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On the group-space of the continuous transformation group with a Riemannian metric

By

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INTRODUCTION. When a group of transformations is semi-simple, it is well known that we can treat the group-space as a Riemannian space being defined the fundamental tensor $g_{\alpha\beta}$ by $g_{ab}A^a_\alpha A^b_\beta$ where $g_{ab}=-c^m_{al}c_{bm}^{\ \ l}$. It is impossible however to assign a Riemannian metric to the group-space when the group considered is not semi-simple because the rank of the matrix $\|c^m_{al}c_{bm}^{\ \ l}\|$ is less than r. However, $g_{\alpha\beta}=g_{ab}A^a_\alpha A^b_\beta$ in which g_{ab} are any constants such that the determinat $|g_{ab}|$ is not zero, may be used to assign a Riemannian metric to the group-space ([1]* p. 206). In this paper, we shall show that defining the funbamental tensor by $\sum_a A^a_\alpha A^a_\beta$ we can treat the group-space of any transformation group as a Riemannian space under the transformations of the parameters. That is, although for different sets of symbols $A_a f$ of the same first parmeter-group there correspond different Riemannian spaces in general, for different choices of the parameters there correspond a same Riemannian space. Principally we shall study the properties of these Riemannian spaces concerning with those of the original first parameter-groups, and give an example as an application of this theory.

We shall make use mainly of the notations of L. P. Eisenhart in his work "Continuous Group of Transformotions [1]."

1. PRELIMINARIES.

Let

$$x'^i=f^i(x, a) \qquad (i=1, \cdots, n)$$

* Numbers in brackets refer to the references at the end of the paper.

be the equations of a continuous transformation group G_r with n independent variables x^i and r essential parameters a^α, and

$$a_3^\alpha = \varphi^\alpha(a_1, a_2) \qquad (\alpha = 1, \cdots, r) \tag{1·1}$$

be the equations of the parameter-group of G_r. That is, the groups defined by (1·1) when a_2^α and a_1^α are considered as the parameters are called respectively the first and second parameter-groups of G_r, and we denote them by $\mathfrak{G}_r^{(+)}$ and $\mathfrak{G}_r^{(-)}$. Hereafter when we say merely "parameter-group $\mathfrak{G}_r$", we mean "first parameter-group $\mathfrak{G}_r^{(+)}$".

Let

$$\frac{\partial x'^i}{\partial a^\alpha} = \xi_b^i(x') A_\alpha^b(a) \qquad \begin{pmatrix} i = 1, \cdots, n \\ b, \alpha = 1, \cdots, r \end{pmatrix},$$

$$\frac{\partial a_3^\alpha}{\partial a_2^\beta} = A_b^\alpha(a_3) A_\beta^b(a_2) \qquad (b, \alpha, \beta = 1, \cdots, r)$$

and

$$\frac{\partial a_3^\alpha}{\partial a_1^\beta} = \bar{A}_b^\alpha(a_3) \bar{A}_\beta^b(a_1) \qquad (b, \alpha, \beta = 1, \cdots, r)$$

be the fundamental equations of G_r, $\mathfrak{G}_r^{(+)}$ and $\mathfrak{G}_r^{(-)}$ respectively, where the determinant $|A_b^\alpha|$ is not zero and the matrix $\|A_\alpha^b\|$ is the inverse one of $\|A_b^\alpha\|$, and $\bar{A}_b^\alpha$, $\bar{A}_\alpha^b$ are the ones of $\mathfrak{G}_r^{(-)}$ corresponding to A_b^α, A_α^b of $\mathfrak{G}_r^{(+)}$ respectively. Hence we have the relations $A_\alpha^b A_b^\beta = \delta_\alpha^\beta$ and $A_\alpha^b A_a^\alpha = \delta_a^b$. Of course $\bar{A}_b^\alpha$ and $\bar{A}_\alpha^b$ satisfy the similar relations. When the parameters are changed, $A_b^\alpha(a)$ and $A_\alpha^b(a)$ are transformed as components of a contravariant and a covariant vector respectively.

Let S be the group-space of a parameter-group. There exist three kinds of affine connections ([1]. p. 199) named (+)-connection, (−)-connection and (0)-connection, and the spaces with these connections have been denoted by $S^{(+)}$, $S^{(-)}$ and $S^{(0)}$ respectively in the paper [2]. The coefficients of the connections are given by

$$\begin{cases} L_{\beta\gamma}^\alpha = A_b^\alpha \dfrac{\partial A_\beta^b}{\partial a^\gamma} = -A_\beta^b \dfrac{\partial A_b^\alpha}{\partial a^\gamma}, \\[2ex] \bar{L}_{\beta\gamma}^\alpha = \bar{A}_b^\alpha \dfrac{\partial \bar{A}_\beta^b}{\partial a^\gamma} = -\bar{A}_\beta^b \dfrac{\partial \bar{A}_b^\alpha}{\partial a^\gamma} \end{cases} \tag{1·2}$$

and

(1·3) $$\Gamma^{\alpha}_{\beta\gamma}=\frac{1}{2}(L^{\alpha}_{\beta\gamma}+\bar{L}^{\alpha}_{\beta\gamma})$$

respectively. Since

$$\bar{L}^{\alpha}_{\beta\gamma}=L^{\alpha}_{\gamma\beta}\,,$$

we have also

(1·4) $$\Gamma^{\alpha}_{\beta\gamma}=\frac{1}{2}(L^{\alpha}_{\beta\gamma}+L^{\alpha}_{\gamma\beta})\,.$$

The curvature tensor of $S^{(+)}$ and $S^{(-)}$ are zero, hence both of them are the spaces of affine connections without curvatures. The coefficients $\Gamma^{\alpha}_{\beta\gamma}$ are the symmetric parts of the coefficients $L^{\alpha}_{\beta\gamma}$. Furthermore let the skew-symmetric parts of $L^{\alpha}_{\beta\gamma}$ be $\Omega^{\alpha}_{\beta\gamma}$, then we have

(1·5) $$\Omega^{\alpha}_{\beta\gamma}=\frac{1}{2}(L^{\alpha}_{\beta\gamma}-L^{\alpha}_{\gamma\beta})=\frac{1}{2}\,c_{ab}^{e}A^{a}_{\beta}A^{b}_{\gamma}A^{\alpha}_{e}\,,$$

where c_{ab}^{e} are the constants of structure of $\mathfrak{G}_r^{(+)}$.

2. THE FUNDAMENTAL TENSOR.

In the papers [2] and [3] we have used a set of repères $R_a(a-\vec{A}_1\cdots\vec{A}_r)$ at every point in S, where $\vec{A}_b$ are the vectors whose components are $A^1_b, \cdots, A^r_b$. $A^{\alpha}_a(a)$ and $A^a_{\alpha}(a)$ are transformed contravariantly and covariantly respectively when the parameters are changed. Accordingly, if we define $g_{\alpha\beta}$ by the relations

(2·1) $$g_{\alpha\beta}=A^{a}_{\alpha}A^{a}_{\beta}{}^{*}\ \ (=\sum_{a=1}^{r}A^{a}_{\alpha}A^{a}_{\beta}),$$

then the quantity

$$ds^2=g_{\alpha\beta}\,da^{\alpha}da^{\beta}$$

is an invariant under any transformation of parameters. We shall define this set of $g_{\alpha\beta}$ as the fundamental tensor with respect to a set of repères R_a. The geometrical meaning of this definition is cleared as follows. By this definition, as

(2·2) $$g_{\alpha\beta}A^{\alpha}_{a}=A^{a}_{\alpha}$$

* Throughout this paper, when the same index appears twice in a term this term stands for the sum of the terms obtained by giving the index each of its values.

we have

$$g_{\alpha\beta} A^{\alpha}_{b} A^{\beta}_{c} = \delta_{bc}.$$

Hence $\vec{A}_b$ of R_a is a unit vector and any two different vectors are orthogonal. Consequently every R_a can be treated as a system of orthogonal ennuples, and the metric induced here when it is represented by " composants relatives ω_i" used by E. Cartan coincides with the metric given by $ds^2 = \sum_i (\omega_i)^2$.

