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Addition and corrections to my paper " A treatise on the 14-th problem of Hilbert "

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Concerning the 14-th problem of Hilbert, Zariski [3] conjectured the following:

Conjecture of Zariski. Let D be a positive divisor on a normal projective variety V defined over a field k and let $R[D]$ be the set of functions f on V defined over k such that $(f)+nD\succ 0$ for some natural number n. Then $R[D]$ will be an affine ring over k.

He proved there that if the answer of this conjecture is affirmative, then the answer of the following problem is affirmative:

The generalized 14-th problem of Hilbert: Let $\mathfrak{o}$ be a normal affine ring over a field k and let L' be a function field contained in the function field of $\mathfrak{o}$. Is then $\mathfrak{o}\cap L'$ an affine ring?

In the present paper, we shall show at first that the generalized 14-th problem of Hilbert is equivalent to the conjecture of Zariski and then we shall give some corrections to my paper [2].

§ 1. The proof of the equivalence.

Since Zariski [3] proved that the affirmative answer of the conjecture of Zariski implies the affirmative answer of the generalized 14-th problem of Hilbert, we have only to prove the converse. The writer proved in [2] that the generalized 14-th problem is equivalent to

Problem A. Let $\mathfrak{a}$ be an ideal of a normal affine ring $\mathfrak{o}$ over a field k. Is then the $\mathfrak{a}$-transform of $\mathfrak{o}$ an affine ring?

Therefore we have only to prove that:

The affirmative answer of Problem A implies the affirmative answer of the conjecture of Zariski.

Now we shall use the notations as in the conjecture of Zariski. Let L be the field of quotients of $R[D]$ and let $\mathfrak{o}$ be a normal affine ring of L contained in $R[D]$. We denote by $\mathfrak{v}$ in general spots which corresponds to k-prime divisors on V which are not components of D. Then obviously $R[D]$ is the intersection of all of $\mathfrak{v}$'s, hence $R[D]=\cap_{\mathfrak{v}}(L\cap\mathfrak{v})$, which shows that $R[D]$ is a Krull ring (see [2, p. 60]) and if $\mathfrak{q}$ is a prime ideal of rank 1 in $R[D]$, then there exists one $\mathfrak{v}$ such that $R[D]_{\mathfrak{q}}=L\cap\mathfrak{v}$. Furthermore, since each $L\cap\mathfrak{v}$ is a spot (see [2, foot-note 3]), $R[D]_{\mathfrak{q}}$ is a spot. Let $\mathfrak{Q}$ be the set of prime ideals $\mathfrak{q}$ of rank 1 in $R[D]$ such that $\mathfrak{q}\cap\mathfrak{o}$ is not of rank 1. Since $\mathfrak{o}_{(\mathfrak{q}\cap\mathfrak{o})}$ is dominated by one $\mathfrak{v}$, $\mathfrak{q}\in\mathfrak{Q}$ means that the spot $\mathfrak{o}_{(\mathfrak{q}\cap\mathfrak{o})}$ is an isolated fundamental spot with respect to V, hence $\mathfrak{Q}$ is a finite set. Since $R[D]_{\mathfrak{q}}$ is a spot, we can reduce easily to the case where $\mathfrak{Q}$ is empty (see [2, Proposition A]). Thus we assume that $\mathfrak{Q}$ is empty. Next, let $\mathfrak{P}$ be the set of prime ideals $\mathfrak{p}$ of rank 1 in $\mathfrak{o}$ such that there exists no prime ideal $\mathfrak{q}$ of rank 1 in $R[D]$ which lies over $\mathfrak{p}$. Then $\mathfrak{o}_{\mathfrak{p}}$ $(\mathfrak{p}\in\mathfrak{P})$ is dominated by none of $\mathfrak{v}$, which shows that $\mathfrak{o}_{\mathfrak{p}}$ corresponds to only components of D, which shows that $\mathfrak{P}$ is a finite set. Let $\mathfrak{a}$ be the intersection of members of $\mathfrak{P}$. Then $R[D]$ is the $\mathfrak{a}$-transform of $\mathfrak{o}$. Therefore the equivalence is proved.

