MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXX, Mathematics No. 2, 1957

# On the arithmetic genera and the effective genera of algebraic curves

By

Heisuke HIRONAKA

(Received January 11, 1957)

---

In the study of algebraic curves in a fixed projective space (or on a non-singular surface) the arithmetic genera and the effective genera of curves are taken as their most important numerical characterizations.

In this paper we want to pick up some of their fundamental properties and to discuss on them without any restriction on the characteristic of the universal domain.

Throughout this paper we shall fix a projective $N$-space $\boldsymbol{P}^N$ ($N \geq 2$) and say briefly *a curve* in-stead of a positive 1-dimensional $\boldsymbol{P}^N$-cycle without multiple components. As usual, for a given curve in $\boldsymbol{P}^N$, we write down $1-p_a$ for the constant term of the Hilbert characteristic function of the curve and we call the integer $p_a$ the *arithmetic genus* of the curve; for an irreducible curve the *effective genus* means the arithmetic genus of a non-singular[1] curve which is birationally equivalent to it, and in general for a reducible curve it is defined by the sum of those of the absolutely irreducible components of the curve. Part (I) will be devoted to preliminary definitions and studies on intersection multiplicity and order of singularity from the local view-point; the intersection multiplicity defined in this paper is a generalization of the usual one which we shall need in Part (II) in order to state a generalized modular property of the arithmetic genera of curves. Also in Part (II) we shall prove a formula which expresses the difference of the arithmetic and the effective genera of a curve in terms of the orders of singularity at singular points, which includes the classical genus formula of a plane curve as a special case.

Finally in Part (III), we shall study the order of newly outcoming singularity by specialization of a curve and show an application of the *Principle of Degeneration*, a topological result which was proved algebraically by O. Zariski,[22] to the effective genera of degenerated curves of an irreducible curve.

As for the global investigations of the Chow-varieties of the curves of an arbitrarily given degree in $\boldsymbol{P}^N$. I hope to discuss in future.

(I)

When we speak of two rings, one containing the other, they will be assumed to contain the same unity. Suppose that a commutative and Noetherian ring $\mathfrak{o}$ contains a field $k$, we may consider $\mathfrak{o}$ as a vector space over the field $k$, whose dimension will be denoted by $\dim_k \mathfrak{o}$ if finite. Moreover, if the other ring $\mathfrak{o}'$ containing $\mathfrak{o}$ is given, we write $(\mathfrak{o}':\mathfrak{o})$ for the factor space of $\mathfrak{o}'$ by $\mathfrak{o}$ both being considered as vector spaces over $k$, whose dimension will be denoted by $\dim_k(\mathfrak{o}':\mathfrak{o})$ if finite.

*Lemma 1. Let* $\mathfrak{o}$ *be semi-local ring of rank 0. Let* $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\cdots$, *and* $\mathfrak{m}_l$ *be the maximal ideals of* $\mathfrak{o}$. *Assume that a field k is contained in* $\mathfrak{o}$ *and that* $[\mathfrak{o}/\mathfrak{m}_i:k]$ *is finite for all i. Then we have*

$$\dim_k \mathfrak{o}=\sum_{i=1}^{l}[\mathfrak{o}/\mathfrak{m}_i:k]\cdot\text{length }\mathfrak{o}_{\mathfrak{m}_i}.$$

(Proof) When $\mathfrak{o}$ is a primary ring with the maximal ideal $\mathfrak{m}$, let $w_i (1\leqq i\leqq g)$ be elements of $\mathfrak{o}$ which forms a linearly independent base of $\mathfrak{o}/\mathfrak{m}$ over $k$ mod. $\mathfrak{m}$ and let $q_0=(0)\subset q_1\subset\cdots\subset q_h=\mathfrak{o}$ be a maximal chain of strictly ascending ideals in $\mathfrak{o}$ so that $q_j=(q_{j-1}, a_j)$ with an element $a_j$. Then the $gh$ elements $w_i a_j$ form a base of $\mathfrak{o}$ over $k$. The general case follows from this, since $\mathfrak{o}=\mathfrak{o}_{\mathfrak{m}_1}\oplus\mathfrak{o}_{\mathfrak{m}_2}\oplus\cdots\oplus\mathfrak{o}_{\mathfrak{m}_l}$.

*Lemma 2. Let* $\mathfrak{o}$ *be a commutative Noetherian ring of rank 1 which contains a field k. Let* $\mathfrak{o}'$ *be a finite* $\mathfrak{o}$*-module in the total ring of quotients of* $\mathfrak{o}$. *If q is the conductor of* $\mathfrak{o}'$ *over* $\mathfrak{o}$, *then we have*

$$q\mathfrak{o}'=q \quad\text{and}\quad \dim_k(\mathfrak{o}':\mathfrak{o})=\dim_k\mathfrak{o}'/q-\dim_k\mathfrak{o}/q\,,$$

*provided that, for each maximal ideal* $\mathfrak{m}$ *of* $\mathfrak{o}$ *containing q*, $[\mathfrak{o}/\mathfrak{m}:k]$ *is finite.*

(Proof) $q\mathfrak{o}'\mathfrak{o}'=q\mathfrak{o}'$, hence every element of $q\mathfrak{o}'$ is contained in the conductor $q$. But the converse is obvious, and we have $q\mathfrak{o}'=q$. Since $\mathfrak{o}'$ is a subring of the total ring of quotients of $\mathfrak{o}$ and finite

$\mathfrak{o}$-module, there exists an element $f \in q$ which is not a zero-divisor in $\mathfrak{o}$. Therefore $\mathfrak{o}'/q$ and $\mathfrak{o}/q$ are both of rank 0, the equality and the finiteness of each term in the right hand side follows from the above results and Lemma 1.

*Lemma 3. In Lemma 2, suppose that $\mathfrak{o}$ is a local ring. Then $\mathfrak{o}'$ is a semilocal ring and we have*

$$\dim_k(\mathfrak{o}'^* : \mathfrak{o}^*) = \dim_k(\mathfrak{o}' : \mathfrak{o}),$$

*where $\mathfrak{o}'^*$ and $\mathfrak{o}^*$ denote the completions of $\mathfrak{o}'$ and $\mathfrak{o}$ respectively.*

(Proof) Since $\mathfrak{o}'$ is finite over $\mathfrak{o}$, $\mathfrak{o}'^*$ is generated by a finite number of elements of $\mathfrak{o}'$ over $\mathfrak{o}^*$. Therefore we have $q\mathfrak{o}'^* \subseteqq q\mathfrak{o}^*$, hence $q\mathfrak{o}'^* = q\mathfrak{o}^*$. But we have $\mathfrak{o}^*/q\mathfrak{o}^* = \mathfrak{o}/q$ and $\mathfrak{o}'^*/q\mathfrak{o}'^* = \mathfrak{o}'/q$, and Lemma 3 follows directly from Lemma 2.

*Lemma 4. Notations being as in Lemma 3, let $k'$ be an overfield of $k$. Consider the tensor product $\mathfrak{o}' \otimes_k k'$ and $\mathfrak{o} \otimes_k k'$. Then we have*

$$\dim_k(\mathfrak{o}' : \mathfrak{o}) = \dim_{k \otimes_k k'}(\mathfrak{o}' \otimes_k k' : \mathfrak{o} \otimes_k k'),$$

*and the conductor of $\mathfrak{o}' \otimes_k k'$ over $\mathfrak{o} \otimes_k k'$ is $q \otimes_k k'$.*

(Proof) We have only to observe that the following sequence are exact: $0 \to \mathfrak{o} \otimes_k k' \to \mathfrak{o}' \otimes_k' \to (\mathfrak{o}' : \mathfrak{o}) \otimes_k k' \to 0$.