Since

$$g_{\alpha\beta} A^{\alpha}_{b} A^{\gamma}_{b} = \delta^{\gamma}_{\beta}$$

we can define $A^{\alpha}_{b} A^{\gamma}_{b}$ as the contravariant components of the fundamental tensor, that is

$$g^{\alpha\beta} = A^{\alpha}_{a} A^{\beta}_{a}.$$

From (2·2) and $g^{\alpha\rho} A^{a}_{\beta} = A^{\alpha}_{a}$, A^{α}_{a} and A^{a}_{α} where a is fixed and $\alpha = 1, \cdots, r$ are regarded as the contravariant and covariant components of the same vector $\vec{A}_a$.

On the other hand when we choose another set of symbols $A'_a f$ instead of the set $A_a f$, being related by

$$A'^{\alpha}_{a}(a) = c^{b}_{a} A^{\alpha}_{b}(a),$$

that is,

$$A'^{a}_{\alpha}(a) = \bar{c}^{a}_{b} A^{b}_{\alpha}(a),$$

where c^{b}_{a} are constants and the determinant $|c^{b}_{a}|$ is not zero, and $\|\bar{c}^{a}_{b}\|$ is the inverse matrix of $\|c^{b}_{a}\|$. The corresponding set of $g'_{\alpha\beta}$ are represented by

$$g'_{\alpha\beta} = \bar{c}^{a}_{b} \bar{c}^{a}_{c} A^{b}_{\alpha} A^{c}_{\beta}. \tag{2·3}$$

Consequently ds'^2 is not equal to ds^2 in general. Thus the metric induced by the above method is useful when we do not change the set of repères. However when the repère, considered an orthogonal ennuples as mentioned above, is subjected to an orthogonal transformation, then $\bar{c}^{a}_{b} \bar{c}^{a}_{c}$ in (2·3) is equal to δ_{bc}. In this case we have $g'_{\alpha\beta} = g_{\alpha\beta}$. Hereafter we denote by $S^{(r)}$ the group-space in which the metric is induced by the fundamental tensor (2·1), and call it the group-space concerned with the repère R_a. Then we have:

THEOREM: *The group-space $S^{(R)}$ concerned with the repère R_a is not different to $S'^{(R)}$ concerned with the repère R_a' which is obtained from R_a by an orthogonal transformation.*

Furthermore, let $\|g_{ab}\|$ be a regular matrix where g_{ab} are the coefficients of any positive definite quadratic form. When we assign a metric to the group-space by using $g_{\alpha\beta}=g_{ab}A^a_\alpha A^b_\beta$ as the fundamental tensor, we can choose a set of symbols $A_1'f, A_2'f, \cdots, A_r'f$ which are related by $A'^\alpha_a=c^b_a A^\alpha_b$ where c^b_a are suitable constants and the determinant $|c^b_a|$ is not zero, so as to have

$$g_{ab}A^a_\alpha A^b_\beta=A'^a_\alpha A'^a_\beta .$$

Consequently, the metric induced by $g_{ab}A^a_\alpha A^b_\beta$ as the fundamental tensor is coincident with the metric concerned with the repère R_a' whose vectors are $\vec{A}_a'$.

The Christoffel symbols of the second kind $\left\{ {\alpha \atop \beta\gamma} \right\}$ of the space are calculated by

$$\left\{ {\alpha \atop \beta\gamma} \right\}=\frac{1}{2}g^{\alpha\delta}\left\{\frac{\partial g_{\beta\delta}}{\partial a^\gamma}+\frac{\partial g_{\gamma\delta}}{\partial a^\beta}-\frac{\partial g_{\beta\gamma}}{\partial a^\delta}\right\}.$$

By (2·1) and (1·2) we have the relations

$$\frac{\partial g_{\alpha\beta}}{\partial a^\gamma}=g_{\lambda\beta}L^{\lambda}_{\alpha\gamma}+g_{\alpha\lambda}L^{\lambda}_{\beta\gamma}. \tag{2·4}$$

Using (2·4) the Christoffel symbols of the first kind $[\beta\gamma, \delta]$ are expressed by

$$\begin{aligned}
[\beta\gamma, \delta]&=\frac{1}{2}\left(\frac{\partial g_{\beta\delta}}{\partial a^\gamma}+\frac{\partial g_{\gamma\delta}}{\partial a^\beta}-\frac{\partial g_{\beta\gamma}}{\partial a^\delta}\right)\\
&=\frac{1}{2}\left\{(g_{\lambda\delta}L^{\lambda}_{\beta\gamma}+g_{\beta\lambda}L^{\lambda}_{\delta\gamma})+(g_{\lambda\delta}L^{\lambda}_{\gamma\beta}+g_{\gamma\lambda}L^{\lambda}_{\delta\beta})-(g_{\lambda\gamma}L^{\lambda}_{\beta\delta}+g_{\beta\lambda}L^{\lambda}_{\gamma\delta})\right\}\\
&=\frac{1}{2}\left\{g_{\lambda\delta}(L^{\lambda}_{\beta\gamma}+L^{\lambda}_{\gamma\beta})+g_{\beta\lambda}(L^{\lambda}_{\delta\gamma}-L^{\lambda}_{\gamma\delta})+g_{\gamma\lambda}(L^{\lambda}_{\delta\beta}-L^{\lambda}_{\beta\delta})\right\}\\
&=g_{\lambda\delta}\Gamma^{\lambda}_{\beta\gamma}+g_{\beta\lambda}\Omega^{\lambda}_{\delta\gamma}+g_{\gamma\lambda}\Omega^{\lambda}_{\delta\beta}.
\end{aligned}$$

Hence

$$\left\{ {\alpha \atop \beta\gamma} \right\}=\Gamma^{\alpha}_{\beta\gamma}+g^{\alpha\delta}g_{\beta\lambda}\Omega^{\lambda}_{\delta\lambda}+g^{\alpha\delta}g_{\gamma\lambda}\Omega^{\lambda}_{\delta\beta}.$$

From (1·5) we have

$$\left\{ {\alpha \atop \beta\gamma} \right\} = \Gamma^{\alpha}_{\beta\gamma} + \frac{1}{2}(c^{e}_{ab} + c^{b}_{ae})\, A^{\alpha}_{a} A^{e}_{\beta} A^{b}_{\gamma}\,. \tag{2·5}$$

If we put

$$D^{\alpha}_{\beta\gamma} = \left\{ {\alpha \atop \beta\gamma} \right\} - \Gamma^{\alpha}_{\beta\gamma} \left(= \frac{1}{2}(c^{e}_{ab} + c^{b}_{ae})\, A^{\alpha}_{a} A^{e}_{\beta} A^{b}_{\gamma} \right), \tag{2·6}$$

these quantities form a tensor in $S^{(R)}$ symmetric with respect to β and γ. We can verify these symbols $\left\{ {\alpha \atop \beta\gamma} \right\}$ satisfy the equations

$$\frac{\partial g_{\alpha\beta}}{\partial a^{\gamma}} = g_{\lambda\beta} \left\{ {\lambda \atop \alpha\gamma} \right\} + g_{\alpha\lambda} \left\{ {\lambda \atop \beta\gamma} \right\}.$$

Of course it is verified that the Christoffel symbols satisfy the relations

$$\frac{\partial^2 a^{\alpha}}{\partial a'^{\lambda} \partial a'^{\mu}} + \left\{ {\alpha \atop \beta\gamma} \right\} \frac{\partial a^{\beta}}{\partial a'^{\lambda}} \frac{\partial a^{\gamma}}{\partial a'^{\mu}} = \left\{ {\lambda \atop \mu\nu} \right\}' \frac{\partial a^{\alpha}}{\partial a'^{\lambda}}$$

when the parameters a^{α} are replaced by a'^{α}.

