be noticed that the well-known theorem[6] on the monotonicity of modulus of any ring-domain can be deduced from the above Corollary in the special case where the set of points β is empty.

6. Variation of an inner boundary component. In this section we consider the case where a given multiply-connected domoin D extends outward beyond any one of the inner boundary components, remaining fixed other boundary components. Let D_1 be a new domain and γ' be the totality of inner boundary components of D_1. Thus we obtain the following

THEOREM VII. *Let D and D_1 $(D \subset D_1)$ be multiply-connected domains described above and let $p(z)$ and $S(z)$ have the same meanings as in Theorem* V. *If $p_1(z)$ is the corresponding function associated with D_1, then there holds the following inequality*

$$\int_{\alpha+\gamma} S\frac{\partial p}{\partial n}ds - \int_{\beta} p\frac{\partial S}{\partial n}ds \geqq \int_{\alpha+\gamma'} S\frac{\partial p_1}{\partial n}ds - \int_{\beta} p_1\frac{\partial S}{\partial n}ds, \tag{19}$$

i.e., the value of the left-hand side of (19) *is a monotone decreasing domain functional, when the domain D increases beyond any one of the inner boundary components of D.*

Proof. We define two functions $u(z)$ and $v(z)$ in the same way as in the proof of Theorem V. Thus we obtain the following relations:

$$(u, u)_{D_1} = \int_{\alpha+\gamma'} S\frac{\partial p_1}{\partial n}ds + \int_{\beta} p_1\frac{\partial S}{\partial n}ds + \oint_{D_1} S\frac{\partial S}{\partial n}ds, \tag{20}$$

$$(v, v)_{D_1} = \int_{\alpha+\gamma} S\frac{\partial p}{\partial n}ds + \int_{\beta} p\frac{\partial S}{\partial n}ds + \oint_{D} S\frac{\partial S}{\partial n}ds + \oint_{D_1-\bar{D}} S\frac{\partial S}{\partial n}ds, \tag{21}$$

$$\begin{aligned} (u, u-v)_{D_1} &= (u, u-v)_D + (u, u-v)_{D_1-\bar{D}} \\ &= \int_{\alpha+\beta+\gamma} (u-v)\frac{\partial u}{\partial n}ds + \int_{\gamma'-\gamma} (u-v)\frac{\partial u}{\partial n}ds \\ &= \int_{\alpha+\beta+\gamma} (p_1-p)\frac{\partial(p_1+S)}{\partial n}ds + \int_{\gamma'-\gamma} p_1\frac{\partial(p_1+S)}{\partial n}ds \\ &= \int_{\beta} p_1\frac{\partial S}{\partial n}ds - \int_{\beta} p\frac{\partial S}{\partial n}ds \quad \left(\begin{matrix} p=\text{const. on } \gamma \text{ and (iv)}, \\ p_1=\text{const. on } \gamma' \text{ and (iv)} \end{matrix}\right). \end{aligned} \tag{22}$$

Inserting (20), (21) and (22) into the inequality

$$(v, v)_{D_1} \geqq (u, u)_{D_1} - 2(u, u-v)_{D_1},$$

we obtain the required result. Q.E.D.

Corresponding to Corollaries 11∼14, some analogous results are obtainable from Theorem VII, but these will be omitted here.

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