As for the conductor, it is obvious that $q \otimes_k k'$ is contained in the conductor and the converse is proved as follows: any element $f$ of $\mathfrak{o} \otimes_k k'$ can be written as $f = \sum_i f_i \otimes u_i$ with linearly independent elements $u_i$'s of $k'$ over $k$. And $f \cdot \mathfrak{o}' \subseteqq \mathfrak{o} \otimes_k k'$ implies that $f_i \mathfrak{o}' \subseteqq \mathfrak{o}$ i. e., $f_i \in q$ for each $i$, hence $f$ is in the $q \otimes_k k'$.

Now let $C_1$ and $C_2$ be two curves in $\boldsymbol{P}^N$ which have no common components. For an arbitrary common point $P$ of $C_1$ and $C_2$, we want to define the intersection multiplicity of $C_1$ and $C_2$ at $P$.

Let $k$ be a field over which $C_1$, $C_2$ and $P$ are rational. Let $R$ be the local ring of $\boldsymbol{P}^N$ at $P$ over $k$. Denote by $A_i$ the ideal of $C_i$ in $R (i=1, 2)$. Since $C_1$ and $C_2$ has no common components, the ideal $(A_1, A_2)R$ is a primary ideal belonging to the maximal ideal of $R$.

Definition 1. *We shall define and denote by $i(P, C_1 \cdot C_2)$ the intersection multiplicity of $C_1$ and $C_2$ at $P$ as follows:*

$$i(P, C_1 \cdot C_2) = \text{length } R/(A_1, A_2)R.$$

*Remark.* The above defined integer $i(P, C_1 \cdot C_2)$ depends only upon the $C_i$'s and $P$, that is, it is independent of $k$.[2]

Next we try to obtain another type of computation of the above $i(P; C_1 \cdot C_2)$. Let $C_1$, $C_2$, $\cdots$, and $C_r (r \geqq 2)$ be curves in $\boldsymbol{P}^N$, no two of which have common components but pass through a point $\boldsymbol{P}$. Let $k$ be a field over which all the $C_i$'s and the point $P$ are rational. Put $\sum_{i=1}^{r} C_i = C$, which is a curve in $\boldsymbol{P}^N$ by the assumption. Let $\mathfrak{o}$ be *the local ring of $C$ at $P$ over the field $k$.*[3] The total ring of quotients $s$ of $\mathfrak{o}$ is the direct sum of the function fields of prime rational components of the $C$'s over $k$. Hence we can find $r$ elements $e_i$ $(1 \leqq i \leqq r)$ in $s$ such that: 1) $e_1 + e_2 + \cdots + e_r = 1$, $e_i^2 = e_i$, $e_i e_j = 0$ if $i \neq j$, 2) the ideal of $C_i$ in $\mathfrak{o}$ is the kernel of the homormophism of $\mathfrak{o}$ onto $\mathfrak{o} e_i$ for $1 \leqq i \leqq r$. Put $\mathfrak{o}' = \mathfrak{o} e_1 + \mathfrak{o} e_2 + \cdots + \mathfrak{o} e_r$, which is a finite $\mathfrak{o}$-module in $s$.

*Proposition 1. Notations being as above, we have* $\dim_k(\mathfrak{o}' : \mathfrak{o}) = \sum_{q=2}^{r} i(P; (\sum_{i=1}^{q-1} C_i) \cdot C_q)$.

(Proof) First consider the case when $r=2$. Denote by $\mathfrak{n}_i$ the ideal of $C_i$ in $\mathfrak{o} (i=1, 2)$. We have $\mathfrak{n}_1 \mathfrak{o}' = \mathfrak{n}_1 \mathfrak{o} e_1 + \mathfrak{n}_1 \mathfrak{o} e_2 = \mathfrak{n}_1 \mathfrak{o} e_2 = \mathfrak{n}_1 \mathfrak{o} (e_1 + e_2) = \mathfrak{n}_1 \mathfrak{o} = \mathfrak{n}_1$. Therefore we have $\dim_k(\mathfrak{o}' : \mathfrak{o}) = \dim_k(\mathfrak{o}'/\mathfrak{n}_1 \mathfrak{o}' : \mathfrak{o}/\mathfrak{n}_1) = \dim_k(\mathfrak{o} e_1 + \mathfrak{o} e_2 / \mathfrak{n}_1 \mathfrak{o} e_2 : \mathfrak{o} e_1) = \dim_k(\mathfrak{o} e_2 / \mathfrak{n}_1 \mathfrak{o} e_2) = \dim_k(\mathfrak{o}/(\mathfrak{n}_1, \mathfrak{n}_2)\mathfrak{o}) =$ length $(\mathfrak{o}/(\mathfrak{n}_1, \mathfrak{n}_2)\mathfrak{o})$, where the last equality follows from Lemma 1.

When $r \geqq 3$, we have $\dim_k(\mathfrak{o}' : \mathfrak{o}) = \sum_{q=2}^{r} \dim_k(\mathfrak{o}(\sum_{i=1}^{q-1} e_i) + \sum_{j=q}^{r} \mathfrak{o} e_j : \mathfrak{o}(\sum_{i=1}^{q} e) + \sum_{j=q+1}^{r} \mathfrak{o} e_j) = \sum_{q=2}^{r} \dim_k(\mathfrak{o}(\sum_{i=1}^{q-1} e_i) + \mathfrak{o} e_q : \mathfrak{o}(\sum_{i=1}^{q} e_i))$. From the property of the $e_i$'s, we can see that $\mathfrak{o}(\sum_{i=1}^{q} e_i)$ is isomorphic to the local ring of $\sum_{i=1}^{q} C_i$ at $P$ over $k$ and that $\dim_k(\mathfrak{o}(\sum_{i=1}^{q-1} e_i) + \mathfrak{o} e_q : \mathfrak{o}(\sum_{i=1}^{q} e_i)) = i(P; (\sum_{i=1}^{q-1} C_i) \cdot C_q)$. These complete the proof.

*Proposition 2. Assume that there exists a surface $S$ in $\boldsymbol{P}^N$ such that $S$ contains all $C_i$'s and that $P$ is a simple point of $S$. Then we have*

$$\dim_k(\mathfrak{o}' : \mathfrak{o}) = \sum_{i \neq j} i(P; C_i \cdot C_j),$$

*Remark.* The assumption that $P$ is simple on $S$ means that the only one irreducible component $S_0$ of $S$ passes through $P$ and that $P$ is sinple on $S_0$[4]. It will be observed in the following proof that our $i(P; C_i \cdot C_j)$ is equal to the usual intersection multiplicity

$i(P;\ C_i \cdot C_j,\ S_0)$ with reference to the ambient variety $S_0$.[5)]

(Proof) By virtue of proposition 1 we have only to prove that $i(P;\ (\sum_{i=1}^{q-1} C_i) \cdot C_q) = \sum_{i=1}^{q-1} i(P;\ C_i \cdot C_q)$. We may assume that $S_0$ is defined over $k$. Let $R$ be the local ring of $S$, or $S_0$, at $P$ over $k$. By assumption $R$ is a regular local ring of rank 2[6)] and it is factorizable.[7)] Hence the ideal of $C_i$ in $R$ is principal, i. e., generated by an element $v_i$ in $R$. Since the $C_i$'s have no common components the ideal of $\sum_{i=1}^{q-1} C_i$ in $R$ is $(\prod_{i=1}^{q-1} v_i)R$. Since any system of parameters in a regular local ring is distinct,[8)] we have $i(P;\ (\sum_{i=1}^{q-1} C_i) \cdot C_q) =$ length $R/(\prod_{i=1}^{q-1} v_i,\ v_q)R = e((\prod_{i=1}^{q-1} v_i,\ v_q)R)$. Applying the associativity formula to this, we have $e((\prod_{i=1}^{q-1} v_i,\ v_q)R) = \sum_{i=1}^{q-1} e((v_i,\ v_q)R)$, which is equal to $\sum_{i=1}^{q-1}$ length $(R/(v_i,\ v_q)R) = \sum_{i=1}^{q-1} i(P;\ C_i \cdot C_q)$.