3. Geodesics.

In a group-space S, the trajectory of one-parameter sub-group with a symbol $e^{a} A_{a} f$, where $(e^{1}, \cdots, e^{r})$ are constants and one of them at least is not zero, is expressed by

$$\frac{da^{\alpha}}{dt} = e^{a} A^{\alpha}_{a}(a) \tag{3·1}$$

where t is a suitable parameter. This trajectory is always a path in $S^{(+)}$, $S^{(-)}$ and $S^{(0)}$. However it is impossible in general that a trajectory is a geodesic in $S^{(R)}$. It was proved by E. Cartan and J. A. Schouten that, in order that the Riemannian geodesics for any group coincide with the trajectory of one-parameter sub-groups, $c^{a}_{ab} = 0$ for $b = 1, \cdots, r$ must hold ([1] p. 207, [4]). This result is however a necessary condition. Let us now study the necessary and sufficient conditions under which a trajectory coincide with a geodesic in $S^{(R)}$. When the arc-length s along the curve (3·1) from a certain point on it is used as a parameter in stead of t, the curve is expressed by

$$\frac{da^\alpha}{ds}=e^b\psi(s)A_b^\alpha .$$

As $\dfrac{da^\alpha}{ds}$ are the components of the unit tangent vector, $\sum\{e^b\psi(s)\}^2$ is equal to 1. Consequently $\psi(s)$ is constant. Denoting $e^a\psi(s)$ by e^α, we have the equations of the curve

$$(3\cdot3) \qquad \frac{da^\alpha}{ds}=e^a A_a^\alpha(a)$$

where $\sum_a(e^a)^2=1$. A necessary and sufficient condition that the curve (3·2) is a geodesic in $S^{(R)}$, is that the relations

$$(3\cdot3) \qquad \frac{d^2a^\alpha}{ds^2}+\begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}\frac{da^\beta}{ds}\frac{da^\gamma}{ds}=0$$

are to be satisfied by (3·2). As

$$\frac{d^2a^\alpha}{ds^2}=e^a\frac{\partial A_a^\alpha}{\partial a^\gamma}\frac{da^\gamma}{ds}=e^a e^b A_b^\gamma\frac{\partial A_a^\alpha}{\partial a^\gamma}=-e^a e^b A_a^\beta A_b^\gamma L_{\beta\gamma}^\alpha ,$$

replacing these results into (3·3), we have

$$(3\cdot4) \qquad \left[\begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}-L_{\beta\gamma}^\alpha\right]e^a e^b A_a^\beta A_b^\gamma=0.$$

Furthermore, exchanging β and γ, and a and b in the above equations and regarding that $\begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}$ are symmetric with respect to β and γ, we have also

$$(3\cdot5) \qquad \left[\begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}-L_{\gamma\beta}^\alpha\right]e^a e^b A_a^\beta A_b^\gamma=0.$$

Adding (3 4) and (3·5), we have from (1.4) and (2·5)

$$(3\cdot6) \qquad e^a e^b A_a^\beta A_b^\gamma D_{\beta\gamma}^\alpha=0 .$$

As (3·4) and (3·6) are equivalent, we have:

THEOREM: *A necessary and sufficient condition that the trajectory of a one-parameter sub-group which is generated by a unit-vector $e^a\vec{A}_a$ where e^a are constants satisfying $\sum_a(e^a)^2=1$ coincides with a geodesic is that the contracted tensor $(e^a A_a^\beta)(e^b A_b^\gamma)D_{\beta\gamma}^\alpha$ is zero.*

Moreover, (3·6) is deformed to

$$(c_{a'b'}^{c'}+c_{a'c'}^{b'})A_{a'}^{\alpha}A_{\beta}^{c'}A_{\gamma}^{b'}\cdot e^{a}e^{b}A_{a}^{\beta}A_{b}^{\gamma}=0.$$

Consequently

$$(c_{a'b}^{a}+c_{a'a}^{b})e^{a}e^{b}A_{a'}^{\alpha}=0.$$

As $|A_{a'}^{\alpha}|\neq 0$, we have

$$(c_{ab}^{e}+c_{ae}^{b})e^{e}e^{b}=0,$$

that is

$$e^{b}e^{e}c_{ab}^{e}=0. \tag{3·7}$$

Hence we have:

THEOREM: *A necessary and sufficient condition that the trajectory of a one-parameter sub-group which is generated by a symbol $e^{a}A_{a}f$ coincides with a geodesic is that $e^{e}e^{b}c_{ab}^{e}$ $(a=1, \cdots, r)$ vanish.*

Furthermore, when the rank of the matrix $\|c_{ab}^{d}\|$ where a denotes the columns and b and d the rows is $s(<r)$, the system of linear equations

$$e^{a}c_{ab}^{d}=0,$$

as e^{a} are considered as unknowns, has $m(=r-s)$ sets of independent solutions $(e_{i}^{1}, \cdots, e_{i}^{r})$, where $i=1, \cdots, m$. Every $e_{i}^{a}A_{a}f$ generates an exceptional sub-group. Hence we have:

THEOREM: *When the rank of the matrix $\|c_{ab}^{d}\|$ where a denotes the columns and b and d the rows is s, in S there exist $r-s$ independent geodesics at least through every point which are the trajectories generated by $r-s$ independent exceptional one-parameter sub-groups.*

If the original parameter-group $\mathfrak{G}_{r}$ has a set of m exceptional one-parameter sub-groups, they form an Abelian sub-group of mth order. When the original group is an Abelian, the corresponding group-space is euclidean because $g_{\alpha\beta}$ are constant. It will be shown also after the time when we calculate the curvature tensor of $S^{(k)}$ in § 5. Accordingly the result of the above theorem is understood on a point of this view.

It is more interesting to notice from the form of (3·7) that there may exists another geodeseic curve being a trajectory of a one- parameter sub-group which is not exceptional. For an example, let $\mathfrak{G}_{r}$ be a direct product of its two invariant sub-groups $\mathfrak{G}_{m}$ and $\mathfrak{G}_{r-m}$ and let their symbol be $A_{1}f, \cdots, A_{m}f$ and $A_{m+1}f, \cdots, A_{r}f$

respectively. Furthermore in this case let $\mathfrak{G}_m$ be Abelian, then $c_{ab}^{d}=0$ for $a, b=1, \cdots, r$ and $d=1, \cdots, m$. Consequently when we choose as a set of constants $(e^1, \cdots, e^r)$ a set $(c^1, \cdots, c^m, 0, \cdots, 0)$, the equations $e^b e^c c_{ab}^{e}=0$ for $a=1, \cdots, r$. Hence the trajectory of the one-parameter sub-group which is generated by this symbol $e^a A_a f$ is a geodesic of $S^{(R)}$.

