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A treatise on the 14-th problem of Hilbert

By

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The following problem is known as the 14-th problem of Hilbert:

Let $x_1, \cdots, x_n$ be algebraically independent elements over a field k and let L' be a subfield of $k(x_1, \cdots, x_n)$ containing k. Is $k[x_1, \cdots, x_n] \cap L'$ an affine ring?

Zariski [6] treated this problem in the following form:

Is $\mathfrak{o} \cap L'$ an affine ring, when $\mathfrak{o}$ is a normal affine ring over a ground field k and L' a function field over k?

And he proved[1] that if dim L' is not greater than 2 then the answer is affirmative.

In the present paper we want to treat the problem in the following form:

Is $\mathfrak{o} \cap L'$ an affine ring, when $\mathfrak{o}$ is a normal affine ring over a ground ring I and L' a function field over I?

And we shall prove also that the answer is affirmative if dim L' is not greater than 2, which is a slight generalization of the result due to Zariski because we need not assume that the ground ring is a field. Our method is based on the notion of the $\mathfrak{a}$-transform which will be explained in § 1. As biproducts of our treatment, we shall show a characterization of affine models contained in an affine model (§ 5) and that if D is a divisorial closed set of a normal affine model A of dimension 2, then $A-D$ is an affine model (§ 6).

Terminology. We shall use the same terminology as in Nagata [4].

Results assumed to be known. Some basic results contained

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1) See foot-note 7) below.

in Nagata [4] and [5] are used freely.

§ 1. The notion of $\mathfrak{a}$-transform.

Let $\mathfrak{a}$ be an ideal of an integral domain $\mathfrak{o}$ and let L be the field of quotients of $\mathfrak{o}$. Then we shall denote by $\mathfrak{a}^{-1}$ the set of elements $x \in L$ such that $x\mathfrak{a} \subseteq \mathfrak{o}$; $\mathfrak{a}^{-n}$ will denote $(\mathfrak{a}^n)^{-1}$. The union of all the $\mathfrak{a}^{-n}$ $(n=1, 2, \cdots)$ is called the *$\mathfrak{a}$-transform* of $\mathfrak{o}$. Obviously $\mathfrak{a}^{-i} \subseteq \mathfrak{a}^{-j}$ if $i<j$. Furthermore, every element of $\mathfrak{o}[\mathfrak{a}^{-i}]$ is contained in some $\mathfrak{a}^{-n}$. Therefore the $\mathfrak{a}$-transform $\mathfrak{S}$ is the union of all the subrings $\mathfrak{o}[\mathfrak{a}^{-n}]$ and $\mathfrak{S}$ is a subring of L.

If $\mathfrak{a}=0$, then $\mathfrak{a}^{-1}=L$ and $\mathfrak{S}=L$. We shall treat hereafter only the case where $\mathfrak{a}\neq 0$.

LEMMA 1. If $a(\neq 0) \in \mathfrak{a}$, then $\mathfrak{a}^{-1}$ is the set of elements b/a with $b \in a\mathfrak{o} : \mathfrak{a}$.

Proof. If $b \in a\mathfrak{o} : \mathfrak{a}$, then $b\mathfrak{a} \subseteq a\mathfrak{o}$ and $(b/a)\mathfrak{a} \subseteqq \mathfrak{o}$. Conversely, if $z \in \mathfrak{a}^{-1}$, then $b=za \in \mathfrak{o}$ because $a \in \mathfrak{a}$. It follows that $b\mathfrak{a} \subseteq a\mathfrak{o}$ and $b \in a\mathfrak{o}:\mathfrak{a}$.

COROLLARY. If $\mathfrak{o}$ is Noetherian, then $\mathfrak{a}^{-1}$ is a finite $\mathfrak{o}$-module and $\mathfrak{o}[\mathfrak{a}^{-1}]$ is finitely generated over $\mathfrak{o}$.

LEMMA 2. $\mathfrak{S}$ is the set of elements $z \in L$ such that $z\mathfrak{a}^n \subseteq \mathfrak{o}$ with some integer n.

Proof. This follows immediately from the fact that $\mathfrak{S}$ is the union of all the $\mathfrak{a}^{-n}$.

LEMMA 3. Let $\mathfrak{o}'$ be either $\mathfrak{S}$ or $\mathfrak{o}[\mathfrak{a}^{-n}]$. Then there exists a one to one correspondence between prime ideals $\mathfrak{p}'$ of $\mathfrak{o}'$ and prime ideals $\mathfrak{p}$ of $\mathfrak{o}$ except those containing $\mathfrak{a}$ such that $\mathfrak{p}'$ corresponds to $\mathfrak{p}=\mathfrak{p}' \cap \mathfrak{o}$. In the case we have $\mathfrak{o}'_{\mathfrak{p}'}=\mathfrak{o}_{\mathfrak{p}}$.

Proof. Let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}$ which does not contain $\mathfrak{a}$. Then there exists an element $a \in \mathfrak{a}$ which is not in $\mathfrak{p}$. Then $\mathfrak{o}'$ is a subring of $\mathfrak{o}[1/a]$. Since $\mathfrak{p}\mathfrak{o}[1/a]$ is prime, $\mathfrak{p}'=\mathfrak{p}\mathfrak{o}[1/a] \cap \mathfrak{o}'$ is a prime ideal of $\mathfrak{o}'$ and $\mathfrak{p}=\mathfrak{p}' \cap \mathfrak{o}$, $\mathfrak{o}'_{\mathfrak{p}'}=\mathfrak{o}_{\mathfrak{p}}$. Conversely, if $\mathfrak{p}'$ is a prime ideal of $\mathfrak{o}'$ which does not contain $\mathfrak{a}$, then $\mathfrak{p}=\mathfrak{p}' \cap \mathfrak{o}$ does not contain $\mathfrak{a}$ and by the above observation we have $\mathfrak{o}'_{\mathfrak{p}'}=\mathfrak{o}_{\mathfrak{p}}$. Thus the assertion is proved completely.

LEMMA 4. If an ideal $\mathfrak{b}$ of $\mathfrak{o}$ has the same radical with $\mathfrak{a}$, then $\mathfrak{S}$ is also the $\mathfrak{b}$-transform of $\mathfrak{o}$, provided that $\mathfrak{a}$ and $\mathfrak{b}$ have finite bases.

This follows immediately from Lemma 2.

LEMMA 5. Let $a_1, \cdots, a_n$ be non-zero elements of $\mathfrak{o}$ which generate $\mathfrak{a}$. Let $t_1, \cdots, t_{n-1}$ be algebraically independent elements over L and let t_n be the element of $L(t_1, \cdots, t_{n-1})$ such that $\sum a_i t_i = 1$. Then $\mathfrak{s} = \mathfrak{o}[t_1, \cdots, t_n] \cap L$. If $\mathfrak{o}$ is a normal ring and if $\mathfrak{o}'$ is the derived normal ring of $\mathfrak{o}[t_1, \cdots, t_n]$, then $\mathfrak{s} = \mathfrak{o}' \cap L$.

