MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXIX, Mathematics No. 2, 1955.

# $S$-extensions of Riemannian spaces

By

Makoto MATSUMOTO

(Recieved, 23, March, 1955)

---

We consider a space $V^n$ with an asymmetric Euclidean connection $L_{jk}^i$. Since the metric of $V^n$ is defined by means of the fundamental tensor $g_{ij}$, we have the differential equations of geodesics, applying the calculus of variations, as well as for the case of Riemannian spaces. But the equations do not coincide generally with that of paths, because the symmetric parts of $L_{jk}^i$ are not equal to the Christoffel symbols $\Gamma_{jk}^i$ constructed by the fundamental tensor $g_{ij}$.

If both of them are identical, then the fundamental tensor $g_{ij}$ is covariant constant with respect to $(L)$ and $(\Gamma)$, so that we may define the covariant differentiations with respect to $(L)$ and $(\Gamma)$. Such a connection will be called $S$-connection and the Riemannian space, whose fundamental tensor is same as that of $V^n$, will be called the space induced by $V^n$. The concept named by $S$-extension of Riemannian space is converse of concept of the induced Riemannian space.

## § 1. Definition of $S$-connection and $S$-extension

Let $V^n$ be an $n$-dimensional space with an asymmetric Euclidean connection and $P$ a current point of $V^n$, whose local co-ördinates are $x^i$. The connection is defined by the equations

$$dP(x)=e_i(x)dx^i,$$

$$de_j(x)=L_{jk}^i(x)e_i(x)dx^k,$$

where $(e_i)$ is the natural frame attached to the point $P(x)$ and $L_{jk}^i(x)$ are the coefficients of the linear connection. The metric of $V^n$ is defined by the quadratic differential form

$$ds^2=e\,g_{ij}dx^i dx^j \qquad (e=\pm 1),$$

where $e$ is taken such that $ds^2$ is non-negative. Then we have

$$e_i e_j = g_{ij}.$$

It is well known that there exist the following relations between the fundamental tensor $g_{ij}$ and the coefficients $L^i_{jk}$ of connection.

$$\frac{\partial g_{ij}}{\partial x^k} - g_{hj} L^h_{ik} - g_{ih} L^h_{jk} = 0. \tag{1}$$

We denote by $\Gamma^i_{jk}$ and $S^i_{\cdot jk}$ the symmetric and skew-symmetric parts of $L^i_{jk}$ respectively, and then $L^i_{jk}$ are written in the form

$$L^i_{jk} = \Gamma^i_{jk} + S^i_{\cdot jk}. \tag{2}$$

Under a transformation of local co-ördinates $(x) \to (\bar{x})$, the coefficients $L^i_{jk}$ of connection are transformed to $\overline{L}^a_{bc}$, which are given by the equations

$$\overline{L}^a_{bc} = L^i_{jk} \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^b} \frac{\partial x^k}{\partial \bar{x}^c} + \frac{\partial^2 x^i}{\partial \bar{x}^b \partial \bar{x}^c} \frac{\partial \bar{x}^a}{\partial x^i}. \tag{3}$$

It is clear that the symmetric parts $\Gamma^i_{jk}$ of $L^i_{jk}$ are subjected to the same transformation (3), while the skew-symmetric parts $S^i_{\cdot jk}$ are components of a tensor, which is usually called the *torsion tensor* of $V^n$.

Substitution in (1) from (2) gives

$$\frac{\partial g_{ij}}{\partial x^k} - g_{hj} \Gamma^h_{ik} - g_{ih} \Gamma^h_{jk} = S_{ijk} + S_{jik}, \tag{4}$$

where by definition $S_{ijk} = g_{ih} S^h_{\cdot jk}$, which will be called the *covariant torsion tensor* of $V^n$.

We see from (4) that the symmetric parts $\Gamma^i_{jk}$ coincide with the Christoffel symbols constructed by the fundamental tensor $g_{ij}$, if and only if the *covariant torsion tensor* $S_{ijk}$ *is skew-symmetric with respect to all the indices.* This condition will be named the *S-condition* and we shall say that the space $V^n$ is of *S-connection* when the $S$-condition is satisfied.