THEOREM: *When the rank of the matrix $\|c_{ab}^{e}\|$ where e denotes the columnes and a and b rows is $p(<r)$, that is, when the order of the derived group of the original group is p, in $S^{(R)}$ there exist $r-p$ independent geodesics at least through every point which are the trajectories generated by independ $r-p$ one-parameter sub-groups.*

EXAMPLE. Let us consider the group of motion in 2-dimentional euclidean space

$$\begin{cases} x'^1=x^1 \cos a^3-x^2 \sin a^3+a^1, \\ x'^2=x^1 \sin a^3+x^2 \cos a^3+a^2. \end{cases}$$

The parameter-group is given by

$$\begin{cases} a_3^1=a_1^1 \cos a_2^3-a_1^2 \sin a_2^3+a_2^1 , \\ a_2^3=a_1^1 \sin a_2^3+a_1^2 \cos a_2^3+a_2^2 , \\ a_3^3=a_1^3+a_2^3 , \end{cases}$$

and both of the matrices $(A_1^\alpha A_2^\alpha A_3^\alpha)$ and $\begin{pmatrix} A_\alpha^1 \\ A_\alpha^2 \\ A_\alpha^3 \end{pmatrix}$ are determined as follows:

$$\left\| \begin{matrix} A_1^1 & A_2^1 & A_3^1 \\ A_1^2 & A_2^2 & A_3^2 \\ A_1^3 & A_2^3 & A_3^3 \end{matrix} \right\| = \left\| \begin{matrix} 1 & 0 & a^2 \\ 0 & 1 & -a^1 \\ 0 & 0 & 1 \end{matrix} \right\|$$

Consequently the constants of structure are

$$c_{12}^1=c_{12}^2=c_{12}^3=0 ; \quad c_{13}^1=c_{13}^3=0 ; \quad c_{13}^2=-1 ; \quad c_{23}^1=1 ; \quad c_{23}^2=c_{23}^3=0.$$

As the rank of the matrix $\|c_{ab}^{e}\|$ where a denotes the columns is 3, the group has no exceptional sub-group. On the other hand, as the rank of the matrix $\|c_{ab}^{e}\|$ where e denotes the columns is 2, we can choose a set of constants $(c, c', 0)$ as (e^1, e^2, e^3) where one of c and c' is not zero at least, so as to have the relations $e^b e^e c_{ab}^{e}=0$. Consequently the trajectory of the one-parameter sub-group $\mathfrak{G}_1$

generated by the symbol $cA_1f+c'A_2f$ is a geodesic. This $\mathfrak{G}_1$ is a group of translations and a sub-group of the translation group $\mathfrak{G}_2$ in euclidean plane.

4. GEODESIC CURVATURES.

In general the trajectory of a one-parameter sub-group $\mathfrak{G}_1$ is not a geodesic in $S^{(R)}$. Let the symbol of $\mathfrak{G}_1$ be $e^a A_a f$ where e^a are constants satisfying $\sum_a (e^a)^2=1$, then the trajectory generated by $\mathfrak{G}_1$ is given by the equations (3·2). Let μ^α and $K_{(g)}$ be the curvature vector and the geodesic curvature of this curve. Then we have from (3·2), (2·5), (1·5) and (1·2)

$$K_{(g)}\mu^\alpha=\frac{d}{ds}(e^a A_a^\alpha)+\left\{{\alpha \atop \beta\gamma}\right\}e^b A_b^\beta\cdot e^c A_c^\gamma$$

$$=\frac{1}{2}(c_{ac}^{\ \ b}+c_{ab}^{\ \ c})e^b e^c A_a^\alpha$$

$$=e^b e^c c_{ab}^{\ \ c} A_a^\alpha .$$

From the property of the curvature vector, μ^α must be perpendicular to the tangent vector of the curve. It is verified as follows.

$$g_{\alpha\beta}(e^a A_a^\alpha)\mu^\beta=A_\alpha^{a'} A_\beta^{a'}(e^a A_a^\alpha)\cdot e^b e^c c_{eb}^{\ \ c} A_e^\beta/K_{(g)}$$

$$=e^a e^b e^c c_{ab}^{\ \ c}/K_{(g)}$$

$$=\frac{1}{2}\,e^c\cdot e^a e^b(c_{ab}^{\ \ c}+c_{ba}^{\ \ c})/K_{(g)}=0$$

The square of the length of the vector $K_{(g)}\mu^\alpha$ is given by

$$g_{\alpha\beta}(e^b e^c c_{ab}^{\ \ c} A_a^\alpha)(e^{b'} e^{c'} c_{a'b'}^{\ \ \ c'} A_{a'}^\beta)$$

$$=(e^b e^c c_{ab}^{\ \ c})(e^{b'} e^{c'} c_{ab'}^{\ \ \ c'})=e^b e^c e^d e^f c_{ab}^{\ \ c} c_{ad}^{\ \ f} .$$

Accordingly we have:

THEOREM: *At every point in $S^{(R)}$, the geodesic curvature of the trajectory through this point which is generated by the one-parameter sub-group whose symbol is $e^a A_a f$ is constant.*

Consequently, through in each one of the spaces $S^{(+)}$, $S^{(-)}$ and $S^{(0)}$ the trajectory of a one-parameter sub-group is always a geodesic, in the space $S^{(R)}$ all the trajectories which are generated by a same symbol are the curves with a constant geodesic curvature.

The geodesic properties in $S^{(R)}$ mentioned in this section and the previous one are held when the orthogonal repère R_a is transformed by any similar transformation. Hence we have:

THEOREM: *When the repère R_a is transformed by any similar transformation, the geodesic properties in the corresponding $S^{(R)}$ is not altered.*

5. CURVATURE TENSORS.

Let $R^{\alpha}_{\beta\gamma\delta}$ be the curvature tensor of the second kind of $S^{(R)}$. As the Christoffel symbols $\left\{ {\alpha \atop \beta\gamma} \right\} = \Gamma'^{\alpha}_{\beta\gamma} + D^{\alpha}_{\beta\gamma}$, we have

$$R^{\alpha}_{\beta\gamma\delta} = \frac{\partial \left\{ {\alpha \atop \beta\gamma} \right\}}{\partial a^{\delta}} - \frac{\partial \left\{ {\alpha \atop \beta\gamma} \right\}}{\partial a^{\delta}} + \left\{ {\sigma \atop \beta\delta} \right\} \left\{ {\alpha \atop \sigma\gamma} \right\} - \left\{ {\sigma \atop \beta\gamma} \right\} \left\{ {\alpha \atop \sigma\gamma} \right\}$$

$$= \frac{\partial (\Gamma'^{\alpha}_{\beta\delta} + D^{\alpha}_{\beta\delta})}{\partial a^{\gamma}} - \frac{\partial (\Gamma'^{\alpha}_{\beta\gamma} + D^{\alpha}_{\beta\gamma})}{\partial a^{\delta}} + (\Gamma'^{\sigma}_{\beta\delta} + D^{\sigma}_{\beta\delta})(\Gamma'^{\alpha}_{\sigma\gamma} + D^{\alpha}_{\sigma\gamma})$$

$$- (\Gamma'^{\sigma}_{\beta\gamma} + D^{\sigma}_{\beta\gamma})(\Gamma'^{\alpha}_{\sigma\delta} + D^{\alpha}_{\sigma\delta}).$$

Hence we have

$$(5\cdot1) \qquad R^{\alpha}_{\beta\gamma\delta} = \Gamma'^{\alpha}_{\beta\gamma\delta} + D^{\alpha}_{\beta\gamma\delta} + \Gamma'^{\sigma}_{\beta\delta} D^{\alpha}_{\sigma\gamma} - \Gamma'^{\sigma}_{\beta\delta} D^{\alpha}_{\sigma\delta} + D^{\sigma}_{\beta\delta} \Gamma'^{\alpha}_{\sigma\gamma} - D^{\sigma}_{\beta\gamma} \Gamma'^{\alpha}_{\sigma\delta}$$

where

$$(5\cdot2) \qquad \begin{aligned} \Gamma'^{\alpha}_{\beta\gamma\delta} &= \frac{\partial \Gamma'^{\alpha}_{\beta\delta}}{\partial a^{\gamma}} - \frac{\partial \Gamma'^{\alpha}_{\beta\gamma}}{\partial a^{\delta}} + \Gamma'^{\sigma}_{\beta\delta} \Gamma'^{\alpha}_{\sigma\gamma} - \Gamma'^{\sigma}_{\beta\gamma} \Gamma'^{\alpha}_{\sigma\delta}, \\ D^{\alpha}_{\beta\gamma\delta} &= \frac{\partial D^{\alpha}_{\beta\delta}}{\partial a^{\gamma}} - \frac{\partial D^{\alpha}_{\beta\gamma}}{\partial a^{\delta}} + D^{\sigma}_{\beta\delta} D^{\alpha}_{\sigma\gamma} - D^{\sigma}_{\beta\gamma} D^{\alpha}_{\sigma\delta}. \end{aligned}$$