*Remark.* As is easily seen in the above proof, the distributive law of intersection multiplicity is verified so far as there exists such a surface $S$ as in proposition 2. But in general it is not. For example: Let $g_1$, $g_2$ and $g_3$ are the three axes in affine 3-space passing through the origin $P$. Then we have $i(P;\ (g_1+g_2) \cdot g_3) = 1$, which is a special case of the following assertion.

*Proposition 3. For two curves $C_1$ and $C_2$, $i(P;\ C_1 \cdot C_2) = 1$ if and only if the tangent spaces of Zariski of $C_1$ and $C_2$ at $P$ have no common points other than $P$.*

(Proof) The tangent space of Zariski[10)] of a curve $C$ at a point $P$ on it is defined as follows: By a projective transformation, we can take the point $P$ as the origin of the affine N-space $A^N$ with the hyperplane: $Y_0 = 0$ at infinity. The tangent space of Zariski is the linear variety in $A_N$, defined by all the linear forms which are linear parts of the polynomials in $k[X_1,\ X_2,\ \cdots,\ X_N]$ with $X_i = Y_i/Y_0 (1 \leq i \leq N)$ which vanish on the curve $C$. The proof of the assertion follows directly from our definition 1.

For our convenience, we introduce the following notations: 1) By virtue of Proposition 1, we can see that the integer $\sum_{q=2}^{r} i(P;\ (\sum_{i=1}^{q-1} C_i) \cdot C_q)$ is independent of the order of the suffices, we shall denote this brifly by $i(P;\ \Lambda_{i=1}^{r} C_i)$. Here if both $\sum_{i=1}^{q-1} C_i$ and $C_q$ not contain $P$ then $i(P;\ (\sum_{i=1}^{q-1} C_i) \cdot C_q) = 0$. 2) We shall denote by $\Lambda_{i=1}^{r} C_i$

the 0-dimensional $\boldsymbol{P}^N$-cycle $\sum_P i(P; \mathop{\Lambda}_{i=1}^{r} C)P$, where $P$ runs over all the points of $\sum_{i=1}^{r} C_i$.

Now we proceed to study the order of singularity of a curve $C$ at a point $P$.

Suppose that all the absolutely irreducible components $C_i (1 \leqq i \leqq r)$ of $C$ are defined and all the points on the derived normal curves $\widetilde{C}_i$ of $C_i (1 \leqq i \leqq r)$ corresponding to $P$ are rational over a common field $k$, we say that $C$ is *totally rational at $P$ over $k$*. Let $\mathfrak{o}$ be the local ring of $C$ at $P$ over such a field $k$.

*Definition 2.*[21] *We define and denote by $\delta(P: C)$ the order of singularity of $C$ along $P$ as follows: $\delta(P: C) = \dim_k(\tilde{\mathfrak{o}} : \mathfrak{o})$, where $\tilde{\mathfrak{o}}$ denotes the integral closure of $\mathfrak{o}$ in its total ring of quotients.*

By virtue of Lemma 2 $\dim_k(\tilde{\mathfrak{o}} : \mathfrak{o})$ is finite and $\delta(P: C)$ is well defined. We shall see that $\delta(P: C)$ depends only upon $C$ and $P$, i. e., it is independent of $k$. It is sufficient to see that, for an arbitrary overfield $k'$ of $k$, $\delta'(P: C)$ defined as above is equal to the above $\delta(P: C)$. Since $C$ and $P$ are rational over $k$, the local ring $\mathfrak{o}'$ of $C$ at $P$ over $k'$ is isomorphic to the ring of quotients of $\mathfrak{o} \otimes_k k'$ with respect to the maximal ideal $\mathfrak{m} \otimes_k k'$, where $\mathfrak{m}$ is the maximal ideal of $\mathfrak{o}$, over the isomorphism $k' \cong k \otimes_k k'$. We shall see that $\tilde{\mathfrak{o}} \otimes_k k'$ is integrally closed. Let $1 = e_1 + e_2 + \cdots + e_r$ be the decomposition of unity in the total ring of quotients $s$ of $\mathfrak{o}$, each $e_i$ corresponding to $C_i$ as remarked before proposition 1. Then each $e_i$ satisfies an equation $e_i^2 - e_i = 0$ and is integral over $\mathfrak{o}$. Hence $e_i \in \tilde{\mathfrak{o}}$ and $\tilde{\mathfrak{o}} = \tilde{\mathfrak{o}} e_1 + \tilde{\mathfrak{o}} e_2 + \cdots + \tilde{\mathfrak{o}} e_r$, where each $\tilde{\mathfrak{o}} e_i$ is isomorphic to the integral closure of $\mathfrak{o} e_i$ and to the semi-local ring of the $\widetilde{C}_i$ at the points $\widetilde{P}_{ij}$'s corresponding to $P$. By assumption $\widetilde{P}_{ij}$'s are rational over $k$ and absolutely simple on $C$, hence $\tilde{\mathfrak{o}} e_i \otimes_k k'$ is integrally closed. Thus $\tilde{\mathfrak{o}} \otimes_k k' = \tilde{\mathfrak{o}} e_1 \otimes_k k' \oplus \tilde{\mathfrak{o}} e_2 \otimes_k k' \oplus \cdots \oplus \tilde{\mathfrak{o}} e_r \otimes_k k'$ integrally closed.

This and the isomorphism $\mathfrak{o}' \cong (\mathfrak{o} \otimes_k k')_{\mathfrak{m} \otimes_k k'}$ imply the equality $\delta'(P: C) = \dim_{k \otimes_k k'}((\tilde{\mathfrak{o}} \otimes_k k')_S : (\mathfrak{o} \otimes_k k')_{\mathfrak{m} \otimes_k k'})$, where $S$ is the complementary set $\mathfrak{o} \otimes_k k' - \mathfrak{m} \otimes_k k'$. But the conductor of $\tilde{\mathfrak{o}} \otimes_k k'$ over $\mathfrak{o} \otimes_k k'$ is $q \otimes_k k'$ with that $q$ of $\tilde{\mathfrak{o}}$ over $\mathfrak{o}$, whose only one prime divisor is $\mathfrak{m} \otimes_k k'$. Therefore, applying our Lemma 2, $\delta'(P: C) = \dim_{k \otimes_k k'}(\tilde{\mathfrak{o}} \otimes_k k' : \mathfrak{o} \otimes_k k')$, which is equal to

$$\dim_k(\tilde{\mathfrak{o}} : \mathfrak{o}), \text{ by Lemma 3, or } \delta(P: C).$$

*Proposition 4. Notations being as above, we have* $\delta(P:C)=\sum_{i=1}^{r}\delta(P:C_i)+i(P;\ \Lambda_{i=1}^{r} C_i)$.

(Proof) $\delta(P:C)=\dim_k(\tilde{\mathfrak{o}}:\mathfrak{o})=\dim_k(\tilde{\mathfrak{o}}:\mathfrak{o}')+\dim_k(\mathfrak{o}':\mathfrak{o})$, where $\mathfrak{o}'=\mathfrak{o}e_1+\mathfrak{o}e_2+\cdots+\mathfrak{o}e_r$. In the above terms, $\dim_k(\tilde{\mathfrak{o}}:\mathfrak{o}')=\sum_{i=1}^{r}\dim_k(\tilde{\mathfrak{o}}e_i:\mathfrak{o}e_i)=\sum_{i=1}^{r}\delta(P:C_i)$ and $\dim_k(\mathfrak{o}':\mathfrak{o})=i(P;\ \Lambda_{i=1}^{r} C_i)$.