If $V^n$ is Riemannian, then the coefficients $L^i_{jk}$ is of course symmetric and so the torsion vanishes. Hence, such a $V^n$ is clearly of $S$-connection. Moreover, we consider $V^n$ with so-called half-symmetric connection, namely

$$S^i_{\cdot jk} = \frac{1}{n-1} (\delta^i_j S_k - \delta^i_k S_j) \qquad (S_k = S^i_{\cdot ik}).$$

If such a space is of $S$-connection, it is easily seen that the space is Riemannian.

We consider $V^n$ of $S$-connection, the underlying $n$-manifold be $M^n$. Then we may define a Riemannian $n$-space $R^n$, the underlying manifold and the fundamental tensor be same as $V^n$. The coefficients $\Gamma^i_{jk}$ of the connection of $R^n$ are the Christoffel symbols constructed from $g_{ij}$ and coincide with the symmetric parts of the coefficients $L^i_{jk}$ of connection of the original space $V^n$. The space will be called the *Riemannian space induced from $V^n$ of $S$-connection.*

Conversely, let $R^n$ be a Riemannian space and $g_{ij}$ the fundamental tensor of $R^n$. We take arbitrarily a skew-symmetric covariant tensor $S_{ijk}$ of the third order and define the function $L^i_{jk}$ by the equation

$$L^i_{jk}=\Gamma^i_{jk}+S^i_{\cdot jk} \qquad (S^i_{\cdot jk}=g^{ih}S_{hjk}), \tag{2'}$$

where $\Gamma^i_{jk}$ are the Christoffel symbols of $R^n$. It is obvious that $L^i_{jk}$ as thus defined are subjected to the law of transformation (3). Therefore we may define a space with asymmetric Euclidean connection $V^n$ on the underlying space of $R^n$, such that the fundamental tensor is common with $R^n$ and the coefficients of the connection are given by (2′). The space as thus defined is clearly of $S$-connection and its induced Riemannian space coincides with the original space $R^n$. The space $V^n$ will be called the *$S$-extension of the Riemannian space $R^n$ with respect to the tensor $S_{ijk}$.* Since we may choose arbitrarily a skew-symmetric tensor $S_{ijk}$ and then construct a $S$-extension, we shall have a number of $S$-extension of $R^n$. Especially any $S$-extension of flat space is a space with absolute parallelism, due to Einstein[1].

The geodesic ($g$) of $R^n$ is defined by the differential equations

$$\frac{d^2x^i}{ds^2}+\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0,$$

where the parameter $s$ is arc-length ($g$). Making use of (2), the above equations are expressed in the forms

$$\frac{d^2x^i}{ds^2}+L^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0.$$

These equations define, as well known, the path of $V^n$, the $S$-extension of $R^n$, and we see that the parameter $s$ is the affine parameter of the path. Thus we have the

THEOREM. *Let $R^n$ be a Riemannian space, the underlying manifold be $M^n$. Then any curve of $M^n$, which expresses a geodesic (g) of $R^n$, is a path of any S-extension of $R^n$, and the arc-length of (g) is the affine parameter of the path.*

It is to be noted finally that the Riemannian space $R^n$ and the $S$-extension $V^n$ have the same topological properties, since they have the same underlying manifold.

## § 2. Curvature and torsion of space with an asymmetric Euclidean connection

In this section, before entering on the main subject of this paper, we shall examine the general properties of $V^n$ with an asymmetric Euclidean connection. Most of the following formulae are already well known[2], so that we shall describe in outline the theories.

The covariant derivatives of a tensor $u^{h}_{\cdot i}$ with respect to the connection $(L)$ are defined as follows:

$$u^{h}_{\cdot i/j}=u^{h}_{\cdot i,j}+u^{a}_{\cdot i}L^{h}_{aj}-u^{h}_{\cdot a}L^{a}_{ij}\,, \tag{5}$$

where comma means the ordinal partial differentiation. The covariant derivatives of a scalar are equal to the ordinal one. By the well known methods we have the Ricci identities

$$v_{/j/k}-v_{/k/j}=-2v_{/a}S^{a}_{\cdot jk}\,, \tag{6}$$

$$u^{h}_{\cdot i/j/k}-u^{h}_{\cdot i/k/j}=u^{a}_{\cdot i}L^{\cdot h}_{a\cdot jk}-u^{h}_{\cdot a}L^{\cdot a}_{i\cdot jk}-2u^{h}_{\cdot i/a}S^{a}_{\cdot jk}\,. \tag{7}$$

where by definition $L^{\cdot h}_{i\cdot jk}$ are components of the curvature tensor of $V^n$ given by the equations

$$L^{\cdot h}_{i\cdot jk}=L^{h}_{ij,k}-L^{h}_{ik,j}+L^{a}_{ij}L^{h}_{ak}-L^{a}_{ik}L^{h}_{aj}\,.$$