Let $L^{\alpha}_{\beta\gamma\delta}$ and $\bar{L}^{\alpha}_{\beta\gamma\delta}$ be the curvature tensors of the spaces $S^{(+)}$ and $S^{(-)}$ respectively. They are zero tensors, as mentioned before, hence

$$L^{\alpha}_{\beta\gamma\delta} \equiv \frac{\partial L^{\alpha}_{\beta\delta}}{\partial a^{\gamma}} - \frac{\partial L^{\alpha}_{\beta\gamma}}{\partial a^{\delta}} + L^{\sigma}_{\beta\delta} L^{\alpha}_{\sigma\gamma} - L^{\sigma}_{\beta\gamma} L^{\alpha}_{\sigma\delta} = 0,$$

and

$$\bar{L}^{\alpha}_{\beta\gamma\delta} \equiv \frac{\partial \bar{L}^{\alpha}_{\beta\delta}}{\partial a^{\gamma}} - \frac{\partial \bar{L}^{\alpha}_{\beta\gamma}}{\partial a^{\delta}} + \bar{L}^{\sigma}_{\beta\delta} \bar{L}^{\alpha}_{\sigma\gamma} - \bar{L}^{\alpha}_{\beta\gamma} \bar{L}^{\alpha}_{\sigma\delta} = 0.$$

By these relations, (1·6) and (1·7) we have

$$\Gamma^{\alpha}_{\beta\gamma\delta}=-\frac{1}{2}(L^{\sigma}_{\beta\delta}L^{\alpha}_{\sigma\gamma}-L^{\sigma}_{\beta\gamma}L^{\alpha}_{\sigma\delta})-\frac{1}{2}(\bar{L}^{\sigma}_{\beta\delta}\bar{L}^{\alpha}_{\sigma\gamma}-\bar{L}^{\sigma}_{\beta\gamma}\bar{L}^{\alpha}_{\sigma\delta})+\Gamma^{\sigma}_{\beta\delta}\Gamma^{\alpha}_{\sigma\gamma}-\Gamma^{\sigma}_{\beta\gamma}\Gamma^{\alpha}_{\sigma\delta}$$

$$=\frac{1}{4}\{-L^{\sigma}_{\beta\gamma}(L^{\alpha}_{\sigma\gamma}-\bar{L}^{\alpha}_{\sigma\gamma})+L^{\sigma}_{\beta\delta}(L^{\alpha}_{\sigma\delta}-\bar{L}^{\alpha}_{\sigma\delta})$$

$$-\bar{L}^{\sigma}_{\beta\delta}(\bar{L}^{\alpha}_{\sigma\gamma}-L^{\alpha}_{\sigma\gamma})+\bar{L}^{\sigma}_{\beta\gamma}(\bar{L}^{\alpha}_{\sigma\delta}-L^{\alpha}_{\sigma\delta})\}$$

$$=\frac{1}{2}\{-L^{\sigma}_{\beta\delta}\Omega^{\alpha}_{\sigma\gamma}+L^{\sigma}_{\beta\gamma}\Omega^{\alpha}_{\sigma\delta}-L^{\sigma}_{\delta\beta}\Omega^{\alpha}_{\gamma\sigma}+L^{\sigma}_{\gamma\beta}\Omega^{\alpha}_{\delta\sigma}\}.$$

Hence we have

$$\Gamma^{\alpha}_{\beta\gamma\delta}=-\Omega^{\sigma}_{\beta\gamma}\Omega^{\alpha}_{\sigma\gamma}+\Omega^{\sigma}_{\beta\gamma}\Omega^{\alpha}_{\sigma\delta}. \tag{5·3}$$

On the other hand from (2·6)

$$\frac{\partial D^{\alpha}_{\beta\gamma}}{\partial a^{\delta}}=\frac{1}{2}(c^{e}_{ab}+c^{b}_{ae})\left\{\frac{\partial A^{\alpha}_{a}}{\partial a^{\delta}}A^{e}_{\beta}A^{b}_{\gamma}+A^{\alpha}_{a}\frac{\partial A^{e}_{\beta}}{\partial a^{\delta}}A^{b}_{\gamma}+A^{\alpha}_{a}A^{e}_{\beta}\frac{\partial A^{b}_{\gamma}}{\partial a^{\delta}}\right\}$$

$$=\frac{1}{2}(c^{e}_{ab}+c^{b}_{ae})\{-A^{\sigma}_{a}A^{e}_{\beta}A^{b}_{\gamma}L^{\alpha}_{\sigma\delta}$$

$$+A^{\alpha}_{a}A^{e}_{\sigma}A^{b}_{\gamma}L^{\sigma}_{\beta\delta}+A^{\alpha}_{a}A^{e}_{\beta}A^{b}_{\sigma}L^{\sigma}_{\gamma\delta}\}$$

$$=-L^{\alpha}_{\sigma\delta}D^{\sigma}_{\beta\gamma}+L^{\sigma}_{\beta\delta}D^{\alpha}_{\sigma\gamma}+L^{\sigma}_{\gamma\delta}D^{\alpha}_{\beta\sigma}.$$

From these results and (5·2) we have

$$D^{\alpha}_{\beta\gamma\delta}=-L^{\alpha}_{\sigma\gamma}D^{\sigma}_{\beta\delta}+L^{\sigma}_{\beta\gamma}D^{\alpha}_{\sigma\delta}+L^{\sigma}_{\delta\gamma}D^{\alpha}_{\beta\sigma}+L^{\alpha}_{\sigma\delta}D^{\sigma}_{\beta\gamma}-L^{\sigma}_{\beta\delta}D^{\alpha}_{\sigma\gamma}$$

$$-L^{\sigma}_{\gamma\delta}D^{\alpha}_{\beta\sigma}+D^{\sigma}_{\beta\delta}D^{\alpha}_{\sigma\gamma}-D^{\sigma}_{\beta\gamma}D^{\alpha}_{\sigma\delta}.$$

Consequently

$$D^{\alpha}_{\beta\gamma\delta}+(\Gamma^{\sigma}_{\beta\delta}D^{\alpha}_{\sigma\gamma}-\Gamma^{\sigma}_{\beta\gamma}D^{\alpha}_{\sigma\delta}+D^{\sigma}_{\beta\delta}\Gamma^{\alpha}_{\sigma\gamma}-D^{\sigma}_{\beta\gamma}\Gamma^{\alpha}_{\sigma\delta}) \tag{5·4}$$

$$=\Omega^{\alpha}_{\sigma\delta}D^{\sigma}_{\beta\gamma}-\Omega^{\alpha}_{\sigma\gamma}D^{\sigma}_{\beta\delta}+\Omega^{\sigma}_{\beta\gamma}D^{\alpha}_{\sigma\delta}-\Omega^{\sigma}_{\beta\delta}D^{\alpha}_{\sigma\gamma}$$

$$+2\Omega^{\sigma}_{\delta\gamma}D^{\alpha}_{\beta\sigma}+D^{\sigma}_{\beta\delta}D^{\alpha}_{\sigma\gamma}-D^{\sigma}_{\beta\gamma}D^{\alpha}_{\sigma\delta}.$$