Proof. Let $c = f(t_1, \cdots, t_n)$ be an element of $\mathfrak{o}[t_1, \cdots, t_n] \cap L$. By the relation $\sum a_i t_i = 1$, $a_i^r c$ is expressed as a polynomial in $t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n$ for an r (for every i). Since $t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n$ are algebraically independent over L and since $a_i^r c \in L$, we have $a_i^r c \in \mathfrak{a}$. It follows that there exists one m such that $c\mathfrak{a}^m \subseteq \mathfrak{o}$ and c is in $\mathfrak{s}$. Conversely, let c be an element of $\mathfrak{s}$. Then there exists an r such that $c\mathfrak{a}^r \subseteq \mathfrak{o}$. Let $m_1, \cdots, m_u$ be the set of monomials in $a_1 \cdots, a_n$ of degree r. Then 1 is expressed as a linear combination of m_i's with coefficients in $\mathfrak{o}[t_1, \cdots, t_n]$: $1 = \sum m_i f_i$. Since $m_i c = d_i$ is in $\mathfrak{o}$, $c = \sum m_i f_i c = \sum d_i f_i$ is in $\mathfrak{o}[t_1, \cdots, t_n]$. Thus $\mathfrak{s} = \mathfrak{o}[t_1, \cdots, t_n] \cap L$. Now we assume that $\mathfrak{o}$ is normal. Let c be an element of $\mathfrak{o}' \cap L$. Then c is integral over $\mathfrak{o}[t_1, \cdots, t_n]$. By the same reason as above, we see that $a_i^r c$ is integral over $\mathfrak{o}[t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n]$ for an r; it follows that $a_i^r c$ is integral over $\mathfrak{o}$ and $a_i^r c \in \mathfrak{o}$. Thus $c \in \mathfrak{s}$ and the assertion is proved.

LEMMA 6. If $\mathfrak{o}$ is a Krull ring[2)], then $\mathfrak{s}$ is the intersection of all $\mathfrak{o}_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all prime ideals of rank 1 in $\mathfrak{o}$ which do not contain $\mathfrak{a}$. Hence $\mathfrak{s}$ is also a Krull ring.

Proof. Let $\mathfrak{d}$ be the intersection of all such $\mathfrak{o}_{\mathfrak{p}}$. By Lemma 3 we have the inclusion $\mathfrak{s} \subseteq \mathfrak{d}$. We denote by $\mathfrak{q}$ prime ideals of rank 1 in $\mathfrak{o}$ containing $\mathfrak{a}$; they are in a finite number because $\mathfrak{o}$ is a Krull ring. Let d be an element of $\mathfrak{d}$. Since $\mathfrak{o}_{\mathfrak{q}}$'s are discrete valuation rings, there exists one n such that $d\mathfrak{a}^n \subseteq \mathfrak{o}_{\mathfrak{q}}$ for every $\mathfrak{q}$. Since $d \in \mathfrak{d}$, we have $d\mathfrak{a}^n \subseteq \mathfrak{o}$ and $d \in \mathfrak{s}$. Thus $\mathfrak{d} = \mathfrak{s}$.

COROLLARY. If $\mathfrak{o}$ is a Krull ring, there exists an ideal $\mathfrak{b}$ of $\mathfrak{o}$ which is generated by two elements such that $\mathfrak{s}$ is the $\mathfrak{b}$-transform of $\mathfrak{o}$.

Proof. Lemma 6 shows that $\mathfrak{s}$ is uniquely determined only by prime ideals of rank 1 which contain $\mathfrak{a}$. The corollary follows from this fact.

§ 2. The 14-th problem of Hilbert.

PROBLEM 1. *Let $\mathfrak{o}$ be a normal affine ring of a function field L over a ground ring I. Is $L' \cap \mathfrak{o}$ an affine ring over I for an*

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2) A Krull ring is an "endlich discrete Hauptordnung" in the sense of Krull [3]. Observe that a Noetherian normal ring is a Krull ring.

an arbitrary function field over I contained in L?

PROBLEM 2. *Let* $\mathfrak{o}^*$ *be a normal affine ring of a function field* L^* *over a ground ring I. Let* $\mathfrak{o}^*[t_1, t_2]$ *be an affiine ring with* t_1, t_2 *such that* $\sum a_i t_i = 1$ *with* $a_i \in \mathfrak{o}^*$ *and that each of* t_i *is transcendental over* $\mathfrak{o}^*$. *Is* $\mathfrak{o}^*[t_1, t_2] \cap L^*$ *an affine ring?*

PROBLEM 3. *Let* $\mathfrak{a}^*$ *be an ideal of a normal affine ring* $\mathfrak{o}^*$ *over a ground ring I. Is the* $\mathfrak{a}^*$*-transform of* $\mathfrak{o}^*$ *an affine ring?*

We shall show here the following

PROPOSITION 1. *The above three problems are equivalent to each other.*

REMARK. The reason why we used two symbols $\mathfrak{o}$ and $\mathfrak{o}^*$ in these problems is that 1) the family of normal affine rings for which Problem 2 is true coincides with that for which Problem 3 is true and 2) the family may not coincide with that for which Problem 1 is true. (See the proof below.)

Proof. It is obvious that if Problem 1 is true in general then so is Problem 2. Lemma 5 and the corollary to Lemma 6 shows that Problem 2 and Problem 3 are equivalent to each other (even for a fixed normal affine ring $\mathfrak{o}^*$). Now we shall show that Problem 1 is affirmative if Problem 3 is affirmative. We shall use the notations as in Problem 1. Let L'' be the field of quotients of $\mathfrak{S} = \mathfrak{o} \cap L'$. Then $\mathfrak{S} = L'' \cap \mathfrak{o}$.

(1) *$\mathfrak{S}$ is a Krull ring.*

Proof. $\mathfrak{o} = \cap \mathfrak{o}_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all prime ideals of rank 1 in $\mathfrak{o}$ and $\mathfrak{S} = \cap (L'' \cap \mathfrak{o}_{\mathfrak{p}})$. Siece $\mathfrak{o}_{\mathfrak{p}}$ is a discrete valuation ring, $L'' \cap \mathfrak{o}_{\mathfrak{p}}$ is also a discrete valuation ring. For an element $a \neq 0$ of $\mathfrak{S}$, there exist only a finite number of $\mathfrak{p}$'s such that a is a non-unit in $\mathfrak{o}_{\mathfrak{p}}$ (hence in $L'' \cap \mathfrak{o}_{\mathfrak{p}}$) and $\mathfrak{S}$ is a Krull ring.

(2) *If* $\mathfrak{q}$ *is a prime ideal of rank* 1 *in* $\mathfrak{S}$, *then there exists a prime ideal* $\mathfrak{p}$ *of rank* 1 *in* $\mathfrak{o}$ *which lies over* $\mathfrak{q}$.

Proof. Since $\mathfrak{S} = \cap (L'' \cap \mathfrak{o}_{\mathfrak{p}})$, there exists one $\mathfrak{p}$ such that $L'' \cap \mathfrak{o}_{\mathfrak{p}} = \mathfrak{S}_{\mathfrak{q}}$ (see [3] or [5]), which proves our assertion.