Since the fundamental tensor $g_{ij}$ is covariant constant, we have

$$g_{ij/k/l}-g_{ij/l/k}=-g_{aj}L^{\cdot a}_{i\cdot kl}-g_{ia}L^{\cdot a}_{j\cdot kl}=0\,.$$

Hence, if we put $L_{ijkl}=g_{aj}L^{\cdot a}_{i\cdot kl}$, then we have the identities satisfied by $L_{ijkl}$ as follows:

$$L_{ijkl}=-L_{jikl}=-L_{ijlk}\,. \tag{8}$$

Further we differentiate (6) covariantly with respect to $x^l$ and sum the equations obtained by cyclic permutation of the indices $j$, $k$, $l$, and make use of the Ricci identities. This process gives the following formulae.

$$
(9)\qquad
\begin{aligned}
L_{(i\cdot jk)}^{\;\;h} &= 2S^{\;h}_{\cdot(ij/k)} + 4S^{\;a}_{\cdot(ij}S^{\;h}_{\cdot k)a}\,, \\
L_{h(ijk)} &= -2S_{h(ij/k)} + 4S_{ha(i}S^{\;a}_{\cdot jk)}\,.
\end{aligned}
$$

We contract (9) with respect to $h$ and $k$, and put $L_{ij}=L_{i\cdot ja}^{\;a}$ and $S_i=S^{\;a}_{\cdot ai}$. Then we have

$$
(10)\qquad \tfrac{1}{2}(L_{ij}-L_{ji}) = S^{\;a}_{\cdot ij/a} + S_{i/j} - S_{j/i} + 2S^{\;a}_{\cdot ij}S_a\,.
$$

The tensor $L_{ij}$ will be called the *Ricci tensor*, $S_i$ the *torsion vector* and $S^{\;a}_{\cdot ij/a}$ the *derived torsion tensor.* The Ricci tensor does not always be symmetric and the above equations give the skew-symmetric parts of the tensor.

Next, we differentiate the equation

$$
u_{i/j/k} - u_{i/k/j} = -u_a L_{i\cdot jk}^{\;a} - 2u_{i/a}S^{\;a}_{\cdot jk}\,,
$$

covariantly with respect to $x^l$ and sum the equations obtained by cyclic permutation of $j$, $k$, $l$, and then we get immediately the Bianchi identities

$$
(11)\qquad
\begin{aligned}
L_{i\cdot(jk/l)}^{\;h} &= 2L_{i\cdot a(j}^{\;h}S^{\;a}_{\cdot kl)}\,, \\
L_{hi(jk/l)} &= 2L_{hia(j}S^{\;a}_{\cdot kl)}\,.
\end{aligned}
$$

## § 3. Curvature and torsion of *S*-connection

Let $V^n$ be a space with *S*-connection and $R^n$ the induced Riemannian space. We denote by semi-colon the covariant differentiation with respect to the Christoffel symbols $\Gamma^i_{jk}$, which are equal to the symmetric parts of the connection $L^i_{jk}$ of $V^n$. Any tensor of $R^n(V^n)$ may be regarded as tensor of $V^n(R^n)$. Hence both of the covariant derivatives $u^{\;h}_{\cdot i;j}$ and $u^{\;h}_{\cdot i/j}$ of the tensor $u^{\;h}_{\cdot i}$ are tensors of $R^n$ as well as of $V^n$. Especially, the fundamental tensor $g_{ij}$ is covariant constant as the covariant differentiations with respect to $(L)$ and $(\Gamma)$.