Next we shall study the process of revealing out the intrinsic singularity of the so-called *neighboring points* along a singular point of a curve by means of quadratic transformations.

Let $\mathfrak{o}$ be a local ring of rank 1 and let $\mathfrak{m}$ be the maximal ideal of $\mathfrak{o}$. An element $x$ of $\mathfrak{o}$ is called a superficial element of degree $s$ if the following conditions are satisfied: 1) $x\in\mathfrak{m}^s$, 2) $\mathfrak{m}^{s+a}:x\mathfrak{o}=\mathfrak{m}^a$ for sufficiently large $a$.

*Lemma 5.*[11] *An element $x$ of $\mathfrak{o}$ is a superficial element of degree $s$ if and only if a) $x$ is not a zero-divisor in $\mathfrak{o}$, b) $x\in\mathfrak{m}^s$ and c) $\mathfrak{o}[\mathfrak{m}^s/x]$ is a finite $\mathfrak{o}$-module.*

(Proof) If $x$ is a superficial element of degree $s$ and if $x\cdot y=0$ for an element $y\in\mathfrak{o}$, then we have $y\in\bigcap_a(\mathfrak{m}^{s+a}:x\mathfrak{o})=\bigcap_a\mathfrak{m}^a=(0)$, i. e., $y=0$, which says a). Therefore we shall assume a) and b) and prove that 2) and c) are equivalent. Since $\mathfrak{o}$ is of rank 1, a) implies that there exists an integer $q$ such that $\mathfrak{m}^q\subseteq x\mathfrak{o}$. Hence, if $s+a\geqq q$, 2) implies that $x\mathfrak{m}^a=\mathfrak{m}^{s+a}$. The converse is also proved using a). Further that $x\mathfrak{m}^a=\mathfrak{m}^{s+a}$ hold for all sufficiently large $a$ is equivalent to that $x\mathfrak{m}^{sb}=\mathfrak{m}^{s(b+1)}$ do for all sufficiently large $b$. This can be rewrited as $(\mathfrak{m}^s/x)^b=(\mathfrak{m}^s/x)^{b+1}$ for each $b$. It is easily to be seen that this is equivalent to c).

*Lemma 6.*[11] *When $\mathfrak{o}$ has two superficial elements $x$ and $y$ respectively of degrees $s$ and $t$. ($s\geqq 1, t\geqq 1$) Then we have* $\mathfrak{o}[\mathfrak{m}^s/x]=\mathfrak{o}[\mathfrak{m}^t/y]$.

(Proof) As is observed in the proof of Lemma 5, we have $x\mathfrak{m}^{n-s}=y\mathfrak{m}^{n-t}=\mathfrak{m}^n$ for sufficiently large $n$. Put $n=sa+t$; we have $x\mathfrak{m}^{(a-1)s+t}=y\mathfrak{m}^{as}$, in particular, $x^a\mathfrak{m}^t\subseteq y\mathfrak{m}^{as}$ for sufficiently large $a$. This can be rewrited as $\mathfrak{m}^t/y\subseteq(\mathfrak{m}^s/x)^a$; from these for all sufficiently large $a$ it follows that $\mathfrak{o}[\mathfrak{m}^s/x]\supseteq\mathfrak{o}[\mathfrak{m}^t/y]$. The converse is the same as above.

Let $P$ be a point of a curve $C$ in $\boldsymbol{P}^N$; we assume that $P$ and $C$ are rational over a field $k$ We may assume that the coordinates of $P$ is $(1;0;\cdots;0)$. Let $H=k[y_0, y_1 \cdots, y_N]$ be the homogeneous coordinate ring of $C$ over $k$, i.e., the quotient ring of $k[Y]=k[Y_0, Y_1, \cdots, Y_N]$ by the homogeneous ideal of $C$ in it. The curve $C'$, whose homogeneous coordinate ring is $H'=k[y_0y_1, y_0y_2, \cdots, y_0y_N, y_1^2, \cdots, y_1y_N, \cdots, y_N^2]$, in a projective space $P^{N(N+3)/2}$ is called the quadratic transform of $C$ with center $P$. A point of $C'$ corresponds to $P$ if and only if $y_1^2=\cdots=y_1y_N=\cdots=y_N^2=0$ in the coordinates of the point. And the points $P_i'(1\leqq i\leqq s)$ of $C'$ which correspond to $P$ are associated in a one to one way to the finite specializations of the ratio $(x_1:x_2:\cdots:x_N)$ over $(x_1, x_2, \cdots, x_N) \xrightarrow{k} (0, 0, \cdots, 0)$[12] where $x_i=y_i/y_0$ $(1\leqq i\leqq N)$; these points $P_i'$'s are called *the first neighboring points* of $C$ along $P$.

The following lemma will explain geometric meanings of superficial element of $\mathfrak{o}$.

*Lemma 7. A form $F(X)$ of degree $s$ in $X_1, X_2, \cdots, X_N$ does not vanish at every finite specializations of $(x_1: x_2: \cdots: x_N)$[12] over $(x) \xrightarrow{k} (0)$ if and only if $F(x)$ is a superficial element of degree $s$ in $\mathfrak{o}$.*

(Proof) Let $M_1(X), M_2(X), \cdots, M_\mu(X)$ be the monomials of degree $s$ in $X_1, \cdots, X_N$. Then $F(X)$ does not vanish at every finite specialization of $(x_1:x_2:\cdots:x_N)$ over $(x) \xrightarrow{k} (0)$ if and only if and only if $M_j(x)/F(x)$'s are integral over $\mathfrak{o}$ for all $j$.[13] Our assertion follows from Lemma 5.

*Remark.* By means of the above Lemma we can see that if the points $P_i'$'s are all rational over $k$ there exists a superficial element of some degree in $\mathfrak{o}$. But even if that's the case there may exist no superficial element of degree 1. When the field $k$ contains infinitely many elements, there always exists a superficial element of degree 1.

For the local ring $\mathfrak{o}$ and a superficial element $x$ of degree $s$, if there exists, the semi-local ring $\mathfrak{o}'=\mathfrak{o}[m^s/x]$ is called *the first neighborhood ring* of $\mathfrak{o}$. By virtue of Lemmas 7 & 6, this $\mathfrak{o}'$ is uniquely determined by $\mathfrak{o}$ and consists of all the functions induced on $C'$ which are defined at every point of $P_i'$'s. Thus the rings $\mathfrak{o}_i'$'s of quotients of $\mathfrak{o}'$ with respect to the maximal ideals in it are the local rings of $C'$ at the first neighboring points $P_i'$'s along $P$. If moreover some of the $P_i'$'s is singular we repeat the above process on $C'$, $P_i'$ and $\mathfrak{o}'$ and obtain the first neighboring points $P_{ij}''$ (neighborhood ring $\mathfrak{o}_i''$) of $C'$ along $P_i'$ (of $\mathfrak{o}_i'$); these points $P_{ij}''$, are called *the second neighboring points* of $C$ along $P$ (these rings $\mathfrak{o}_i''$'s the *second neighborhood rings* of $\mathfrak{o}$) ; and so on. The neighboring points, thus obtained by successive quadratic transformations beginning with the center $P$, are always of a finite number which will be seen in the proof of the following Theorem.