PROPOSITION A. *There exists a normal affine ring* $\mathfrak{o}^*$ *of* L'' *such that* 1) $\mathfrak{o}^* \subseteq \mathfrak{S}$ *and* 2) *for every prime ideal* $\mathfrak{q}$ *of rank* 1 *in* $\mathfrak{S}$, $\mathfrak{q} \cap \mathfrak{o}^*$ *is of rank* 1.

Proof. Let A be the affine model defined by $\mathfrak{o}$. Let $\mathfrak{o}'$ be a normal affine ring of L'' contained in $\mathfrak{S}$ and let Ω be the set of prime ideals $\mathfrak{q}$ of rank 1 in $\mathfrak{S}$ such that rank $(\mathfrak{q} \cap \mathfrak{o}') > 1$. By (2), $\mathfrak{o}'_{(\mathfrak{q} \cap \mathfrak{o}')}$ corresponds to a spot of rank 1 in A, which shows that

$\mathfrak{o}'_{(\mathfrak{q}\cap\mathfrak{o}')}$ is an isolated fundamental spot with respect to A and the number of such spots are in a finite number and $\mathfrak{Q}$ is a finite set by the finiteness of isolated transforms (see [4, IV]). Since there exists a prime ideal $\mathfrak{p}$ of rank 1 in $\mathfrak{o}$ such that $\mathfrak{s}_{\mathfrak{q}}=\mathfrak{o}_{\mathfrak{p}}\cap L''$, $\dim \mathfrak{s}_{\mathfrak{q}}=\dim L''-1$ (for each $\mathfrak{q}$).[3] Since $\mathfrak{o}'_{(\mathfrak{q}\cap\mathfrak{o}')}\neq\mathfrak{s}_{\mathfrak{q}}$, $\dim \mathfrak{s}_{\mathfrak{q}}$ is greater than $\dim \mathfrak{o}'_{(\mathfrak{q}\cap\mathfrak{o}')}$ and there exists an element $x\in\mathfrak{s}$ such that the residue class of x modulo $\mathfrak{q}$ is algebraically independent over $\mathfrak{o}'/(\mathfrak{q}\cap\mathfrak{o}')$. Let $\mathfrak{o}''$ be the derived normal ring of the affine ring generated by these x's over $\mathfrak{o}'$. By the finiteness of $\mathfrak{Q}$, after a finite steps of this procedure we reach to the case where $\mathfrak{Q}$ is empty, which proves Proposition A.

(3) *With the* $\mathfrak{o}^*$ *in Proposition A, there exist only a finite number of prime ideals* $\mathfrak{p}^*$ *of rank* 1 *in* $\mathfrak{o}^*$ *such that there exists no prime ideal of rank* 1 *in* $\mathfrak{s}$ *which lies over* $\mathfrak{p}^*$.

Proof. By (2), such $\mathfrak{o}^*_{\mathfrak{p}^*}$ does not correspond to any spot in the affine model A defined by $\mathfrak{o}$ and we see the finiteness of $\mathfrak{p}^*$'s (see [3, IV]).

PROPOSITION B. *Let* $\mathfrak{a}^*$ *be the intersection of the prime ideals* $\mathfrak{p}^*$ *in (3). Then* $\mathfrak{s}=\mathfrak{o}\cap L'$ *is the* $\mathfrak{a}^*$*-traesform of* $\mathfrak{o}^*$

This follows immediately from Lemma 6.

Now this Proposition B settles Proposition 1.

§ 3. Some properties of the $\mathfrak{a}$-transform.

PROPOSITION 2. *Let* $\mathfrak{o}$ *and* $\mathfrak{o}'$ *be rings which contain the same ring* I *and set* $\mathfrak{o}''=\mathfrak{o}\otimes_I\mathfrak{o}'$. *Assume that for every ideal* $\mathfrak{a}$ *of* $\mathfrak{o}$, $\mathfrak{a}\mathfrak{o}''=\mathfrak{a}\otimes_I\mathfrak{o}'$. *Then for every pair of ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *in* $\mathfrak{o}$, $(\mathfrak{a}\cap\mathfrak{b})\mathfrak{o}''=\mathfrak{a}\mathfrak{o}''\cap\mathfrak{b}\mathfrak{o}''$. *Furthermore, if* $\mathfrak{b}$ *is generated by a finite number of non-zero-divisors in* $\mathfrak{o}''$, *then* $(\mathfrak{a}:\mathfrak{b})\mathfrak{o}''=\mathfrak{a}\mathfrak{o}'':\mathfrak{b}\mathfrak{o}''$.

Proof. Since $0\to\mathfrak{a}\to\mathfrak{a}+\mathfrak{b}\to\mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b})\to0$ is exact, we see that $\mathfrak{a}\otimes\mathfrak{o}'\to(\mathfrak{a}+\mathfrak{b})\otimes\mathfrak{o}'\to(\mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b}))\otimes\mathfrak{o}'\to0$ is exact. By our assumption, it follows that $0\to\mathfrak{a}\mathfrak{o}''\to\mathfrak{a}\mathfrak{o}''+\mathfrak{b}\mathfrak{o}''\to\mathfrak{b}\mathfrak{o}''/(\mathfrak{a}\cap\mathfrak{b})\mathfrak{o}''\to0$ is exact. But it is obvious that $0\to\mathfrak{a}\mathfrak{o}''\to\mathfrak{a}\mathfrak{o}''+\mathfrak{b}\mathfrak{o}''\to\mathfrak{b}\mathfrak{o}''/(\mathfrak{a}\mathfrak{o}''\cap\mathfrak{b}\mathfrak{o}'')\to0$ is exact and

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3) Let $x_1,\cdots,x_r$ be a maximal set of elements of $\mathfrak{o}$ such that 1) they are algebraically independent over L'' and 2) their residue classes modulo $\mathfrak{p}$ are also algebraically independent over $\mathfrak{s}/\mathfrak{q}$. Then the function field L of $\mathfrak{o}$ is algebraic over the field of quotients of $\mathfrak{s}_{\mathfrak{q}}(x_1,\cdots,x_r)$ and $\mathfrak{s}_{\mathfrak{q}}(x_1,\cdots,x_r)$ is domidated by $\mathfrak{o}_{\mathfrak{p}}$. Since $\mathfrak{s}_{\mathfrak{q}}(x_1,\cdots,x_r)$ and $\mathfrak{o}_{\mathfrak{p}}$ are discrete valuation rings, $\dim \mathfrak{o}_{\mathfrak{p}}=\dim \mathfrak{s}_{\mathfrak{q}}(x_1,\cdots,x_r)=\dim \mathfrak{s}_{\mathfrak{q}}+r$. But $r=\dim L-\dim L''$ and $\dim \mathfrak{o}_{\mathfrak{p}}=\dim L-1$. Therefore $\dim \mathfrak{s}_{\mathfrak{q}}=\dim L''-1$. For the notation $\mathfrak{s}_{\mathfrak{q}}(x_1\cdots,x_r)$, see [4, I] or foot-note 6) below.

we have $(\mathfrak{a}\cap\mathfrak{b})\mathfrak{o}''=\mathfrak{a}\mathfrak{o}''\cap\mathfrak{b}\mathfrak{o}''$. Now we assume that $\mathfrak{b}$ is generated by a finite number of non-zero-divisors $\{b_i\}$. For each $b_i=b$, $(\mathfrak{a}\cap b\mathfrak{o})\mathfrak{o}''=\mathfrak{a}\mathfrak{o}''\cap b\mathfrak{o}''$. But $b(\mathfrak{a}:b\mathfrak{o})=\mathfrak{a}\cap b\mathfrak{o}$ and $b(\mathfrak{a}\mathfrak{o}'':b\mathfrak{o}'')=\mathfrak{a}\mathfrak{o}''\cap b\mathfrak{o}''$. Therefore we have $\mathfrak{a}\mathfrak{o}'':b\mathfrak{o}''=(\mathfrak{a}:b\mathfrak{o})\mathfrak{o}''$. Since $\mathfrak{a}:\mathfrak{b}=\cap_i(\mathfrak{a}:b_i\mathfrak{o})$ and since $\mathfrak{a}\mathfrak{o}'':\mathfrak{b}\mathfrak{o}''=\cap_i(\mathfrak{a}\mathfrak{o}'':b_i\mathfrak{o}'')$, we have the required equality.