The curvature tensor $R_{i\cdot jk}^{\;h}$ of $R^n$ defined by

$$
R_{i\cdot jk}^{\;h} = \Gamma^h_{ij,k} - \Gamma^h_{ik,j} + \Gamma^a_{ij}\Gamma^h_{ak} - \Gamma^a_{ik}\Gamma^h_{aj}
$$

is a tensor of $V^n$, which will be called *the curvature tensor of the second kind* of $V^n$; while the curvature tensor $L_{i\cdot jk}^{\;h}$ of $V^n$ is called *of the first kind.* The Ricci identities for the covariant differentiation (;) are given by

$$
(12)\qquad u^{\;h}_{\cdot i;j;k} - u^{\;h}_{\cdot i;k;j} = u^{\;a}_{\cdot i}R_{a\cdot jk}^{\;h} - u^{\;h}_{\cdot a}R_{i\cdot jk}^{\;a}\,.
$$

Making use of (2) we have

$$(13)\qquad \begin{aligned} L_{i\cdot jk}^{h} &= R_{i\cdot jk}^{h} + S_{\cdot ij;k}^{h} - S_{\cdot ik;j}^{h} + S_{\cdot ij}^{a} S_{\cdot ak}^{h} - S_{\cdot ik}^{a} S_{\cdot aj}^{h}\,, \\ L_{hijk} &= R_{hijk} - S_{hi;jk} + S_{hik;j} + S_{\cdot ij}^{a} S_{ahk} + S_{\cdot ik}^{a} S_{ahj}\,. \end{aligned}$$

There exist the following relations between the covariant derivatives (;) and (/) of the torsion tensor $S_{\cdot ij}^{h}$, which are easily obtained.

$$(14)\qquad S_{\cdot ij;k}^{h} = S_{\cdot ij/k}^{h} - S_{\cdot ij}^{a} S_{\cdot ak}^{h} + S_{\cdot aj}^{h} S_{\cdot ik}^{a} - S_{\cdot ai}^{h} S_{\cdot jk}^{a}\,.$$

It is a remarkable property of $S$-connection that the *torsion vector* $S_i$ *vanishes identically*, by means of skew-symmetry of the covariant torsion tensor $S_{ijk}$. Hence we obtain from (14) the interesting identities

$$(15)\qquad S_{\cdot ij;a}^{a} = S_{\cdot ij/a}^{a}\,.$$

The equations (13) and (14) give the equations

$$(16)\qquad \begin{aligned} L_{i\cdot jk}^{h} &= R_{i\cdot jk}^{h} + S_{\cdot ij/k}^{h} - S_{\cdot ik/j}^{h} + S_{\cdot ij}^{a} S_{\cdot ka}^{h} - S_{\cdot ik}^{a} S_{\cdot ja}^{h} - 2S_{\cdot ai}^{h} S_{\cdot jk}^{a}\,, \\ L_{hijk} &= R_{hijk} - S_{hij/k} + S_{hik/j} + S_{\cdot hj}^{a} S_{aik} - S_{\cdot hk}^{a} S_{aij} - 2S_{\cdot hi}^{a} S_{ajk}\,. \end{aligned}$$

It is well known that the curvature tensor $R_{hijk}$ of the Riemannian space $R^n$ satisfies the identities

$$R_{hijk} - R_{jkhi} = 0\,.$$

Making use of this we have from (16)

$$(17)\qquad \begin{aligned} L_{hijk} - L_{jkhi} &= -S_{ijk/h} + S_{hjk/i} + S_{hik/j} - S_{hij/k} \\ &= -S_{ijk;h} + S_{hjk;i} + S_{hik;j} - S_{hij;k}\,. \end{aligned}$$

From (10) we have immediately

$$(18)\qquad \tfrac{1}{2}(L_{ij} - L_{ji}) = S_{\cdot ij/a}^{a}\,.$$

Therefore the Ricci tensor $L_{ij}$ of the space $V^n$ of $S$-connection is not generally symmetric and the skew-symmetric parts of $L_{ij}$ are the components of the derived torsion tensor. On the other hand, we obtain the expression of components of the Ricci tensor as follows:

$$(19)\qquad L_{ij} = R_{ij} + S_{\cdot ij/a}^{a} + S_{\cdot bi}^{a} S_{\cdot aj}^{b}\,,$$

in virture of (16), where $R_{ij}$ are the components of the Ricci tensor of $R^n$, namely $R_{ij} = R_{i\cdot ja}^{a}$, which is symmetric, as well known.