From (5·1), (5·3) and (5·4), we have

$$R^{\alpha}_{\beta\gamma\delta}=(-\Omega^{\sigma}_{\beta\delta}\Omega^{\alpha}_{\sigma\gamma}+\Omega^{\sigma}_{\beta\gamma}\Omega^{\alpha}_{\sigma\delta})+(\Omega^{\alpha}_{\sigma\delta}D^{\sigma}_{\beta\gamma}-\Omega^{\alpha}_{\sigma\gamma}D^{\sigma}_{\beta\delta}) \tag{5·5}$$

$$+(\Omega^{\sigma}_{\beta\gamma}D^{\alpha}_{\sigma\delta}-\Omega^{\sigma}_{\beta\delta}D^{\alpha}_{\sigma\gamma})+2\Omega^{\sigma}_{\delta\gamma}D^{\alpha}_{\beta\sigma}+(D^{\sigma}_{\beta\delta}D^{\alpha}_{\sigma\gamma}-D^{\sigma}_{\beta\gamma}D^{\alpha}_{\sigma\delta}).$$

As

$$\Omega_{\beta\delta}^{\sigma}\Omega_{\sigma\gamma}^{\alpha}=\frac{1}{4}\,c_{ab}^{e}A_{\beta}^{a}A_{\delta}^{b}A_{e}^{\sigma}\cdot c_{a'b'}^{e'}A_{\sigma}^{a'}A_{\gamma}^{b'}A_{e'}^{\alpha}=\frac{1}{4}\,c_{ab}^{e}c_{eb'}^{e'}A_{e'}^{\alpha}A_{\beta}^{a}A_{\gamma}^{b'}A_{\delta}^{b},$$

we have, using the Jacobi relations,

$$(5\cdot6)\qquad -\Omega_{\beta\delta}^{\sigma}\Omega_{\sigma\gamma}^{\alpha}+\Omega_{\beta\gamma}^{\sigma}\Omega_{\sigma\delta}^{\alpha}=\frac{1}{4}\,c_{dc}^{e}c_{eb}^{a}A_{a}^{\alpha}A_{\beta}^{b}A_{\gamma}^{c}A_{\delta}^{d}.$$

Using (2·6) we have

$$(5\cdot7)\qquad \Omega_{\sigma\delta}^{\alpha}D_{\beta\gamma}^{\sigma}-\Omega_{\sigma\lambda}^{\alpha}D_{\beta\delta}^{\sigma}=\frac{1}{4}\{c_{ed}^{a}(c_{ec}^{b}+c_{eb}^{c})-c_{ec}^{a}(c_{el}^{b}+c_{eb}^{d})\}A_{a}^{\alpha}A_{\beta}^{b}A_{\gamma}^{c}A_{\delta}^{d},$$

$$(5\cdot8)\qquad \Omega_{\beta\gamma}^{\sigma}D_{\sigma\delta}^{\alpha}-\Omega_{\beta\delta}^{\sigma}D_{\sigma\gamma}^{\alpha}=\frac{1}{4}\{c_{bc}^{e}(c_{al}^{e}+c_{ae}^{d})-c_{bd}^{e}(c_{ac}^{e}+c_{ae}^{c})\}A_{a}^{\alpha}A_{\beta}^{b}A_{\gamma}^{c}A_{\delta}^{d},$$

$$(5\cdot9)\qquad 2\Omega_{\delta\gamma}^{\sigma}D_{\beta\sigma}^{\alpha}=\frac{1}{2}\,c_{db}^{e}(c_{ae}^{b}+c_{ab}^{e})A_{a}^{\alpha}A_{\beta}^{b}A_{\gamma}^{c}A_{\delta}^{d},$$

and

$$(5\cdot10)\qquad D_{\beta\delta}^{\sigma}D_{\sigma\gamma}^{\alpha}-D_{\beta\gamma}^{\sigma}D_{\sigma\delta}^{\alpha}=\frac{1}{4}\{(c_{ed}^{b}+c_{eb}^{d})(c_{ac}^{e}+c_{ae}^{c})$$
$$-(c_{ec}^{b}+c_{eb}^{c})(c_{ad}^{e}+c_{ae}^{d})\}A_{a}^{\alpha}A_{\beta}^{b}A_{\gamma}^{c}A_{\delta}^{d}.$$

Replacing the terms of the second members of (5·5) by the results obtained from (5·6) to (5·10), we have

$$R_{\beta\gamma\delta}^{\alpha}=\frac{1}{4}\,C_{abcd}A_{a}^{\alpha}A_{\beta}^{b}A_{\gamma}^{c}A_{\delta}^{d}$$

and consequently the curvature tensor of the first kind $R_{\alpha\beta\gamma\delta}$ is given by

$$(5\cdot11)\qquad R_{\alpha\beta\gamma\delta}=\frac{1}{4}\,C_{abcd}A_{a}^{\alpha}A_{\beta}^{b}A_{\gamma}^{c}A_{\delta}^{d}$$

where

$$(5\cdot12)\qquad C_{abcd}=c_{dc}^{e}c_{eb}^{a}+c_{ed}^{a}(c_{ec}^{b}+c_{eb}^{c})+c_{ce}^{a}(c_{ed}^{b}+c_{eb}^{d})+c_{bc}^{e}(c_{ad}^{e}+c_{ae}^{d})$$
$$+c_{db}^{e}(c_{ac}^{e}+c_{ae}^{c})+2c_{dc}^{e}(c_{ae}^{b}+c_{ab}^{e})$$
$$+(c_{ed}^{b}+c_{eb}^{d})(c_{ac}^{e}+c_{ae}^{c})+(c_{be}^{c}+c_{be}^{c})(c_{ad}^{e}+c_{ae}^{d}).$$

From the type of $R_{\beta\gamma\delta}^{\alpha}$ and the fact that the constants of structure of any Abelian group are zero, it is evident that the group-space for an Abelian group is euclidean. Conversely, if $S^{(R)}$ is euclidean, $g_{\alpha\beta}$ are constant everywhere, hence $\vec{A}_a$ constant. Consequently we have:

THEOREM: *The group-space $S^{(R)}$ is euclidean when and only when the group is Abelian.*

It is verified that the quantities C_{abcd} are skew-symmetric with respect to the first pair of indices a and b and to the last pair c and d, accordingly, as a mater of course, $R_{\alpha\beta\gamma\delta}$ are skew-symmetric with respect to α and β, and also γ and δ. It is convenient to separate C_{abcd} into three groups of sums, that is, the sum of $c^{\cdot\cdot e}_{\cdot\cdot}c^{\cdot\cdot e}_{\cdot\cdot}$-types, $c^{\cdot\cdot e}_{\cdot\cdot}c^{\cdot\cdot}_{e\cdot}$-types and $c^{\cdot\cdot}_{e\cdot}c^{\cdot\cdot}_{e\cdot}$-types as follows.

Sum of $c^{\cdot\cdot e}_{\cdot\cdot}c^{\cdot\cdot e}_{\cdot\cdot}$-types: $c^{e}_{bc}c^{e}_{ad}+c^{e}_{db}c^{e}_{ac}+2c^{e}_{dc}c^{e}_{ab}$.

Sum of $c^{\cdot\cdot e}_{\cdot\cdot}c^{\cdot\cdot}_{e\cdot}$-types: $c^{e}_{dc}c^{a}_{eb}+c^{e}_{cd}c^{b}_{ea}+c^{e}_{ba}c^{c}_{ed}+c^{e}_{ab}c^{d}_{ec}$.

(where the Jacobi relations have been used)

Sum of $c^{\cdot\cdot}_{e\cdot}c^{\cdot\cdot}_{e\cdot}$-types: $c^{a}_{ed}c^{b}_{ec}+c^{a}_{ce}c^{b}_{ed}+c^{a}_{ed}c^{c}_{eb}+c^{a}_{ce}c^{d}_{eb}$

$+c^{b}_{ed}c^{c}_{ae}+c^{b}_{ce}c^{d}_{ae}+c^{d}_{eb}c^{c}_{ae}+c^{c}_{be}c^{d}_{ae}$.