*Lemma 8. The tangent space of Zasiski of $C$ at $P$ is of dimension $\leqq 2$ if and only if there exists a surface $S$ containing $C$ such that $P$ is a simple point on $S$.*

(Proof) Let $k$ be a field over which $C$ and $P$ are rational. The local ring $R$ of $\boldsymbol{P}^N$ at $P$ over $k$ is regular local ring of rank $N$. Denote by $M$ the maximal ideal of $R$ and by $a$ the ideal of $C$ in it; $\mathfrak{o}=R/a$ is the local ring of $C$ at $P$ and $\mathfrak{m}=M/s$ is the maximal ideal of it. We have $\dim_k \mathfrak{m}/\mathfrak{m}^2=\dim_k M/M^2-\dim_k(a, M^2)/M^2=N-\dim_k(a, M^2)/M^2$. There fore $\dim_k \mathfrak{m}/\mathfrak{m}^2\leqq 2$ if and only if $\dim_k(a, M^2)/M^2\geqq N-2$, the latter of which means that there exist $(N-2)$ elements $f_1, f_2, \cdots, f_{N-2}$ in $a$ which can be extended to a regular system of parameters of $R$. Consider the surface $S$ which is defined by $f_1=f_2=\cdots=f_{N-2}=0$ locally at $P$. Conversely such a surface as mentioned in this Lemma is always obtained as above.

*Theorem 1. Let $C$ be a curve in $\boldsymbol{P}^N$ and $P$ be a point of $C$. Assume that there exists a surface $S$, containing $C$, in $\boldsymbol{P}^N$ such that $P$ is a simple pont of $S$. Then we have $\delta(P:C)=\sum_r r(r-1)/2$, where $r$ runs over all the multiplicities of the neighboring points of $C$ along $P$, inclusively of $P$ itself.*

(Proof) Let $\mathfrak{o}$ be the local ring of $C$ at $P$ over such a field $k$ that $C$ is totally rational along $P$ over $k$. The Theorem asserts the following equality $\dim_k(\tilde{\mathfrak{o}}:\mathfrak{o})=\sum_{\mathfrak{o}'} m(\mathfrak{o}')(m(\mathfrak{o}')-1)/2$ where $\tilde{\mathfrak{o}}$ is the integral closure of $\mathfrak{o}$ and $\mathfrak{o}'$ runs over all the local rings in neighborhood rings of $\mathfrak{o}$. We shall prove it by induction on $\dim_k(\tilde{\mathfrak{o}}:\mathfrak{o})$. When it is equal to $0$, $P$ is a simple point and $m(\mathfrak{o})=1$, and the equality holds. By the assumption on $S$ there exist two elements $z$ and $w$ in $\mathfrak{o}$ which generate the maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$. We may assume that $z$ is a superficial element of degree 1 of $\mathfrak{o}$. Then $\mathfrak{o}_1=\mathfrak{o}[w/z]$ is the first neighborhood ring of $\mathfrak{o}$. If $\mathfrak{m}'$ be any of the maximal ideals of $\mathfrak{o}_1$ then $\mathfrak{m}'$ contains $z$ and $w/z-a$ with some $a\in k$. But the ideal $(z, w/z-a)\mathfrak{o}_1$ is maximal in $\mathfrak{o}_1$, hence, equal to $\mathfrak{m}'$; thus every local ring of the first neighborhood ring of $\mathfrak{o}$ contains two elements which generate the maximal ideal, or, the quadratic transform $C'$ of $C$ with center $P$ satisfies the assumption in our Theorem at every point corresponding to $P$. On the other hand we have $\dim_k(\tilde{\mathfrak{o}}:\mathfrak{o})=\dim_k(\tilde{\mathfrak{o}}:\mathfrak{o}_1)+\dim_k(\mathfrak{o}_1:\mathfrak{o})$ while $\dim_k(\tilde{\mathfrak{o}}:\mathfrak{o}_1)=\dim_k(\tilde{\mathfrak{o}}^*:\mathfrak{o}_1^*)=\sum_i\dim_k(\tilde{\mathfrak{o}}_i^*:\mathfrak{o}_i^*)=\sum_i\dim_k(\tilde{\mathfrak{o}}_i:\mathfrak{o}_i)$, where the $\mathfrak{o}_i'$'s are the local rings in $\mathfrak{o}_1$, or those of the $C'$ at the points corresponding to $P$, and the $\tilde{\mathfrak{o}}_i$ is the integral closure of $\mathfrak{o}_i'$. Thus, by induction, we have only to prove that $\dim_k(\mathfrak{o}_1:\mathfrak{o})=m(\mathfrak{o})(m(\mathfrak{o})-1)/2$. Since $\mathfrak{o}'$ is a finite $\mathfrak{o}$-module $\mathfrak{o}^*$ is canonically imbedded in $\mathfrak{o}_1^*$ and $\mathfrak{o}_1^*=\mathfrak{o}^*[w/z]$. Now take a subring $r=k\{z\}$ in $\mathfrak{o}^*$. Since $\mathfrak{r}$ is a regular local ring of rank 1 and does not contain any zero-divisor of $\mathfrak{o}^*$ and of $\mathfrak{o}_1^*$, $\mathfrak{o}^*$ and $\mathfrak{o}_1^*$ have linearly independent bases as $\mathfrak{r}$-modules. Since we can write as $\mathfrak{o}^*=\mathfrak{r}[w]$, and $\mathfrak{o}_1^*=\mathfrak{r}[w/z]$, we can take such module-bases as $\mathfrak{o}^*=\sum_{i=0}^{s-1}\mathfrak{r}w^i$ and $\mathfrak{o}_1^*=\sum_{i=0}^{s-1}\mathfrak{r}(w/z)^i$ where $s=[\mathfrak{o}^*:\mathfrak{r}]=[\mathfrak{o}_1^*:\mathfrak{r}]$. Using these expressins and the independency of the base-elements with respect to $\mathfrak{r}$, we can prove that the $s(s-1)/2$ elements $u_{ij}=w^i/z^j(1\leqq j\leqq i,\ 1\leqq i\leqq s-1)$ of $\mathfrak{o}_1^*$ forms a linearly independent base of the factor space $(\mathfrak{o}_1^*:\mathfrak{o}^*)$ with respect to $k$. Therefore we have proved $\dim_k(\mathfrak{o}_1:\mathfrak{o})=\dim_k(\mathfrak{o}_1^*:\mathfrak{o}^*)=s(s-1)/2$. On the other hand by the extension formula[14] $rm(z\mathfrak{o}^*:\mathfrak{r})=[\mathfrak{o}^*:\mathfrak{r}]e(z\mathfrak{r})$ ; the left hand side is equal to $e(z\mathfrak{o}^*)=e(z\mathfrak{o})=\mathfrak{m}(\mathfrak{o})$ and the right hand side is equal to $[\mathfrak{o}^*:\mathfrak{r}]=s$. Thus we have $\dim_k(\mathfrak{o}':\mathfrak{o})=m(\mathfrak{o})(m(\mathfrak{o})-1)/2$, and the proof is completed.

*Remark.* The condition of the existence of such $S$ can not be omitted. *In general, we have the inequality: $\delta(P;C)\leqq\sum_r r(r-1)/2$ and the equality bolds if and only if there exists such a $S$ as in the Theorem 1.* This can be proved by taking a module base of $\mathfrak{o}$ over $\mathfrak{r}$ consisting of monomials $w_1^i w_2^j\cdots w_d^l$ where $w_1, w_2, \cdots, w_d$ togethered with $z$ form a minimal base of $\mathfrak{m}$.

(II)

We want to derive the genus formula and the generalized modular property of the arithmetic genera of curves.