Further we put $L = g^{ij} L_{ij}$ and $R = g^{ij} R_{ij}$. These scalars are respectively called the *scalar curvature of the first and the second* *kind* of $V^n$, the latter be the scalar curvature of $R^n$. From (19) we obtain

$$L=R-S^{abc}S_{abc}. \tag{20}$$

If the fundamental form of $V^n$ is positive-definite, the scalar $S^{abc}S_{abc}$ is non-negative and hence we have the

THEOREM. *If the fundamental form of $V^n$ of S-connection is positive-definite, then the scalar curvature of the first kind is not greater than that of the second kind. In the other words, the scalar curvature of the first kind of any S-extension of the Riemannian space $R^n$, the fundamental form of which is positive-definite, is not greater than the scalar curvature of $R^n$.*

It is concluded from (15) and (18) that the Ricci tensor $L_{ij}$ is symmetric, if and only if the skew-symmetric tensor $S_{ijk}$ satisfies the equations

$$S^{a}_{\cdot ij;a}=0, \tag{21}$$

namely, the derived torsion tensor vanishes identically. If $S_{ijk}$ is a harmonic tensor, the above equations are satisfied by means of the definition of harmonic tensors[3]. Therefore

THEOREM. *If the tensor $S_{ijk}$ is harmonic, then the Ricci tensor $L_{ij}$ of the S-extension with respect to $S_{ijk}$ is symmetric.*

Next, we take a Killing tensor $S_{ijk}$, so that the covariant derivatives $S_{ijk;l}$ are skew-symmetric[4]. In this case, we see from (17) that the curvature tensor $L_{hijk}$ satisfies the identities

$$L_{hijk}=L_{jkhi}. \tag{22}$$

Consequently the curvature tensor satisfies the identities, which are satisfied by the curvature tensor of Riemannian space. Further the equation (21) are clearly satisfied for the Killing tensor $S_{ijk}$. Thus we have the

THEOREM. *If the tensor $S_{ijk}$ is a Killing tensor, then the curvature tensor $L_{hijk}$ of the first kind of the S-extension with respect to $S_{ijk}$ satisfies* (8), (9) *as well as* (22), *and the Ricci tensor $L_{ij}$ is symmetric.*

## § 4. S-extensions of completely harmonic Riemannian spaces

The Riemannian space $R^n$ is centroharmonic, when the following equation is satisfied.

(23) $$g^{ij}\Omega_{;i;j}=f(\Omega),$$

where $\Omega(x_0, x)$ is the characteristic function with respect to the point $(x_0)$ and $(x)$, which is named so by Synge[5] and is given by $\Omega=(e/2)s^2$, $s$ be the geodesic distance from $(x_0)$ to $(x)$. The covariant differentiation (;) in (23) is taken at the point $(x)$, and the point $(x_0)$ is called the base point. If the equation (23) holds for all choices of the base point, the space is called completely harmonic[6].

Let $V^n$ be a $S$-extension of the Riemannian space $R^n$ with respect to the tensor $S_{ijk}$, then we have

$$\Omega_{;i;j}=\Omega_{|i|j}+\Omega_{|a}S^{a}_{\cdot ij}$$

Therefore (23) is expressed in the form

(24) $$g^{ij}\Omega_{|i|j}=f(\Omega)$$

Now we shall call *completely harmonic* a space with an asymmetric Euclidean connection $V^n$, such that the equation (24) holds for all choices of the base point. If $V^n$ is of general asymmetric Euclidean connection, the methods used in the 7th section of the paper by Copson and Ruse[6], does not be applicable. Because the equations (7·4), ...., (7·10) in their paper have been found by Synge[5], obtained by the successive covariant differentiations of the equation $\Omega=(e/2)s^2$ along the geodesic. In our case, the geodesic of $V^n$

$$\frac{d^2x^i}{ds^2}+\begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{ds}\frac{dx^k}{ds}=0,$$

is obtained by the calculus of variation by means of the fundamental form and $\begin{Bmatrix} i \\ jk \end{Bmatrix}$ in the above equation are the Christoffel symbols constructed by the fundamental tensor. But the symmetric parts $\Gamma^i_{jk}$ of the coefficients $L^i_{jk}$ of the connection are not always identical with $\begin{Bmatrix} i \\ jk \end{Bmatrix}$, as shown in the first section. Besides, $\Omega_{ij....k}$ in their paper is the covariant derivatives with respect to $\begin{Bmatrix} i \\ jk \end{Bmatrix}$

Now, if $V^n$ is of $S$-connection, the symmetric parts $\Gamma^i_{jk}$ coincide with $\begin{Bmatrix} i \\ jk \end{Bmatrix}$ and the equation

$$g^{ij}\Omega_{|i|j}=g^{ij}\Omega_{;i;j}$$

is satisfied, and hence the process, by which they gave the complete condition for harmonic space, may be applicable equally well to $V^n$ of $S$-connection. Consequently we have the