When we put $d=a$ we have the next results.

(5·13)
- from the sum of $c^{\cdot\cdot e}_{\cdot\cdot}c^{\cdot\cdot e}_{\cdot\cdot}$-types: $3c^{e}_{ab}c^{e}_{ac}$.
- from the sum of $c^{\cdot\cdot e}_{\cdot\cdot}c^{\cdot\cdot}_{e\cdot}$-types: $2c^{e}_{ac}c^{a}_{eb}+c^{e}_{ba}c^{b}_{ea}+c^{e}_{ba}c^{c}_{ea}$.
- from the sum of $c^{\cdot\cdot}_{e\cdot}c^{\cdot\cdot}_{e\cdot}$-types: $2c^{a}_{ea}(c^{b}_{eb}+c^{c}_{eb})+c^{e}_{ca}c^{b}_{ae}$ $+c^{e}_{ca}c^{e}_{ab}+c^{b}_{ea}c^{c}_{ae}+c^{e}_{ab}c^{c}_{ea}$.

Denoting C_{abca} by C_{bc}, we have the Ricci tensor as follows.

$$R_{\beta\gamma}=\frac{1}{4}A^{b}_{\beta}A^{c}_{\gamma}C_{bc}$$

where C_{bc} are expressed by adding the results of (5·13) as follows.

$$C_{bc}=2\{c^{e}_{ab}c^{e}_{ac}+c^{e}_{ac}c^{a}_{eb}+c^{a}_{ea}c^{b}_{ec}+c^{a}_{ea}c^{c}_{eb}\}+c^{b}_{ea}c^{c}_{ae}.$$

It is verified that the Ricci tensor is symmetric with respect to its indices from the fact that C_{bc} is symmetric.

Furthermore the scalar curvature is obtained as follows.

$$R=g^{\beta\gamma}R_{\beta\gamma}=\frac{1}{4}C_{bb} \tag{5·14}$$

$$=\frac{1}{4}\{c_{ab}^{\ \ e}c_{ab}^{\ \ e}+4c_{ea}^{\ \ a}c_{eb}^{\ \ b}+2c_{ab}^{\ \ e}c_{eb}^{\ \ a}\}\,.$$

Accordingly we have:

THEOREM: *The scalar curvature R is constant at every point in $S^{(R)}$ and the value of it is* $\frac{1}{4}(c_{ab}^{\ \ e}c_{ab}^{\ \ e}+4c_{ea}^{\ \ a}c_{eb}^{\ \ b}+2c_{ab}^{\ \ e}c_{eb}^{\ \ a})$.

When the group is integrable we can choose the symbols A_1f, $\cdots$, A_rf so as to have the relations $c_{ab}^{\ \ e}=0$ for $e>a$ or b, consequently $c_{ab}^{\ \ e}c_{eb}^{\ \ a}=0$ $(a, b, e=1, \cdots, r)$. From (5·14)

$$R=\frac{1}{4}\{c_{ab}^{\ \ e}c_{ab}^{\ \ e}+4c_{ea}^{\ \ a}c_{eb}^{\ \ b}\}$$

$$=\frac{1}{4}\sum_{a,b,c}(c_{ab}^{\ \ c})^2+\sum_{e}(c_{e1}^{\ \ 1}+\cdots+c_{er}^{\ \ r})^2\geqq 0\,.$$

Hence we have:

THEOREM: *When the group is non-Abelian and integrable, the repére can be chosen so that the scalar curvature is positive.*

Let $e^a\vec{A}_a$ and $f^a\vec{A}_a$ be different vectors at a same point a^α, and let K be Gaussian curvature of the surface generated by the geodesics which pass the point a^α and are tangent to the plane spaned by the two vectors $e^a\vec{A}_a$ and $f^a\vec{A}_a$. Then we have

$$K=\frac{\frac{1}{4}C_{abcd}e^af^be^cf^d}{(\delta_c^a\delta_d^b-\delta_d^a\delta_c^b)e^af^be^cf^d}\,.$$

Consequently we have the next theorem.

THEOREM: *In $S^{(R)}$ The Gaussian curvature which is determined by two vectors $e^a\vec{A}_a$ and $f^a\vec{A}_a$ is constant.*

6. EXAMPLE.

Let us consider the group of linear substitutions

$$x'=ax+b \qquad (a>0)\,.$$

The parameter-group of this group is given by

$$\begin{cases} a''=a'a\,, \\ b''=a'b+b', \end{cases}$$

and the matrices of the vectors of repère by

$$(6\cdot1)\qquad \begin{aligned}(A_1^\alpha A_2^\alpha)&=\begin{Vmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2\end{Vmatrix}=\begin{Vmatrix} a & 0 \\ b & 1\end{Vmatrix},\\ \begin{pmatrix}A_\alpha^{\cdot 1}\\ A_\alpha^{\cdot 2}\end{pmatrix}&=\begin{Vmatrix} A_1^{\cdot 1} & A_2^{\cdot 1} \\ A_1^{\cdot 2} & A_2^{\cdot 2}\end{Vmatrix}=\begin{Vmatrix} \frac{1}{a} & 0 \\ -\frac{b}{a} & 1\end{Vmatrix},\end{aligned}$$

where we used the parameters, a, b instead of a^1, a^2 respectively.

As $(A_1, A_2)f=-A_2f$, the constants of structure are given by

$$(6\cdot2)\qquad c_{12}^{\cdot 2}=-1,\quad c_{21}^{\cdot 2}=1,\quad c_{12}^{\cdot 1}=0,\quad c_{21}^{\cdot 1}=0.$$

As the space $S^{(R)}$ of this group is 2-dimensional, it can be represented as a surface in 3-dimentional euclidean space.

The fundamental tensor of this surface is given by

$$\begin{Vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{Vmatrix}=\begin{Vmatrix} \frac{1+b^2}{a^2} & -\frac{b}{a} \\ -\frac{b}{a} & 1\end{Vmatrix}$$

and

$$g=\frac{1}{a^2}.$$

From (1·3)

$$\begin{cases} L_{11}^{\cdot 1}=-\frac{1}{a}, & L_{12}^{\cdot 1}=0, \quad L_{21}^{\cdot 1}=0, \quad L_{22}^{\cdot 1}=0, \\ L_{11}^{\cdot 2}=0, & L_{12}^{\cdot 2}=-\frac{1}{a}, \quad L_{21}^{\cdot 2}=0, \quad L_{22}^{\cdot 2}=0, \end{cases}$$

and from (1·5)

$$(6\cdot3)\qquad \begin{cases} \Gamma_{11}^{\cdot 1}=-\frac{1}{a}, & \Gamma_{12}^{\cdot 1}=\Gamma_{21}^{\cdot 1}=0, \quad \Gamma_{22}^{\cdot 1}=0, \\ \Gamma_{11}^{\cdot 2}=0, & \Gamma_{12}^{\cdot 2}=\Gamma_{21}^{\cdot 2}=-\frac{1}{2a}, \quad \Gamma_{22}^{\cdot 2}=0. \end{cases}$$

From (2·6) and (6·2) we have

$$D_{\beta\gamma}^{\alpha}=\frac{1}{2}\{-A_1^\alpha A_\beta^{\cdot 2}A_\gamma^{\cdot 2}-A_1^\alpha A_\beta^{\cdot 2}A_\gamma^{\cdot 2}\}+\frac{1}{2}\{A_2^\alpha A_\beta^{\cdot 2}A_\gamma^{\cdot 1}+A_2^\alpha A_\beta^{\cdot 1}A_\gamma^{\cdot 2}\}$$

$$=-A_1^{\alpha}A_\beta^2A_\gamma^2+\frac{1}{2}A_2^{\alpha}(A_\beta^2A_\gamma^1+A_\beta^1A_\gamma^2).$$