REMARK. In the above Proposition 2, if I is a Dedekind domain, $\mathfrak{o}$ and $\mathfrak{o}'$ are integral domains, then it holds $\mathfrak{a}\mathfrak{o}''=\mathfrak{a}\otimes\mathfrak{o}'$ for every ideal $\mathfrak{a}$ of $\mathfrak{o}$. For, $\mathfrak{a}$ is torsion-free and so is $\mathfrak{a}\otimes\mathfrak{o}'$, and $\mathfrak{a}\otimes\mathfrak{o}'$ is contqined in $\mathfrak{o}''$ (see [4, II]). In this case, we need not assume that b_i's are non-zero-divisors, as is easily seen.

Let $\mathfrak{a}$ be an ideal of an integral domain $\mathfrak{o}$. We say that the $\mathfrak{a}$-transform $\mathfrak{S}$ is *finite* if there exists an integer n such that $\mathfrak{S}=\mathfrak{o}[\mathfrak{a}^{-n}]$. When $\mathfrak{o}$ is Noetherian and when $\mathfrak{a}\neq 0$, this is equivalent to say that $\mathfrak{S}$ is finitely generated over $\mathfrak{o}$.

LEMMA 7. Let $\mathfrak{o}$ and I' be integral domains containing a Dedekind domain I and let $\mathfrak{a}$ be an ideal of $\mathfrak{o}$ which has a finite base. Assume that $\mathfrak{o}\otimes_I I'$ is an integral domain. Set $\mathfrak{o}_i=\mathfrak{o}[\mathfrak{a}^{-i}]$, and $\mathfrak{a}'=\mathfrak{a}\mathfrak{o}'$. Then $\mathfrak{o}_i\otimes I'=(\mathfrak{o}\otimes I')[\mathfrak{a}'^{-i}]$.

Proof. If $\mathfrak{a}=0$, the assertion is obvious. Assume that $a\neq 0\in\mathfrak{a}$. Then $a^i\mathfrak{o}':\mathfrak{a}^i\mathfrak{o}'=(a\mathfrak{o}:\mathfrak{a}^i)\mathfrak{o}'$, with $\mathfrak{o}'=\mathfrak{o}\otimes I'$ which proves our assertion by virtue of Lemma 1.

LEMMA 8. Let $\mathfrak{a}$ be an ideal of a Noetherian integral domain $\mathfrak{o}$, $\mathfrak{S}$ the $\mathfrak{a}$-transform $\mathfrak{o}$ and $\mathfrak{o}'$ a subring of $\mathfrak{S}$ containing $\mathfrak{o}$. Then $\mathfrak{S}$ is also the $\mathfrak{a}\mathfrak{o}'$-transform of $\mathfrak{o}'$.

Proof. Let $\mathfrak{S}'$ be the $\mathfrak{a}\mathfrak{o}'$-transform of $\mathfrak{o}'$. Since $\mathfrak{a}^{-n}\subseteq\mathfrak{a}\mathfrak{o}'^{-n}$ for every n, $\mathfrak{S}'$ contains $\mathfrak{S}$. Let s be an element of $\mathfrak{S}'$. Then there exists n such that $s\mathfrak{a}^n\subseteq\mathfrak{o}'$. Let $a_1,\cdots,a_t$ be a base of $\mathfrak{a}^n$. Then there exists one n' such that $sa_i\mathfrak{a}^{n'}\subseteq\mathfrak{o}$ and $s\mathfrak{a}^{n+n'}\subseteq\mathfrak{o}$, which shows that $s\in\mathfrak{S}$ and $\mathfrak{S}'\subseteq\mathfrak{S}$. Thus $\mathfrak{S}=\mathfrak{S}'$.

COROLLARY. With the same notations as above, if $a\in\mathfrak{a}$ then $a\mathfrak{S}:\mathfrak{a}\mathfrak{S}=a\mathfrak{S}$ and $\mathfrak{a}\mathfrak{S}$ is not of rank 1.

Let M be a model over a ground ring I. An affine model A over I is called an associated affine model of M if A satisfies the following two conditions:

(1) M is a subset of A and (2) the set of spots of rank 1 in M coincides with that of A.

THEOREM 1. *Let D be a closed set of an affine model A which is different from A and let $\mathfrak{a}$ be an ideal which defines D in the affine ring $\mathfrak{o}$ of A. Then $A-D$ has an associated affine model if* *and only if the $\mathfrak{a}$-transform $\mathfrak{s}$ of $\mathfrak{o}$ is finite; in this case $\mathfrak{s}$ defines an associated affine model.*

Proof. If $\mathfrak{s}$ is finite, then $\mathfrak{s}$ is an affine ring and defines an associated affine model of $A-D$ by Lemma 3 and by the corollary to Lemma 8. Conversely, assume that A' is an associated affine model of $A-D$ and let $\mathfrak{o}'$ be the affine ring of A'. For an element x of $\mathfrak{o}'$ let $\mathfrak{a}_x$ be the set of elements b of $\mathfrak{o}$ such that $bx \in \mathfrak{o}$. Since x is in every spots in $A-D$, the ideal $\mathfrak{a}_x$ is not contained in any prime ideal $\mathfrak{p}$ of $\mathfrak{o}$ such that $\mathfrak{o}_{\mathfrak{p}} \in A-D$ and therefore $\mathfrak{a}_x$ contains $\mathfrak{a}^n$ for an integer n and $x \in \mathfrak{o}[a^{-n}] \subseteq \mathfrak{s}$. Thus $\mathfrak{o}'$ is a subring of $\mathfrak{s}$ and $\mathfrak{s}$ is the $\mathfrak{a}\mathfrak{o}'$-transform of $\mathfrak{o}'$ by Lemma 8. Since every spot of rank 1 in A' is in $A-D$, $\mathfrak{a}\mathfrak{o}'$ is not of rank 1 and $\mathfrak{s}$ is integral over $\mathfrak{o}'$, which shows that $\mathfrak{s}$ is an affine ring.