THEOREM. *If the Riemannian space $R^n$ is completely harmonic, any S-extension of $R^n$ is also completely harmonic. Conversely, if $V^n$ is of S-connection and completely harmonic, then the Riemannian space induced from $V^n$ is completely harmonic.*

## § 5. Subspaces of spaces of $S$-connection

We consider a variety $V^n$ of the space $V^m$ of $S$-connection*, which is given by the equation $y^\alpha = y^\alpha(x)$. When a current point $P$ of $V^m$ displaces along $V^n$, we have

$$dP = e_\alpha dy^\alpha = e_\alpha B_i^\alpha dx^i \quad \left(B_i^\alpha = \frac{\partial y^\alpha}{\partial x^i}\right).$$

Hence we put

$$e_i = e_\alpha B_i^\alpha \tag{25}$$

it follows

$$dP = e_i dx^i. \tag{26}$$

We see that $(e_i)$ are $n$ vectors of frame attached to $P$ of $V^n$. We take further $m-n$ vectors $e_P$ $(P=n+1,\dots,m)$, which are orthogonal to $e_i$ and matually orthogonal, whose lengthes are unit. Then we have

$$e_i e_P = 0\,, \quad e_P e_Q = \delta_{PQ} \tag{27}$$

Now we put

$$\begin{aligned} de_i &= (L_{ik}^j e_j + L_{ik}^P e_P)\, dx^k, \\ de_P &= (L_{Pk}^j e_j + L_{Pk}^Q e_Q)\, dx^k, \end{aligned} \tag{28}$$

$$e_P = e_\alpha B_P^\alpha\,. \tag{29}$$

The equations (28), (25), (29) and

$$de_\alpha = L_{\alpha\gamma}^\beta e_\beta dy^\gamma \tag{30}$$

give the equations

$$B_{i,k}^\alpha + B_i^\beta B_k^\gamma L_{\beta\gamma}^\alpha - B_j^\alpha L_{ik}^j = L_{ik}^P B_P^\alpha\,, \tag{31}$$

$$B_{P,k}^\alpha + B_P^\beta B_k^\gamma L_{\beta\gamma}^\alpha = L_{Pk}^j B_j^\alpha + L_{Pk}^Q B_Q^\alpha\,. \tag{32}$$

---

* In this section we assume that the fundamental form of $V^m$ is positive-definite.

The equations (31) give

$$S^{\alpha}_{\cdot\beta\gamma}B^{\beta}_{i}B^{\gamma}_{k}=\tfrac{1}{2}(L^{j}_{ik}-L^{j}_{ki})B^{\alpha}_{j}+\tfrac{1}{2}(L^{P}_{ik}-L^{P}_{ki})B^{\alpha}_{P}\,.$$

Hence we denote by $\Gamma^{j}_{ik}$ and $S^{j}_{\cdot ik}$ the symmetric and skew-symmetric parts of $L^{j}_{ik}$, and also by $H^{P}_{ik}$ and $S^{P}_{ik}$ the symmetric and skew-symmetric parts of $L^{P}_{ik}$. Then we have

$$(33)\qquad S^{\alpha}_{\cdot\beta\gamma}B^{\beta}_{i}B^{\gamma}_{k}=S^{j}_{\cdot ik}B^{\alpha}_{j}+S^{P}_{ik}B^{\alpha}_{P}\,.$$

and from (31)

$$(34)\qquad B^{\alpha}_{i,k}+B^{\beta}_{i}B^{\gamma}_{k}\Gamma^{\alpha}_{\beta\gamma}-B^{\alpha}_{j}\Gamma^{j}_{ik}=H^{P}_{ik}B^{\alpha}_{P}\,,$$

where $\Gamma^{\alpha}_{\beta\gamma}$ are the symmetric parts of the coefficients $L^{\alpha}_{\beta\gamma}$ of the connection of $V^m$, that is, the Christoffel symbols constructed by the fundamental tensor $g_{\alpha\beta}$ of $V^m$, and $S^{\alpha}_{\cdot\beta\gamma}$ are components of the torsion tensor of $V^m$.