Let $C$ be a curve in $\boldsymbol{P}^N$ which is rational over a field $k$. Let $H=k[y_0, y_1, \cdots, y_N]$ be the homogeneous coordinate ring of $C$ over $k$, that is, the quotients ring of that $k[Y_0, Y_1, \cdots, Y_N]$ by the homogeneous ideal $C$ in it. The Hilbert characteristic function $\chi(C, m)$ of $C$ is a polynomial in $m$ which gives the number of linearly independent forms of degree $m$ in the $y_i$'s over $k$ for sufficiently large $m$. The constant term of $\chi(C; m)$ is written as $1-p_a(C)$, where $p_a(C)$ is the so-called arithmetic genus of $C$. It is easily to be seen that $\chi(C, m)$, hence, $p_a(C)$, is independent of

$k$ so far as $C$ is rational over $k$. In the following we shall assume that $k$ is algebraically closed; for our purpose no loss of generality comes from this assumption. Let $\tilde{H}$ be the integral closure of $H$ in its total ring of quotients. As is remarked before, we can take tho decomposition of unity, say $1=e_1+e_2+\cdots+e_r$, in $\tilde{H}$ such that the kernel of the homomorphism of $H$ to $He_i$ is the ideal of one of the absolutely irreducible components of $C$, say $C_i$, or $He_i$ is isomorphic to the homogeneous coordinate ring of $C_i$ over $k$. We have $\tilde{H}=\tilde{H}e_1+\cdots+\tilde{H}e_r$; each $\tilde{H}e_i$ is the integral closure of $He_i$ in its field of quotients. We shall denote by $\tilde{H}_m$ and $H_m$ the $k$-modules of the homogeneous elements of degree $m$ in $\tilde{H}$ and $H$ respectively; we have $\tilde{H}_m=(He_1)_m+(He_2)_m+\cdots+(He_r)_m$. (Observe that each $e_i$ is homogeneous of degree 0.) For sufficiently large $m$, $\dim_k \tilde{H}_m$ and $\dim_k(\tilde{H}e_i)_m$ are polynomials in $m$[15], which will be denoted by $\tilde{\chi}(C, m)$ and $\tilde{\chi}(C_i, m)$ respectively. We have $\tilde{\chi}(C, m) =\sum_{i=1}^{r}\tilde{\chi}(C_i, m)$.

*Lemma 9.* $\tilde{\chi}(C, m)-\chi(C, m)$ *is a constant in* $m$.

(Proof) Since $\tilde{H}$ has a base as $H$-module consisting of a finite number of homogeneous elements, there exists a homogeneous element $f$ of $H$ which is not a zero-divisor in $H$ and is in the conductor of $\tilde{H}$ over $H$. Put $d$=degree of $f$. Since $f\tilde{H}_m\subseteq H_{d+m}$, we have $\dim_k\tilde{H}_m=\dim_k f\tilde{H}_m\leqq\dim_k H_{d+m}$, which implies that $\tilde{\chi}(C, m)\leqq\chi(C, d+m)$ for sufficiently large $m$. But obviously $\tilde{\chi}(C, m)\geqq\chi(C, m)$ for sufficiently large $m$. These two inequalities shows that $\tilde{\chi}(C, m)$ and $\chi(C, m)$ have the same degree and the same coefficient in their terms of the highest degree. As is well known, $\chi(C, m)$ is a polynomial of degree 1, and therefore $\tilde{\chi}(C, m)-\chi(C, m)$ must be a constant.

*Lemma 10. The constant term of* $\tilde{\chi}(C_i, m)$ *is equal to* $1-p_a(\tilde{C}_i)$ *where* $C_i$ *is the derived non-singular curve of the* $C_i$.

(Proof) $\tilde{H}e_i$ is the integral closure of $He_i$ and has a base as $He_i$-module consisting of a finite number of homogeneous elements. Put $d$=the maximum of degrees of those elements. If we take a system of generators, say $z_0, z_1, \cdots, z_M$ of $(\tilde{H}e_i)_d$ as $k$-module, $k[z] =k[z_0, z_1, \cdots, z_M]$ is integrally closed for we have $k[z]_m=(\tilde{H}e_i)_{dm}$ for all $m\geqq1$ and integrally dependent upon $H$. Therefore $k[z]$

is the homogeneous coordinate ring of a curve which is biregularly equivalent to $\widetilde{C}_i$. Denote by $\chi(\widetilde{C}_i, m)$ the polynomial in $m$ which is equal to $\dim_k k[z]_m$ for sufficiently large $m$. Then the constant term of it is $1-p_a(\widetilde{C}_i)$. But $\tilde{\chi}(C_i, dm)=\chi(\widetilde{C}_i, m)$ for all sufficiently large $m$, and therefore $\tilde{\chi}(C_i, m)$ has the same constant term $1-p_a(\widetilde{C}_i)$.

By the above Lemmas, we see that the constant $\tilde{\chi}(C, m)-\chi(C, m)$ is equal to $\sum_{i=1}^{r}(1-p_a(\widetilde{C}_i))-(1-p_a(C))$, or, to $p_a(C)-\sum_{i=1}^{r}p_a(C_i)+(r-1)$.

In order to estimate the constant in terms of orders of singularity locally at all the points of $C$, we need the following assertions.

Let $V$ be an arbitrary variety (or, a bunch of subvarieties) in $\boldsymbol{P}^N$, the irreducible components of which are defined over a field $k$. Denote by $k[y]=k[y_0, y_1, \cdots, y_N]$ the homogeneous coordinate ring of $V$, i. e., the quotient ring of $k[Y]=k[Y_0, Y_1, \cdots, Y_N]$ by the ideal of $V$ in it.

*Lemma 11. Let $F_j(Y)$ $(1\leqq j\leqq t)$ be forms of the same degree in $k[Y]$. Then $k[y]$ is integrally dependent over $k[F(y)]$ if and only if the hypersurfaces: $F(Y)=0$ $(1\leqq j\leqq t)$ in $\boldsymbol{P}^N$ have no common points with the variety $V$.*

(Proof) The proof is completed in a *usual manner*, and it is omitted here.[16]

*Lemma 12*[17]. *Assume that the hypersurfaces: $F(Y)=0$ $(1\leqq j\leqq t)$ have no common points with $V$. Denote by $\widetilde{H}$ the integral closure of $H=k[y]$ in its total ring quotients. Let $z$ be an element of $\widetilde{H}_m$ such that $F_i(y)^q z\in H$ for some $q$ and for all $j$. Then $z$ is in $H$, provided that $m$ is not less than a fixed integer independent of $z$.*

(Proof) Put $H'=\{f\in\widetilde{H};\ F_j(y)^q f\in H$ *for some* $q$, *for all* $j\}$. Since $\widetilde{H}$ is a finite $H$-module, hence, a finite $k[F(y)]$-module by Lemma 11, the $H'$ is also a finite $k[F(y)]$-module. Obviously $H'$ is graded and has a base as $k[F(y)]$-module consisting of a finite number of homogeneous elements, say $w_1, w_2, \cdots, w_g$. For each $j$, there exists an integer $q(j)$ such that $F_j(y)^{q(j)}w_i\in H$ for all $i$. Take an integer $q$ such that $(F_1(y), F_2(y), \cdots, F_t(y))^q\ k[F(y)]\subseteq(F_1(y)^{q(1)}, F_2(y)^{q(2)}, \cdots, F_t(y)^{q(t)})\,k[F(y)]$. Then we see that $H_m'\subseteq\sum_{\mu\geqq q}(H_\mu w_1+$

$\cdots+H_{\mu}w_g)\subseteqq H$, or $H_m'=H_m$ for all integer $m\geqq dq+e$, where $d=\deg F_j$ and $e=\max\{\deg w_\nu\}$.

Now we return to our case. Let $C$, $k$, $H$, and $\widetilde{H}$ be as before. Since $k$ contains an infinitely many elements there exists a linear form $L(Y)$ in $Y_0,\cdots,Y_N$ with coefficients in $k$ such that the hyperplane: $L(Y)=0$ does not pass through any component and any multiple point of $C$. Put $\bar{y}=L(y)$ and $A=k[y_0/\bar{y}, y_1/\bar{y},\cdots, y_N/\bar{y}]$. For sufficiently large $d$, $\widetilde{A}=k[\widetilde{H}_d/\bar{y}^d]$ is the integral closure of $A$.