COROLLARY 1. *Let $\mathfrak{a}$ be an ideal of an affine ring $\mathfrak{o}$. If the $\mathfrak{a}$-transform $\mathfrak{s}$ of $\mathfrak{o}$ is finite, then the $\mathfrak{a}\mathfrak{o}'$-transform of $\mathfrak{o}'$ is also finite for an arbitrary finite integral extension $\mathfrak{o}'$ of $\mathfrak{o}$.*

Proof. Let A^* be the affine model defined by $\mathfrak{s}[\mathfrak{o}']$. Then A^* defines an associated affine model of $A'-D'$, where A' is the affine model defined by $\mathfrak{o}'$ and D' is the closed set of A' defined by $\mathfrak{a}\mathfrak{o}'$. It follows that the $\mathfrak{a}\mathfrak{o}'$-transform of $\mathfrak{o}'$ is finite.

COROLLARY 2. *Let $\mathfrak{a}$ be an ideal of a normal affine ring $\mathfrak{o}$. If there exists a finite integral extension $\mathfrak{o}'$ of $\mathfrak{o}$ such that the $\mathfrak{a}\mathfrak{o}'$-transform of $\mathfrak{o}$ is finite, then the $\mathfrak{a}$-transform of $\mathfrak{o}$ is also finite.*

Proof. By Corollary 1, we may assume that $\mathfrak{o}'$ is a normal extension of $\mathfrak{o}$ (i.e., the integral closure of $\mathfrak{o}$ in a normal extension of the function field L of $\mathfrak{o}$). Since $\mathfrak{o}$ is a normal ring, we see that $(\mathfrak{a}\mathfrak{o}')^{-n} \cap L = \mathfrak{a}^{-n}$, which shows that the $\mathfrak{a}$-transform $\mathfrak{s}$ of $\mathfrak{o}$ is the intersection of L with the $\mathfrak{a}\mathfrak{o}'$-transform $\mathfrak{s}'$ of $\mathfrak{o}'$. Since $\mathfrak{o}'$ is a normal extension of $\mathfrak{o}$, $\mathfrak{s}'$ is integral over $\mathfrak{s}$ and $\mathfrak{s}$ is an affine ring.

§ 4. Local observation.

THEOREM 2. *Let $\mathfrak{a}$ be an ideal of an affine ring $\mathfrak{o}$ and let D be the closed set defined by $\mathfrak{a}$ in the affine model A defined by $\mathfrak{o}$. Then the $\mathfrak{a}$-transform of $\mathfrak{o}$ is finite if and only if the $\mathfrak{a}P$-transform of P is finite for every spot $P \in D$.*

Proof. Let $\mathfrak{s}$ be the $\mathfrak{a}$-transform of $\mathfrak{o}$. If $\mathfrak{s}$ is finite, it is obvious that $\mathfrak{a}P$-transform of P is finite for every $P \in D$. Conversely, assume that $\mathfrak{s}$ is not finite. Set $\mathfrak{o}_0 = \mathfrak{o}$ and define $\mathfrak{o}_i$ to be one $\mathfrak{o}[\mathfrak{a}^{-n}]$ containing $\mathfrak{o}_{i-1}[(\mathfrak{a}\mathfrak{o}_{i-1})^{-1}]$ by induction on i. Then $\mathfrak{s}$ is the union of all the $\mathfrak{o}_i$. If there exists one i such that $\mathfrak{a}\mathfrak{o}_i$ is not of rank 1, then $\mathfrak{S}$ is integral over $\mathfrak{o}_i$ and $\mathfrak{S}$ is an affine ring, which is not the case. Let $\mathfrak{a}_i$ be the intersection of prime divisors of $\mathfrak{a}\mathfrak{o}_i$ of rank 1. Then $\mathfrak{a}_i \subseteq \mathfrak{a}_{i+1}$ for every i and the union $\mathfrak{a}^*$ of all the $\mathfrak{a}_i$ is an ideal of $\mathfrak{S}$; $\mathfrak{a}^*$ does not contain 1 because $1 \notin \mathfrak{a}_i$ for every i. Let $\mathfrak{p}^*$ be a prime ideal containing $\mathfrak{a}^*$ and set $\mathfrak{p} = \mathfrak{p}^* \cap \mathfrak{o}$. Then $P = \mathfrak{o}_\mathfrak{p}$ is in D and the $\mathfrak{a}P$-transform of P is not finite.

§ 5. Affine models contained in an affine model.

Let A be the affine model defined by an affine ring $\mathfrak{o}$ and let D be the closed set of A defined by an ideal $\mathfrak{a}$ of $\mathfrak{o}$.

We say that D is divisorial if every irreducible component of D is the locus of a spot of rank 1.

THEOREM 3. *$A-D$ is an affine model if and only if $1 \in \mathfrak{a}\mathfrak{S}$, where $\mathfrak{S}$ is the $\mathfrak{a}$-transform of $\mathfrak{o}$. In this case, D is a divisorial closed set of A.*

Proof. If $A-D$ is an affine model, then it is an associated affine model of $A-D$ and $\mathfrak{S}$ is finite and $\mathfrak{S}$ is integral over the affine ring of $A-D$ by Theorem 1 (and by the proof of Theorem 1). Since the affine model defined by $\mathfrak{S}$ contains $A-D$, $\mathfrak{S}$ defines $A-D$ and $\mathfrak{a}\mathfrak{S}$ contains 1. Conversely, assume that $1 \in \mathfrak{a}\mathfrak{S}$. Then there exists one n such that $\mathfrak{a}\mathfrak{a}^{-n}$ contains 1. Then $\mathfrak{o}[\mathfrak{a}^{-n}]$ defines $A-D$ and $A-D$ is an affine model. We shall show now that D is divisorial under the assumption that $A-D$ is an affine model. Assuming the contrary, let D' be an irreducible component of D which is not divisorial. Let x be an element of $\mathfrak{o}$ which is not zero on D' but is zero on every other component of D. Then considering $\mathfrak{o}[1/x]$, we may assume that $D=D'$. Then $\mathfrak{S}$ is integral over $\mathfrak{o}$ and $1 \notin \mathfrak{a}\mathfrak{S}$, which is a contradiction.

THEOREM 3a. *$A-D$ is an affine model if and only if $\mathfrak{a}(\mathfrak{a}P)^{-n(P)}$ contains for every spot $P \in D$ (with an integer $n(P)$ which may depend on P).*

Proof. Only if part is an immediate consequedce of Theorem 3. We shall prove the if part. The condition $1 \in \mathfrak{a}(\mathfrak{a}P)^{-n(P)}$ is equivalent to $1 \in \mathfrak{a}\mathfrak{S}_P$ where $\mathfrak{S}_P$ is the $\mathfrak{a}P$-transform of P. Therefore by Theorem 2 we have the finiteness of the $\mathfrak{a}$-transform $\mathfrak{S}$ of $\mathfrak{o}$. Then we have $1 \in \mathfrak{a}\mathfrak{S}$ and $A-D$ is an affine model.

COROLLARY 1. *$A-D$ is an affine model if and only if there* *are affiine models* A_i *such that* 1) A *is the union of all the* A_i*'s and* 2) $A_i-(A_i\cap D)$ *is an affine model for every* A_i.