§ 2. Corrections.

In [2, Theorem 4] we asserted that if D is a closed set of an affine model A of dimension 2 ($A\neq D$), then $A-D$ has an associated affine model. This is correct under the additional assumption that A is normal and in the non-normal case the assertion is not true as will be shown by an example in § 3. One error in the proof exists in *l.* 4, p. 67 of the paper.[1] Namely, we stated that from $\mathfrak{o}_{\mathfrak{q}}=\mathfrak{S}_{\mathfrak{q}'}$ it follows that $\mathfrak{q}''=\mathfrak{q}\mathfrak{S}_{\mathfrak{m}'}:\mathfrak{q}'\mathfrak{S}_{\mathfrak{m}'}$ is a primary ideal belonging to $\mathfrak{m}'\mathfrak{S}_{\mathfrak{m}'}$.[2] But we needed really the normality in that conclusion. In fact, the example which will be shown in § 3 shows the non-validity of this conclusion in the non-normal case. Since, even in the normal case, that conclusion may not be obvious, we shall give a detailed proof of that conclusion in § 4.

By this reason, in that Theorem 4, we must assume that A is normal. Under the assumption of normality, the proof of Theorem 4 is valid and there remains no difficulty (except the fact which we shall prove in § 4).

On the other hand, Proposition 5 (p. 69) should be asserted also under the additional assumption that $\mathfrak{o}$ is a normal ring.

§ 3. An example.

Let x, y and z be indeterminates and let k be a field. Let f be an element of $k[x, y, z]$ such that

(1) f is irreducible, and

(2) $f=y(z+yt)+x(u_1y^2+u_2yz+u_3z^2)$ with $t \in k[x, y]$ and u_1, u_2, $u_3 \in k[x, y, z]$.

Set $\mathfrak{o}=k[x, y, z]/(f)$. Then x, y generate a prime ideal $\mathfrak{p}$ of rank 1 in $\mathfrak{o}$; y, z generate a prime ideal $\mathfrak{q}$ of rank 1 in $\mathfrak{o}$. $\mathfrak{o}_{\mathfrak{q}}$ is not normal. Let $\mathfrak{S}$ be the $\mathfrak{p}$-transform of $\mathfrak{o}$. We first consider $\mathfrak{p}^{-1}$. It is obviously generated by 1 and $z_1=(z+yt)/x$. Therefore $\mathfrak{o}[\mathfrak{p}^{-1}]$ is generated by x, y, z_1 satisfying a relation similar to f stated in (2) as is easily seen. Thus $\mathfrak{S}$ is obtained by successive adjunction of elements $z_1, z_2, \ldots, z_n, \ldots$ such that $z_i=(z_{i-1}+yt_{i-1})/x$ with $t_{i-1} \in k[x, y]$. Though we have already seen in essential that $\mathfrak{S}$ is not an affine ring, we shall see a little more. Since $xz_i=z_{i-1}+yt_{i-1}$ $(z_0=z)$, we see that $z_{i-1} \in \mathfrak{p}\mathfrak{S}$. Thus x and y generate a maximal ideal $\mathfrak{m}$ of $\mathfrak{S}$. Therefore if $\mathfrak{S}_{\mathfrak{m}}$ is Noetherian, $\mathfrak{S}_{\mathfrak{m}}$ must be a regular local ring. Let $\mathfrak{q}'$ be the uniquely determined prime ideal of rank 1 in $\mathfrak{S}$ such that $\mathfrak{S}_{\mathfrak{q}'}=\mathfrak{o}_{\mathfrak{q}}$. Since y, $z \in \mathfrak{q}$ and $x \notin \mathfrak{q}$, $z_1=(z+yt)/x$ must be in $\mathfrak{q}'$. By the same reason, we have $z_i \in \mathfrak{q}'$ for every i. Therefore $\mathfrak{q}'$ is generated by $z, z_1, z_2, \ldots, z_n, \ldots$. Therefore $\mathfrak{q}'$ is contained in $\mathfrak{m}$. Since $\mathfrak{o}_{\mathfrak{q}}=\mathfrak{S}_{\mathfrak{q}'}$, we see that $\mathfrak{S}_{\mathfrak{m}}$ is not a normal ring and $\mathfrak{S}_{\mathfrak{m}}$ cannot be a regular local ring and $\mathfrak{S}_{\mathfrak{m}}$ cannot be a Noetherian ring. Now, if $\mathfrak{q}\mathfrak{S}_{\mathfrak{m}} : \mathfrak{q}'\mathfrak{S}_{\mathfrak{m}}$ is a primary ideal belonging to $\mathfrak{m}\mathfrak{S}_{\mathfrak{m}}$, then the treatment in [2, p. 69] shows that $\mathfrak{q}'$ is generated by a finite number of elements. But we see now easily that $\mathfrak{q}'$ cannot be generated by any finite number of the z_i's. Thus $\mathfrak{q}\mathfrak{S}_{\mathfrak{m}} : \mathfrak{q}'\mathfrak{S}_{\mathfrak{m}}$ is not a primary ideal belonging to $\mathfrak{m}\mathfrak{S}_{\mathfrak{m}}$ but is contained in $\mathfrak{q}'\mathfrak{S}_{\mathfrak{m}}$.