Let $q$ be the conductor of $\widetilde{A}$ over $A$. Since $\widetilde{A}/q$ and $A/q$ finite $k$-modules, we can see that $\widetilde{A}/q\cong(\widetilde{H}_m/\bar{y}^m)/q\cap(\widetilde{H}_m/\bar{y}^m)$ and $A/q\cong(H_m/\bar{y}^m)/q\cap(H_m/\bar{y}^m)$ for sufficiently large $m$. Thus we have

$$\begin{aligned}\dim_k(\widetilde{A}:A)&=\dim_k(\widetilde{A}/q)-\dim_k(A/q)\\&=\dim_k(\widetilde{H}_m/\bar{y}^m)-\dim_k(H_m/\bar{y}^m)+\dim_k(q\cap H_m/\bar{y}^m)\\&\qquad-\dim_k(q\cap\widetilde{H}_m/\bar{y}^m)\cdots\cdots(*)\end{aligned}$$

We want to prove that for sufficiently large $m$, $q\cap\widetilde{H}_m/\bar{y}^m=q\cap H_m/\bar{y}^m$; it is sufficient to prove this that we see the following.

*Lemma 13. For sufficiently large $m$, we have $A\cap\widetilde{H}_m/\bar{y}^m=H_m/\bar{y}^m$.*

(Proof) For a sufficiently large $d$, there exists a hypersurface; $F(Y)=0$ of degree $d$ which passes through all the multiple points of $C$ but none of the common points of $C$ and the hyperplane: $L(Y)=0$. Then $\bar{y}^d$ and $F(y)$ satisfy the assumption in Lemma 12, and for sufficiently large $m$ if $z$ of $H_m$ satisfies the conditions: $\bar{y}^{qd}z\in H$ and $F(y)^q z\in H$ for some integer $q$ then $z\in H$. But $F(Y)=0$ contains all the multiple points of $C$ and therefore $k[H_d/F(y)]$ is integrally closed, hence, coincides with $k[H_d/F(y)]$. This means that for any element $z$ of $H$ there exists an integer $q$ such that $F(y)^q z\in H$. Thus, for sufficiently large $m$, if $z/\bar{y}^m\in A$ with $z\in H_m$ then there exists an integer $q$ such that $\bar{y}^q z\in H_{m+q}$ and therefore $z\in H_m$, i.e., $z/\bar{y}^m\in H_m/\bar{y}^m$. The converse is obvious.

From this Lemma and the previous remark on the equality (*), it follows that we have $\dim_k(\widetilde{A}:A)=\dim_k(\widetilde{H}_m/\bar{y}^m)-\dim_k(H_m/\bar{y}^m)$, for sufficiently large $m$. Since the hyperplane; $L(Y)=0$ does not contain any of the irreducible components of $C$, $\bar{y}$ is not a zero-divisor in $H$ and also in $\widetilde{H}$. Therefore we have $\dim_k(H_m/\bar{y}^m)=\dim_k H_m$ and $\dim_k(\widetilde{H}_m/\bar{y}^m)=\dim_k\widetilde{H}_m$.

Thus we have obtained the following equality:

$$p_a(C)-\sum_{i=1}^{r}p_a(\widetilde{C}_i)+(r-1)=\dim_k(\widetilde{A}:A).$$

We shall consider on the above $\dim_k(\widetilde{A}:A)$. The prime divisors, say $p_1, p_2, \cdots$, and $p_\delta$, in $A$ of the conductor $q$ are associated to the multiple points, say $P_1, P_2, \cdots$, and $P_\delta$, of $C$ in such a way that $\mathfrak{o}_i=A_{p_i}$ is the local ring of $C$ at $P_i$ over $k$ for all $i$. Put $S_i=A-p_i$ for each $i$ and $S=\bigcup_{i=1}^{\delta}S_i$. Since $S$ is multiplicatively closed and does not intersect with $q$, we have $\widetilde{A}_S/q\widetilde{A}_S=\widetilde{A}/q$ and $A_S/qA_S=A/q$. By Lemma 2 we have $\dim_k(\widetilde{A}:A)=\dim_k(\widetilde{A}_S:A_S)$, which is equal to $\dim_k(\widetilde{A}_S{}^*:A_S{}^*)$ by Lemma 3. As is well known, the completion $A_S{}^*$ is the direct sum of the completions $A_{S_i}{}^*$'s and therefore $\widetilde{A}_S{}^*$ is so of the $\widetilde{A}_{S_i}{}^*$. By this we have $\dim_k(\widetilde{A}_S{}^*:A_S{}^*)=\sum_{i=1}^{\delta}\dim_k(\widetilde{A}_{S_i}{}^*:A_{S_i}{}^*)$, which is equal to $\sum_{i=1}^{\delta}\dim_k(\widetilde{A}_{S_i}:A_{S_i})$ by Lemma 3. But, easily to be seen, $\widetilde{A}_{S_i}$ is the integral closure of $A_{S_i}$ and $\dim_k(\widetilde{A}_{S_i}:A_{S_i})=\delta(P_i:C)$ for each $i$. Hence we have proved that $\dim_k(\widetilde{A}:A)=\sum_{P\in C}\delta(P:C)$.

We want to sum up the above result in the following.

*Theorem 2. (The general Genus Formula) For an arbitrary curve C in $\boldsymbol{P}^N$ we have:*

$$p_a(C)=\pi(C)+\sum_{P\in C}\delta(P:C)-(r-1),$$

*where $\pi(C)$ is the effective genus of C and r is the number of absolutely irreducible components of C.*

By virtue of Theorem 1, we see that Theorem 2 includes the classical genus formula for plane curves as one of special cases. The following Theorem may be considered as a generalization of the modular property of the arithmetic genera of curves on a non-singular surface (or a surface for which any intersecting points of curves are simple), which is observed through the Proposition 1 and its remark; in the following we have no restriction in this sense.

*Theorem 3. (The general Modular Property of the arithmetic genera of curves) Let $C_i(1\leqq i\leqq r)$ be curves in $\boldsymbol{P}^N$, no two of which have any common component. Then we have*

$$p_a(C_1+C_2+\cdots+C_r)=\sum_{i=1}^{r}p_a(C_i)+\deg(\bigwedge_{i=1}^{r}C)-(r-1).$$

(Proof) By virtue of Proposition 4, for each point $P$ of $C=\sum_{i=1}^{r}C_i$, $\delta(P:C)=\sum_{i=1}^{r}\delta(P:C_i)+i(P,\overset{r}{\underset{i=1}{\Lambda}}C_i)$. Using this and applying Theorem 2 to the $C$ and each of the $C_i$'s, we can easily construct the proof.

(III)

We want to show an application of the Principle of Degeneration which was proved by O. Zariski to the effective genera of degenerated curves.

For two curves $C$ and $C'$ in $\boldsymbol{P}^N$ and a field $k_0$, we shall say that $C'$ is a specialization of $C$ over $k_0$ if the Chow-point $c'$ of $C'$ is a specialization of that $c$ of $C$ over $k_0$. In the field $k=k_0(c)$ we can take a ground place $\phi$ which maps $c$ to $c'$; a ground place means a place whose valuation ring, denote by $\mathfrak{v}$, is discrete of rank 1 and satisfies the finiteness condition. For such a place $\phi$, a point $P'$ of $\boldsymbol{P}^N$ lies on $C'$ if and only if $P'$ is a specialization of some point of $C$ over $\mathfrak{v}$, or, if and only if in the local ring $\mathfrak{R}$ of $\boldsymbol{P}^N$ at $P'$ over $\mathfrak{v}$ the ideal $\mathfrak{A}$ of $C$ does not contain the unity. The quotient ring $\mathfrak{o}=\mathfrak{R}/\mathfrak{A}$ is called the local ring of $C$ at $P'$ of $C'$ over $\mathfrak{v}$.