We contract (33) by $g_{\alpha\delta}B^{\delta}_{h}$ and then we have

$$(35)\qquad S_{hik}=S_{\alpha\beta\gamma}B^{\alpha}_{h}B^{\beta}_{i}B^{\gamma}_{k}\qquad (S_{hik}=g_{hj}S^{j}_{\cdot ik})\,.$$

It follows from (35) that $S_{hik}$ is the projection of the torsion tensor $S_{\alpha\beta\gamma}$ of the enveloping space $V^m$ on $V^n$ on hence $S_{hik}$ is skew-symmetric tensor. Similarly we have from (33), contracting by $g_{\alpha\delta}B^{\delta}_{Q}$

$$(36)\qquad S_{Qik}=S_{\alpha\beta\gamma}B^{\alpha}_{Q}B^{\beta}_{i}B^{\gamma}_{k}\qquad (S_{Qik}=\delta_{PQ}S^{P}_{ik}=S^{Q}_{ik})\,.$$

It follows from (36) that $S^{P}_{ik}$ ($P$: fixed) are components of a skew-symmetric tensor of $V^n$.

The induced metric of $V^n$ from $V^m$ is given by the fundamental tensor

$$(37)\qquad g_{ij}=g_{\alpha\beta}B^{\alpha}_{i}B^{\beta}_{j}\,.$$

Differentiation of (37) gives by means of (31)

$$(38)\qquad g_{ij,k}-g_{aj}L^{a}_{ik}-g_{ia}L^{a}_{jk}=0\,.$$

We know now from (38) and skew-symmetry of $S_{ijk}$ that $L^{i}_{jk}$ are the coefficients of the induced connection of $V^n$ from $V^m$ and $S^{i}_{\cdot jk}$ are the components of the torsion tensor of $V^n$, and further $\Gamma^{i}_{jk}$ are the Christoffel symbols constructed by the fundamental tensor $g_{ij}$ of $V^n$. Thus we have the

THEOREM. *If $V^m$ is a space of S-connection, then a subspace of $V^m$ is also of S-connection.*

The symmetric parts $H^P_{ij}$ of $L^P_{ij}$ are the second fundamental tensors of $R^n$, which is the Riemannian space induced from $V^n$. Hence $H^P_{ij}$ are the second fundamental tensors of $R^n$, which is the subspace of Riemannian space $R^m$ induced from $V^m$.

The asymptotic directions of $R^n$ are defined by the differential equations

$$H^P_{ij} B^\lambda_P dx^i dx^j = 0 .$$

Similarly we define the *asymptotic directions* of $V^n$ by the equations

$$L^P_{ij} B^\alpha_P dx^i dx^j = 0 .$$

Since $H^P_{ij}$ are symmetric parts of $L^P_{ij}$, the asymptotic directions of $V^n$ coincide with that of $R^n$. We see easily that the similar results hold for the lines of curvature of $V^n$ and $R^n$.

Now, from (27) and (28) we have

$$L^i_{Pj} = -g^{ia} L^Q_{aj} \delta_{PQ} = -g^{ia} L^P_{aj} , \qquad L^Q_{Pi} = -L^P_{Qi} . \tag{39}$$

The conditions of integrability of (31) and (32) are given by

$$L_{\alpha\beta\gamma\delta} B^\alpha_i B^\beta_j B^\gamma_k B^\delta_l = L_{ijkl} - L^P_{ik} L^P_{jl} + L^P_{il} L^P_{jk} . \tag{40}$$

$$L_{\alpha\beta\gamma\delta} B^\alpha_P B^\beta_i B^\gamma_j B^\delta_k = -L^P_{ij/k} + L^P_{ik/j} - L^a_{ij} L^P_{Qk} + L^Q_{ik} L^P_{Qj} - 2L^P_{ia} S^a_{\cdot jk} , \tag{41}$$

$$L_{\alpha\beta\gamma\delta} B^\alpha_P B^\beta_Q B^\gamma_i B^\delta_j = L^Q_{Pi/j} - L^Q_{Pj/i} + L^a_{Pi} L^Q_{aj} - L^a_{Pj} L^Q_{ai} + L^R_{Pi} L^Q_{Rj} - L^R_{Pj} L^Q_{Ri} + 2L^Q_{Pa} S^a_{\cdot ij} , \tag{42}$$

where $L_{ijkl}$ are components of the curvature tensor of $V^n$. These equations are respectively the generalizations of the Gauss, Cadazzi and Ricci equations in the case of Riemannian space.