COROLLARY 2. *If there exists an ideal* $\mathfrak{a}(P)$ *of* $\mathfrak{o}$ *which defines* D *such that* $\mathfrak{a}(P)P$ *is principal for every* $P\in D$, *then* $A-D$ *is an affine model.*[4] *In particular, if every spot in* D *is a unique factorization ring and if* D *is a divisorial closed set, then* $A-D$ *is an affine model.*[5]

COROLLARY 3. *If* A *is of dimension* 1, *then* $A-D$ *is an affiine model.*

Proof. We have only to treat when D is consisted of one spot P. If P is normal, then the assertion follows from Theorem 3*a*. Let P' be the derived normal ring of P and let $\mathfrak{c}$ be the conductor of P' with respect to P. Then there exists one n such that $\mathfrak{a}^n\subseteq\mathfrak{c}$ and therefore considering $\mathfrak{o}[\mathfrak{a}^{-n}]$ (by virtue of Lemma 8) we can reduce to the case where D is consisted of normal spots, which proves our assertion.

LEMMA 9. Let $\mathfrak{a}$ be an ideal of an integral domain $\mathfrak{o}$ and let $\mathfrak{S}$ be the $\mathfrak{a}$-transform of $\mathfrak{o}$. If $b\in\mathfrak{o}$ generates a prime ideal in $\mathfrak{o}$ and if $\mathfrak{a}\nsubseteq b\mathfrak{o}$, then $b\mathfrak{S}$ is also a prime ideal.

Proof. Let a be a non-zero element of $\mathfrak{a}$ which is not in $b\mathfrak{o}$. Then an arbitrary element of $\mathfrak{S}$ is expressed in the form q/a^n with $q\in a^n\mathfrak{o}:\mathfrak{a}^n$ (with a suitable n) by Lemma 1. Assume that $(q/a^n)(q'/a^n)\in b\mathfrak{S}$ $(q, q'\in a^n\mathfrak{o}:\mathfrak{a}^n)$. Then we may assume that $qq'/a^{2n}=bq''/a^{2n}$ $(q''\in a^{2n}\mathfrak{o}:\mathfrak{a}^{2n})$ (by a suitable choice of a sufficiently large n). Since $b\mathfrak{o}$ is prime and since $a\notin b\mathfrak{o}$, we have one of q, q', say q, is in $b\mathfrak{o}$. Since $(a^n\mathfrak{o}:\mathfrak{a}^n):b\mathfrak{o}=a^n\mathfrak{o}:b\mathfrak{a}^n=(a^n\mathfrak{o}:b\mathfrak{o}):\mathfrak{a}^n=a^n\mathfrak{o}:\mathfrak{a}^n$, we have $q=bq^*$ with $q^*\in a^n\mathfrak{o}:\mathfrak{a}^n$, which shows that $q/a^n\in b\mathfrak{S}$ and $b\mathfrak{S}$ is a prime ideal.

LEMMA 10. Let $\mathfrak{a}$ be an ideal of an integral domain $\mathfrak{o}$, let $\mathfrak{S}$ be the $\mathfrak{a}$-transform of $\mathfrak{o}$ and let $\mathfrak{o}'$ be the integral closure of $\mathfrak{o}$ in $\mathfrak{S}$. Assume that $\mathfrak{o}'$ is Noetherian. Then for every element x of the derived normal ring $\mathfrak{o}^*$ of $\mathfrak{o}$ which is not in $\mathfrak{o}'$, the conductor $\mathfrak{c}$ of $\mathfrak{o}'[x]$ with respect to $\mathfrak{o}'$ has no prime divisor containing $\mathfrak{a}\mathfrak{o}'$.

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4) Prof. J-P. Serre told me a proof of the fact that if a positive divisor D on an affine variety A is everywhere locally linearly equivalent to zero, then the complement A' of the carrier of D in A is affine. Though his result is a special case of this statement (i. e., it corresponds to the case where $\mathfrak{a}(P)$ can be chosen to be independent on P), his proof gave the writer a good hint and the writer want to express his thanks to Prof. J-P. Serre.

5) Cf. a result on non-singular varieties in Zariski [6, p. 163].

Proof. This follows immediately from Lemma 8.

COROLLARY. With the same notations as in Lemma 10, we assume further that $\mathfrak{o}'$ is Noetherian. Then for every prime divisor $\mathfrak{p}'$ of $\mathfrak{a}\mathfrak{o}'$ of rank 1, 1) there exists an $a(\neq 0)$ of $\mathfrak{p}'$ such that $a\mathfrak{o}'$ has no imbedded prime divisor and 2) $\mathfrak{o}'_{\mathfrak{p}'}$ is a discrete valuation ring.

Proof. The existence of a follows from a result in [5] and 2) is easy.

REMARK. If a finite number of prime ideals of rank 1 in $\mathfrak{o}'$ are given and if none of them coincides with $\mathfrak{p}'$, then we can choose a so that a is not in any of the prime ideals.

§ 6. Affine models of dimension 2.

THEOREM 4. *If D is a closed set of an affine model A of dimension 2 $(A\neq D)$, then $A-D$ has an associated affine model.*

Proof. Let $\mathfrak{o}$ be the affine ring of A and let $\mathfrak{a}$ be an ideal of $\mathfrak{o}$ which defines D. Let $\mathfrak{S}$ be the $\mathfrak{a}$-transform of $\mathfrak{o}$. We have only to show that $\mathfrak{S}$ is an affine ring by Theorem 3. Since the integral closure of $\mathfrak{o}$ in $\mathfrak{S}$ is a finite $\mathfrak{o}$-module, we may assume that every prime divisor $\mathfrak{p}$ of rank 1 of $\mathfrak{a}$ contains an element $a(\neq 0)$ such that $a\mathfrak{o}$ has no imbedded prime divisor and $\mathfrak{o}_{\mathfrak{p}}$ is a discrete valuation ring by the corollary to Lemma 10. On the other hand, we may assume that every prime divisor of $\mathfrak{a}$ is of rank 1. If $1\in\mathfrak{a}\mathfrak{S}$, then $\mathfrak{S}$ is finite and we assume that $1\notin\mathfrak{a}\mathfrak{S}$. Let $\mathfrak{m}'$ be a prime ideal of $\mathfrak{S}$ containing $\mathfrak{a}\mathfrak{S}$ and set $\mathfrak{m}=\mathfrak{m}'\cap\mathfrak{o}$. Since $\mathfrak{a}\mathfrak{S}$ is not of rank 1 by the corollary to Lemma 8, $\mathfrak{m}'$ is not of rank 1 and $\mathfrak{m}$ is of rank 2. Let x be a transcendental element over $\mathfrak{o}$ and let b and c be elements of $\mathfrak{m}$ such that $b\mathfrak{o}$ and $c\mathfrak{o}$ have no common prime divisor. Then $bx+c$ generates a prime ideal in $I(x)[\mathfrak{o}]$, where I is the ground ring of $\mathfrak{o}$.[6] By virtue of Lemma 7, we may assume that there exists an element $b(\neq 0)$ of $\mathfrak{m}$ such that 1) $\mathfrak{a}\nsubseteq b\mathfrak{o}$ and 2) $b\mathfrak{o}$ is a prime ideal. Then by Lemma 9 $b\mathfrak{S}$ is also a prime ideal and $\mathfrak{S}/b\mathfrak{S}$ is a subring of the field of quotients of $\mathfrak{o}/b\mathfrak{o}$, which shows that $\mathfrak{S}/b\mathfrak{S}$ is a Noetherian ring of rank 1 and $\mathfrak{S}/\mathfrak{m}'$ is a finite algebraic extension of $\mathfrak{o}/\mathfrak{m}$ by Krull-Akizuki's theorem (see [2] or

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6) When x is a transcendental element over a ring I, $I(x)$ denotes the ring $I[x]_S$ with the intersection S of complements of prime ideals generated by elements of $\mathfrak{o}$.