§ 4. A lemma on Krull ring.

In order to verify the statement in [2, p. 67, *l.* 4] in the normal case, it will be sufficient to prove the following lemma.[3)]

Lemma. Let $\mathfrak{q}$ be a prime ideal of rank 1 in a Krull ring $\mathfrak{S}$. If $\mathfrak{a}$ is an ideal contained in $\mathfrak{q}$ such that $\mathfrak{a}\mathfrak{S}_{\mathfrak{q}}=\mathfrak{q}\mathfrak{S}_{\mathfrak{q}}$, then $\mathfrak{a} : \mathfrak{q}$ is not contained in $\mathfrak{q}$.

Proof. Since $\mathfrak{S}$ is a Krull ring, $\mathfrak{S}_{\mathfrak{q}}$ is a discrete valuation ring. Therefore there exists an element $a \in \mathfrak{a}$ such that $a\mathfrak{S}_{\mathfrak{q}}=\mathfrak{q}\mathfrak{S}_{\mathfrak{q}}$ (because

$\mathfrak{a}\mathfrak{S}_{\mathfrak{q}}=\mathfrak{q}\mathfrak{S}_{\mathfrak{q}}$). Since $\mathfrak{S}$ is a Krull ring, $a\mathfrak{S}$ is the intersection of a finite number of primary ideals and we see easily that $a\mathfrak{S}:\mathfrak{q}$ is not contained in $\mathfrak{q}$.

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BIBLIOGRAPHY

[1] M. Nagata, On the derived normal rings of Noetherian integral domains, Memoirs Kyoto University, vol. 29, No. 3 (1955).

[2] M. Nagata, A treatise on the 14-th problem of Hilbert, Memoirs Kyoto University, vol. 30, No. 1 (1956).

[3] O. Zariski, Interprétations algébrico-géometriques du quatoziéme probléme de Hilbert, Bull. Sci. Math., vol. 78 (1954).

Notes

1) There is one more error concerning non-normal case in p. 67. Namely, we constructed the ring $\mathfrak{S}^*$; then $\mathfrak{S}^*$ may have a maximal ideal $\mathfrak{m}^*$ of rank 1. This is the reason why proposition 5 should be asserted under an additional condition (see the end of this section).

2) There is a case where $\mathfrak{q}''=\mathfrak{S}_{\mathfrak{m}'}$. In such a case, we have obviously $\mathfrak{q}\mathfrak{S}_{\mathfrak{m}'}=\mathfrak{q}'\mathfrak{S}_{\mathfrak{m}'}$ and $\mathfrak{q}'\mathfrak{S}_{\mathfrak{m}'}$ has a finite base. Therefore we disregarded such a simple case.

3) This lemma was used in the first step of the proof of [1, Theorem 3].