Differentiation of (33) gives

$$\begin{aligned} & S^\alpha_{\cdot\beta\gamma/\delta} B^\beta_i B^\gamma_j B^\delta_k + S^\alpha_{\cdot\beta\gamma} B^\beta_P (L^P_{ik} B^\gamma_j + L^P_{jk} B^\gamma_i) \\ & = S^a_{\cdot ij/k} B^\alpha_a + S^a_{\cdot ij} L^P_{ak} B^\alpha_P + S^P_{ij/k} B^\alpha_P \\ & \quad + S^P_{ij} (L^a_{Pk} B^\alpha_a + L^Q_{Pk} B^\alpha_Q) \end{aligned} \tag{43}$$

from which we have by means of (36)

$$S_{\alpha\beta\gamma/\delta} B^\alpha_i B^\beta_j B^\gamma_k B^\delta_l = S_{ijk/l} - S^P_{jk} L^P_{il} + S^P_{ik} L^P_{jl} - S^P_{ij} L^P_{kl} , \tag{44}$$

$$S_{\alpha\beta\gamma/\delta} B^\alpha_P B^\beta_i B^\gamma_j B^\delta_k = S^P_{ij/k} + S^a_{\cdot ij} L^P_{ak} + S^Q_{ij} L^P_{Qk} + S_{PQi} L^Q_{jk} - S_{PQj} L^Q_{ik} , \tag{45}$$

where we put

$$S_{PQi} = -S_{QPi} = S_{\alpha\beta\gamma} B^{\alpha}_{P} B^{\beta}_{Q} B^{\gamma}_{i}, \tag{46}$$

In $R^n$, instead of (28), we put

$$\begin{aligned} de_i &= (\Gamma^{j}_{ik} e_j + H^{P}_{ik} e_P)\,dx^k, \\ de_P &= (H^{j}_{Pk} e_j + H^{Q}_{Pk} e_Q)\,dx^k, \end{aligned} \tag{47}$$

where $\Gamma^{j}_{ik}$ are the Christoffel symbols constructed by the fundamental tensor $g_{ij}$ of $R^n(V^n)$, and $H^{P}_{ij}$ are the symmetric parts of $L^{P}_{ik}$, and that $H^{j}_{Pk} = -g^{ja} H^{P}_{ak}$. From (47) we obtain (34) and further

$$B^{\alpha}_{P,k} + B^{\beta}_{P} B^{\gamma}_{k} \Gamma'^{\alpha}_{\beta\gamma} = H^{j}_{Pk} B^{\alpha}_{j} + H^{Q}_{Pk} B^{\alpha}_{Q}. \tag{48}$$

Subtraction (48) from (34) gives

$$S^{\alpha}_{\cdot\beta\gamma} B^{\beta}_{P} B^{\gamma}_{k} = -g^{ij} S^{P}_{ik} B^{\alpha}_{j} + \tfrac{1}{2}(L^{Q}_{Pk} - H^{Q}_{Pk}) B^{\alpha}_{Q}.$$

Therefore $(m-n)(m-n-1)/2$ vectors $S_{PQi}$ defined by (46) are given by

$$S_{PQi} = H^{Q}_{Pi} - L^{Q}_{Pi}. \tag{49}$$

If $V^n$ is a hypersurface of $V^m$, that is, $n = m-1$, then vectors $S_{PQi}$ are obviously equal to zero.

## REFERENCES

1) A. Einstein: *Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelisms.* S.-B. Preuss. Akad., 1928, pp. 217-221, Also see, L. P. Eisenhart: *Spaces admitting complete absolute parallelism,* Bull. Amer. Math. Soc., vol. 39 (1933), pp. 217-226.

2) L. P. Eisenhart: Non-Riemannian geometry, Amer. Math. Colloq. VIII, (1927).

3) S. Bochner: *Curvature and Betti numbers,* Annals Math., vol. 49 (1948), pp. 379-390, the equation (26).

4) K. Yano: *Some remarks of tensor fields and curvature,* Annals Malh., vol. 55 (1952), pp. 328-347, the equation (5·1).

5) J. L. Synge: *A characteristic function in Riemannian space and its application to the solution of geodesic triangles,* Proc. London Math. Soc. (2), vol. 32 (1930), pp. 241-258.

6) E. T. Copson and H. S. Ruse: *Harmonic Riemannian spaces,* Proc. Royal Soc. Edinburgh, vol. 60 (1939-1940), pp. 117-133.