Consequently

$$(6\cdot4)\qquad \begin{cases} D_{11}^1=-\dfrac{b^2}{a}, & D_{12}^1=D_{21}^1=b, \quad D_{22}^1=-a\,, \\ D_{11}^2=-\dfrac{b^3}{a^2}-\dfrac{b}{a^2}, & D_{12}^2=D_{21}^2=\dfrac{b^2}{a}+\dfrac{1}{2a}, \quad D_{22}^2=-b\,. \end{cases}$$

From (2·5), (6·3) and (6·4) we have the Christoffel symbols as follows.

$$\begin{cases} \left\{\begin{matrix}1\\1\;1\end{matrix}\right\}=-\dfrac{1+b^2}{a}, & \left\{\begin{matrix}1\\1\;2\end{matrix}\right\}=\left\{\begin{matrix}1\\2\;1\end{matrix}\right\}=b, & \left\{\begin{matrix}1\\2\;2\end{matrix}\right\}=-a\,, \\ \left\{\begin{matrix}2\\1\;1\end{matrix}\right\}=-\dfrac{b(1+b^2)}{a^2}, & \left\{\begin{matrix}2\\1\;2\end{matrix}\right\}=\left\{\begin{matrix}2\\2\;1\end{matrix}\right\}=\dfrac{b^2}{a}, & \left\{\begin{matrix}2\\2\;2\end{matrix}\right\}=-b\,. \end{cases}$$

From (5·11), (5·12) and (6·2) we have

$$\begin{aligned} R_{1212}&=\frac{1}{4}A_1^aA_2^bA_1^cA_2^dC_{abcd} \\ &=\frac{1}{4}A_1^1A_2^2A_1^1A_2^2C_{1212} \\ &=-\frac{1}{a^2}. \end{aligned}$$

Consequently the Gaussian curvature of this surface is given by

$$K=\frac{R_{1212}}{g}=-1\,.$$

It is verified another calculation as follows.

$$\begin{aligned} K=\frac{1}{2\sqrt{g}}\Big\{&\frac{\partial}{\partial a}\Big(\frac{g_{12}}{g_{11}\sqrt{g}}\frac{\partial g_{11}}{\partial b}-\frac{1}{\sqrt{g}}\frac{\partial g_{22}}{\partial a}\Big) \\ &+\frac{\partial}{\partial b}\Big(\frac{2}{\sqrt{g}}\frac{\partial g_{12}}{\partial a}-\frac{1}{\sqrt{g}}\frac{\partial g_{11}}{\partial b}-\frac{g_{12}}{g_{11}\sqrt{g}}\frac{\partial g_{11}}{\partial a}\Big)\Big\} \end{aligned}$$

$$=\frac{a}{2}\Big\{\frac{\partial}{\partial a}\Big(\frac{-2b^2}{1+b^2}\Big)+\frac{\partial}{\partial b}\Big(\frac{2b}{a}-\frac{2b}{a}-\frac{2b}{a}\Big)\Big\}=-1\,.$$

Hence the Gaussian curvature of the surface is negative constant and its value is -1.

From Fig. 1 at the end of the paper we can imagine that a family of curves $b=ma$ where m is a parameter is seemed to be a family of geodesic on the surface. It will be verified as follows.

When we change the parameters, the metric is not altered. Hence the intrinsic properties of the surface is not altered. Let us take new parameters u, v which are given by the relations

$$(6\cdot5) \qquad \begin{cases} a=e^u, \\ b=ce^u v \qquad (c>0). \end{cases}$$

The fundamental tensor with respect to these parameters is given by

$$\|\bar{g}_{\alpha\beta}\|=\begin{Vmatrix} \bar{g}_{11} & \bar{g}_{12} \\ \bar{g}_{21} & \bar{g}_{22} \end{Vmatrix}=\begin{Vmatrix} 1 & 0 \\ 0 & c^2e^{2u} \end{Vmatrix},$$

hence ds^2 is expressed by

$$ds^2=du^2+c^2e^{2u}dv^2 .$$

Furthermore we know that a surface with negative constant Gaussian curvature is applicable to a one kind of pseudo-spheres. Let this pseudo-sphere be given by

$$(6\cdot6) \qquad \begin{cases} x=r\cos\theta , \\ y=r\sin\theta , \\ z=\varphi(r), \end{cases}$$

then

$$ds^2=[1+\{\varphi'(r)\}^2]\,dr^2+r^2d\theta^2.$$

Putting

$$(6\cdot7) \qquad \theta=v$$

we have the relations

$$(6\cdot8) \qquad \begin{cases} r=ce^u, \\ \{\varphi'(r)\,dr\}^2=(1-c^2e^{2u})\,du^2 , \end{cases}$$

hence

$$\varphi(r)=\left|\int\sqrt{1-c^2e^{2u}}\,du\right|.$$

Calculating this equation, we have

$$(6\cdot9)\qquad \varphi(r)=\log\frac{1+\sqrt{1-c^2e^{2u}}}{ce}-\sqrt{1-c^2e^{2u}}.$$

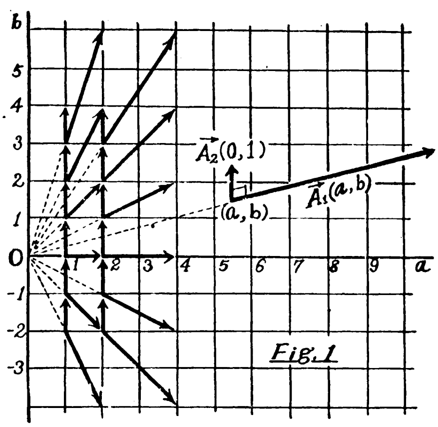
By the equations from (6·5) to (6·9), we have

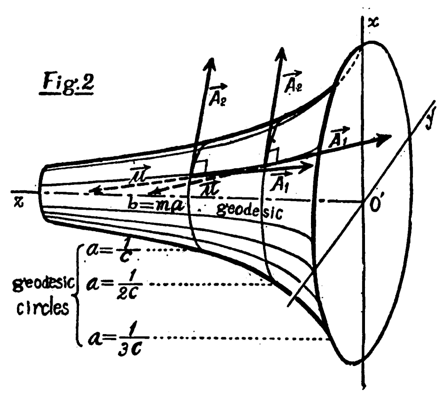
$$\begin{cases} x=ca\cos v\,, \\ y=ca\sin v\,, \\ z=\log\dfrac{1+\sqrt{1-c^2a^2}}{ca}-\sqrt{1-c^2a^2}\,. \end{cases}$$

The intersection of this rotational surface and zx-plane is given by

$$z=\log\frac{1+\sqrt{1-x^2}}{x}-\sqrt{1-x^2},$$

which is a tractrix. Accordingly the surface is applicable to the pseudo spherical surface of revolution of the parabolic type as shown at Fig. 2. It is observed that a family of curves $b=ma$ is the one of geodesics and a family of curves $a=$const. the one orthogonal to it. This surface corresponds to one of the separated part of $S^{(K)}$ with respect to a certain modulus of the parameter b.





From (6·1), (6·2) and (4·2) the geodesic curvature of a family of the curves a=const. of which differential equations are $\frac{da^{\alpha}}{ds}=A^{\alpha}_{2}$, is given by

$$K_{(g)}\mu^{\alpha}=\delta^{b}_{2}\delta^{c}_{2}c_{ab}^{\cdot\cdot c}A^{\alpha}_{a}=-A^{\alpha}_{1}.$$

Hence the length of the vector $K_{(g)}\mu^{\alpha}$ is unit and its sense of direction is opposite to A^{α}_{1}. From the property of the tractrix (6·10), the end point of this curvature vector $-\vec{A}_{1}$ at every point on the surface is situated on z-axis. These vectors are represented by $\vec{\mu}$ in Fig. 2.

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