[5]). In particular, $\mathfrak{m}'$ has a finite base. We shall shows that $\mathfrak{S}_{\mathfrak{m}'}$ is Noetherian. Since $\mathfrak{m}'$ has a finite base, we have only to prove that $\mathfrak{q}'\mathfrak{S}_{\mathfrak{m}'}$ has a finite base for every prime ideal $\mathfrak{q}'$ of rank 1 contained in $\mathfrak{m}'$ by virtue of a theorem of Cohen [2] (cf. [5]). Set $\mathfrak{q}=\mathfrak{q}'\cap\mathfrak{o}$. By Lemma 3, $\mathfrak{o}_{\mathfrak{q}}=\mathfrak{S}_{\mathfrak{q}'}$. Therefore $\mathfrak{q}''=\mathfrak{q}\mathfrak{S}_{\mathfrak{m}'}:\mathfrak{q}'\mathfrak{S}_{\mathfrak{m}'}$ is a primary ideal belonging to $\mathfrak{m}'\mathfrak{S}_{\mathfrak{m}'}$. Since $\mathfrak{m}'$ has a finite base, $\mathfrak{q}''$ contains a power of $\mathfrak{m}'$ and $\mathfrak{q}''$ has a finite base. Since $\mathfrak{S}/\mathfrak{q}'$ is a subring of the field of quotients of $\mathfrak{o}/\mathfrak{q}$, $\mathfrak{S}/\mathfrak{q}'$ is Noetherian by Krull-Akizuki's theorem. Set $\mathfrak{q}^*=\mathfrak{q}'\mathfrak{S}_{\mathfrak{m}'}$. Since $\mathfrak{q}''/\mathfrak{q}^*\mathfrak{q}''$ is a finite $(\mathfrak{S}_{\mathfrak{m}'}/\mathfrak{q}^*)$-module, $(\mathfrak{q}^*\cap\mathfrak{q}'')/\mathfrak{q}^*\mathfrak{q}''$ is a finite module and $\mathfrak{q}^*\cap\mathfrak{q}''$ is finite modulo $\mathfrak{q}\mathfrak{S}_{\mathfrak{m}'}$, which shows that $\mathfrak{q}^*\cap\mathfrak{q}''$ has a finite base. Since $\mathfrak{S}_{\mathfrak{m}'}/\mathfrak{q}^*$ and $\mathfrak{S}_{\mathfrak{m}'}/\mathfrak{q}''$ are Noetherian, $\mathfrak{S}_{\mathfrak{m}'}/(\mathfrak{q}^*\cap\mathfrak{q}'')$ is Noetherian and $\mathfrak{q}^*$ is finite modulo $\mathfrak{q}^*\cap\mathfrak{q}''$, which shows that $\mathfrak{q}^*$ has finite base. Thus $\mathfrak{S}_{\mathfrak{m}'}$ is Noetherian. On the other hand, since $\mathfrak{m}'$ has a finite base, we may assume that $\mathfrak{m}'$ is generated by $\mathfrak{m}$ by virtue of Lemma 7 (and repeat the same reduction as in the beginning of the present proof). Let $\mathfrak{o}^*$ be the derived normal ring of $\mathfrak{o}_{\mathfrak{m}}$ and set $\mathfrak{S}^*=\mathfrak{S}_{\mathfrak{m}'}[\mathfrak{o}^*]$, $\mathfrak{t}^*=\mathfrak{o}_{\mathfrak{m}}[\mathfrak{o}^*]$. $\mathfrak{S}^*$ and $\mathfrak{o}^*$ are semi-local rings and for exery maximal ideal $\mathfrak{m}^*$ of $\mathfrak{S}^*$, the local ring $\mathfrak{S}^*_{\mathfrak{m}^*}$ dominates $\mathfrak{o}^*_{(\mathfrak{m}^*\cap\mathfrak{o}^*)}$, $\mathfrak{o}^*_{(\mathfrak{m}^*\cap\mathfrak{o}^*)}$ is a normal spot hence is analytically irreducible (see [4, I]), $\mathfrak{S}^*_{\mathfrak{m}^*}/(\mathfrak{m}^*\cap\mathfrak{o}^*)\mathfrak{S}^*_{\mathfrak{m}^*}$ is a finite $\mathfrak{o}^*/(\mathfrak{m}^*\cap\mathfrak{o}^*)$-module and rank $(\mathfrak{m}^*\cap\mathfrak{o}^*)=$ rank $\mathfrak{m}^*(=2)$. Therefore we have $\mathfrak{o}^*_{(\mathfrak{m}^*\cap\mathfrak{o}^*)}=\mathfrak{S}^*_{\mathfrak{m}^*}$.

Now, if $\mathfrak{o}$ is normal (observe that if the original $\mathfrak{o}$ is normal, then $\mathfrak{S}$ is normal by Lemma 6 and our reduction of $\mathfrak{o}$ does not less the normality of $\mathfrak{o}$), the above equality shows a contradiction because $\mathfrak{m}$ contains $\mathfrak{a}$. Thus

(*) *If $\mathfrak{o}$ is normal, then $\mathfrak{S}$ is an affine ring.*

Next we consider the general case. Let $\mathfrak{o}''$ be the derived normal ring of $\mathfrak{o}$ and set $\mathfrak{S}''=\mathfrak{S}[\mathfrak{o}'']$. Since rank $\mathfrak{a}\mathfrak{S}\neq1$, dim $\mathfrak{S}/\mathfrak{p}=0$ for every prime ideal $\mathfrak{p}$ of $\mathfrak{S}$ containing $\mathfrak{a}$. Since $\mathfrak{S}''$ is integral over $\mathfrak{S}$, dim $\mathfrak{S}''/\mathfrak{p}''=0$ for every prime ideal $\mathfrak{p}''$ of $\mathfrak{S}''$ which contains $\mathfrak{a}$. Therefore $\mathfrak{p}''$ is a maximal ideal of $\mathfrak{S}''$ and the above equality shows that $\mathfrak{o}''(\mathfrak{p}''\cap\mathfrak{o}'')=\mathfrak{S}''_{\mathfrak{p}''}$. Since $\mathfrak{o}''$ is an affine ring, it follows that rank $\mathfrak{p}''=2$ from the fact that dim $\mathfrak{S}''/\mathfrak{p}''=0$. Thus we have rank $\mathfrak{a}\mathfrak{S}''\neq1$ and therefore the $\mathfrak{a}\mathfrak{o}''$-transform of $\mathfrak{o}''$ is integral over $\mathfrak{S}$ (obviously $\mathfrak{S}$ and $\mathfrak{o}''$ are contained the $\mathfrak{a}\mathfrak{o}''$-transform of $\mathfrak{o}''$ and therefore $\mathfrak{S}''$ is a subring of the $\mathfrak{a}\mathfrak{o}''$-transform of $\mathfrak{o}''$). Since $\mathfrak{o}''$ is a normal ring, $\mathfrak{S}''$ is an affine ring by (*). Since $\mathfrak{S}''$ is a finite integral extension of $\mathfrak{S}$, we see that $\mathfrak{S}$ is also an affine ring. Thus the theorem is proved completely.

THEOREM 5. *If D is a divisorial closed set of a normal affine model A of dimension 2, then $A-D$ is an affine model.*

Proof. Let A' be an associated affine model of $A-D$ (Theorem 4). If a spot $P \varepsilon A$ is fundamental with respect to A', then P corresponds to a spot of rank 1 which is not in $A-D$, which is impossible because A' is an associated affine model of $A-D$. Therefore $A'=A-D$ because A is normal.

THEOREM 6. *Problem* 1 *is affirmative if dim L' is not greater than* 2. (Cf. Zariski [6].)[7]

Proof. This follows from Proposition B, Corollary 3 to Theorem 3a and Theorem 4.

§ 7. Supplementary remarks.

I) *Change of ground rings.*

Let $\mathfrak{a}$ be an ideal of a normal affine ring $\mathfrak{o}$ and let $\mathfrak{s}$ be the $\mathfrak{a}$-transform of $\mathfrak{o}$. Then $\mathfrak{s}$ is finite if and only if there exists a finite integral extension $\mathfrak{o}'$ of $\mathfrak{o}$ such that the $\mathfrak{a}\mathfrak{o}'$-transform is finite. Let I be a ground ring of $\mathfrak{o}$ and let I' be the integral closure of I in $\mathfrak{o}'$. We can choose $\mathfrak{o}'$ so that $\mathfrak{o}'$ is a regular extension of I' (that is, the field of quotients of $\mathfrak{o}'$ is a regular extension of that of I'; see [4, II]). Thus

PROPOSITION 3. *In order to discuss Problem* 1 (*or* 2 *or* 3), *we may assume that the affine ring* $\mathfrak{o}$ (*or* $\mathfrak{o}^*$) *is a regular extension of the ground ring of consideration.*

Next, let $\mathfrak{a}$ be an ideal of an affine ring $\mathfrak{o}$ over a ground ring I. Assume that $\mathfrak{o}$ is a regular extension of I. Let I' be a ground ring which is an integral extension of $I(T)$ with a set T of algebraically independent elements over I. Then Lemma 7 and

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7) Zariski [6] proved really the following result:

Let $\mathfrak{o}$ be a normal affine ring of a function field L over a ground field k and let L' be a subfield of L containing k. If dim $L' \leq 2$ and if k is of characteristic zero, then $\mathfrak{o} \cap L'$ is an affine ring.

He needed the assumption that k is of characteristic zero only for the validity of the local uniformization theorem. Since the local uniformization theorem for surfaces over an arbitrary field was proved by Abhyankar [1], his proof is valid also for non-zero characteristic case.

Theorem 1 shows that the finiteness of the $\mathfrak{a}$-transform of $\mathfrak{o}$ is equivalent to the finiteness of the $\mathfrak{a}I'[\mathfrak{o}]$-transform of $I'[\mathfrak{o}]$. In particular,

PROPOSITION 4. *In order to discuss Problem 1 (or 2 or 3), we can extend ground rings to those of type of I' above. In particular, if the ground ring is a field, then we may assume that the ground ring is algebraically closed.*

II) *An easy consequence of Proposition A.*

THEOREM 7. *Let $\mathfrak{o}$ be a normal affine ring over a ground field k. Let L' be a function field over k and set $\mathfrak{s}=L'\cap\mathfrak{o}$. Let L'' be the field of quotients of $\mathfrak{s}$. Then there exists a function field L^* contained in L'' such that 1) L'' is a finite algebraic extension of L^* and 2) $\mathfrak{o}\cap L^*$ is a polynomial ring over k and is an affine ring of L^*.*

Proof. We use the same notations an in Proposition A. We may assume that no non-unit in $\mathfrak{o}^*$ is a unit in $\mathfrak{s}$. Let $x_1, \cdots, x_r (\in \mathfrak{o}^*)$ be algebraically independent elements over k such that $\mathfrak{o}^*$ is integral over $k[x_1, \cdots. x_r]$, Let L^* be the field of quotients of this polynomial ring and the assertion is easy.

III) *A remark to the proof of Theorem 4.*

Our proof of Theorem 4 depends only to the fact that i) $\mathfrak{s}_{\mathfrak{m}}$ is a Noetherian ring and ii) rank $\mathfrak{m}=2$ for every maixmal ideal $\mathfrak{m}$ of $\mathfrak{s}$ which contains the ideal $\mathfrak{a}$. Therefore a similar argument shows the following assertion:

PROPOSITION 5. *Let $\mathfrak{a}$ be an ideal of an affine ring $\mathfrak{o}$ and let $\mathfrak{s}$ be the $\mathfrak{a}$-transform of $\mathfrak{o}$. If i) $\mathfrak{s}_{\mathfrak{m}}$ is a Noetherian ring and ii) rank $\mathfrak{m}$ + dim $\mathfrak{s}/\mathfrak{m}$ = (dimension of the function field of $\mathfrak{o}$) for every minimal prime divisor $\mathfrak{m}$ of $\mathfrak{a}\mathfrak{s}$ (or, for every maximal ideal $\mathfrak{m}$ of $\mathfrak{s}$ containing $\mathfrak{a}$), then $\mathfrak{s}$ is an affine ring.*

IV) *A remark to Theorem 5.*

Theorem 5 cannot be generalized in the same form to higher dimensional case. For example, let k be a field and let $\mathfrak{o}$ be the affine ring $k[x, y, z, w]/(xy+zw)$. Let $\mathfrak{p}$ be the prime ideal of $\mathfrak{o}$ generated by x and z. Then setting $u=w/x$, $\mathfrak{o}[\mathfrak{p}^{-1}]=k[x, z, u]$ and $\mathfrak{p}\mathfrak{o}[\mathfrak{p}^{-1}]$ is of rank 2. Therefore the $\mathfrak{p}$-transform of $\mathfrak{o}$ is $\mathfrak{o}[\mathfrak{p}^{-1}]$ but $1 \notin \mathfrak{p}\mathfrak{o}[\mathfrak{p}^{-1}]$. (If we denote by A and A' the affine models defined by $\mathfrak{o}$ and $\mathfrak{o}[\mathfrak{p}^{-1}]$ respectively, then the spot $\mathfrak{o}_{(x, y, z, w)}$ is the unique isolated fundamental spot with respect to A' and the locus of the spot $k[x, z, u]_{(x, z)}$ in A' is the transform of the spot $\mathfrak{o}_{(x, y, z, w)}$. Thus, even when $A-D$, A being a normal affine model and D a divisorial closed set of A, has an associated affine model, $A-D$ itself may not be an affine model because the locus of an isolated transform of an isolated fundamental spot may not be a divisorial closed set.)

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