We shall denote by $\mathfrak{p}$ the prime ideal of $\mathfrak{v}$. Since $\mathfrak{v}$ is imbedded isomorphically in $\mathfrak{R}$ and in $\mathfrak{o}$, the same notations $\mathfrak{v}$ and $\mathfrak{p}$ will be used in $\mathfrak{R}$ and in $\mathfrak{o}$. Denote by $\kappa$ the image field of $k$ by the place $\phi$. We can see that if $\mathfrak{J}$ is the J-radical of the ideal $\mathfrak{p}\mathfrak{o}$ then $\mathfrak{o}/\mathfrak{J}$ is isomorphic to the local ring $\mathfrak{o}'$ of $C'$ at $P'$ over $\kappa$ over the isomorphism $\mathfrak{v}/\mathfrak{p}\simeq\kappa$. In this sense the minimal prime divisors $\mathfrak{p}_i(1\leq i\leq r)$ of $\mathfrak{p}\mathfrak{o}$ correspond in a one to one way to the prime rational components $C_i'$'s of $C'$ which contain the point $P'$. From the assumption that $C'$ is a curve, i. e., it has no multiple components, the local ring $\mathfrak{o}_{\mathfrak{p}_i}$ is a valuation ring whose maximal ideal is generated by $\mathfrak{p}$. We shall denote by $\Pi$ a fixed prime element of $\mathfrak{v}$, which is not a zero-divisor in $\mathfrak{o}$. Consider the integral closure $\hat{\mathfrak{o}}$ of $\mathfrak{o}$ in $\mathfrak{o}[1/\Pi](=k[\mathfrak{o}])$.

*Lemma 14. $\Pi\hat{\mathfrak{o}}$ is an unmixed ideal and $\Pi\hat{\mathfrak{o}}\cap\mathfrak{o}=\mathfrak{J}$; moreover $\Pi\mathfrak{o}$ is unmixed if and only if $\hat{\mathfrak{o}}=\mathfrak{o}$.*

(Proof) If we apply Lemma 2 of § 4 in [2],[18] observing that $\Pi$ is not a zero-divisor in $\mathfrak{o}$ and in $\hat{\mathfrak{o}}$ and that $\mathfrak{o}_{\mathfrak{p}_i}$ is normal with the maximal ideal generated by $\Pi$ for all minimal prime divisor $\mathfrak{p}_i$ of $\Pi\mathfrak{o}$, we see the assertion easily.

We shall denote by $\tilde{\mathfrak{o}}$ the integral closure of $\mathfrak{o}$ in its total ring of quotients, which is a finite $\mathfrak{o}$-module. Since $\tilde{\mathfrak{o}}$ is integrally closed, hence direct sum of normal rings, $\Pi\tilde{\mathfrak{o}}$ is an unmixed ideal and $\Pi\tilde{\mathfrak{o}}\cap\hat{\mathfrak{o}}=\Pi\hat{\mathfrak{o}}$. Put $\tilde{\mathfrak{o}}'=\tilde{\mathfrak{o}}/\Pi\tilde{\mathfrak{o}}$, $\hat{\mathfrak{o}}'=\hat{\mathfrak{o}}/\Pi\hat{\mathfrak{o}}$ and $\mathfrak{o}'=\mathfrak{o}/\mathfrak{J}$; $\tilde{\mathfrak{o}}'$, and $\hat{\mathfrak{o}}'$ are also finite $\mathfrak{o}'$-modules. Further $\tilde{\mathfrak{o}}'$, hence $\hat{\mathfrak{o}}'$, is a subring of the total ring of quotients of $\mathfrak{o}'$, which follows from the fact that $\mathfrak{o}_{\mathfrak{p}_i}$ is normal for all minimal prime divisor $\mathfrak{p}_i$ of $\Pi\mathfrak{o}$. Therefore we have the well-defined $\dim_\kappa(\hat{\mathfrak{o}}':\mathfrak{o}')$ associated to each point $P'$ of $C'$.

*Lemma 15. We have* $\dim_\kappa(\hat{\mathfrak{o}}':\mathfrak{o}')=0$ *if and only if* $\Pi\mathfrak{o}$ *is unmixed.*

(Proof) Since $\Pi\hat{\mathfrak{o}}\cap\mathfrak{o}=\mathfrak{J}$, we see $\Pi\hat{\mathfrak{o}}\supseteq\mathfrak{J}\hat{\mathfrak{o}}$ but the converse is obvious; we have $\Pi\hat{\mathfrak{o}}=\mathfrak{J}\hat{\mathfrak{o}}$. Therefore observing that $\hat{\mathfrak{o}}$ is a finite o-module, we conclude that $\hat{\mathfrak{o}}'=\hat{\mathfrak{o}}/\mathfrak{J}\hat{\mathfrak{o}}$ and $\mathfrak{o}'=\mathfrak{o}/\mathfrak{J}$ coincide if and only if $\hat{\mathfrak{o}}=\mathfrak{o}$,[19] which is equivalent to that $\Pi\mathfrak{o}$ is unmixed by Lemma 14.

We shall denote by $\delta(P':C/C')$ instead of the above $\dim_\kappa(\hat{\mathfrak{o}}':\mathfrak{o}')$; we can prove that this integer depends only upon $P'$, $C$ and $C'$, or, it is independent of the ground place, but we need not this fact and the proof is omitted.

*Theorem 4. Notations being as above, we have* $p_a(C')-p_a(C)=\sum_{P'\in C'}\delta(P':C/C')$.

In order to prove this we must have some further preliminary considerations. Let $\mathfrak{H}=\mathfrak{v}[y_0, y_1, \cdots, y_N]$ be the homogeneous coordinate ring of $C$ over $\mathfrak{v}$, i. e., the quotient ring of $\mathfrak{v}[Y]$ by the ideal generated by all the forms in it which vanish on $C$. We shall denote by $\tilde{\mathfrak{H}}$ and by $\hat{\mathfrak{H}}$ the integral closures of $\mathfrak{H}$ respectively in the total ring of quotients of $\mathfrak{H}$ and in $\mathfrak{H}[1/\Pi]=k[y]$; then $\tilde{\mathfrak{H}}$, and therefore $\hat{\mathfrak{H}}$, is a finite $\mathfrak{H}$-module. In a usual manner,[20] we can prove that $\tilde{\mathfrak{H}}$, hence $\hat{\mathfrak{H}}$, has a finite number of homogeneous generators as $\mathfrak{H}$-module. Therefore, if we take a set of generators $\zeta_i(0\leqq i\leqq S)$ of the $\mathfrak{v}$-module $\tilde{\mathfrak{H}}_m$ and also such $\omega_i(0\leqq i\leqq T)$ of $\hat{\mathfrak{H}}_m$ for sufficiently large $m$, then $\mathfrak{v}[\zeta]=\mathfrak{v}[\zeta_0, \zeta_1, \cdots, \zeta_S]$ is integrally closed in its total ring of xuotients and $\mathfrak{v}[\omega]=\mathfrak{v}[\omega_0, \omega_1, \cdots, \omega_T]$ is so in $k[\omega]$. We shall fix such $\mathfrak{v}[\zeta]$ and $\mathfrak{v}[\omega]$, and denote by $\tilde{C}$ the curve in $\boldsymbol{P}^S$ with $k[\zeta]$ as its homogeneous coordinate ring and by $\hat{C}$ the curve in $\boldsymbol{P}^T$ with $k[\omega]$. Further we shall fix a homogeneous ring $\mathfrak{v}[\eta]$ generated by a set of generaters $\eta_i$'s of $\mathfrak{v}$-module $\mathfrak{H}_m$ for the above fixed integer $m$. Let $P'$ be an arbitrary point of $C'$

and let $\mathfrak{P}$ be the ideal of $P'$ in $\mathfrak{H}$, i. e., the ideal generated by all such elements $F(y)$ as the $\phi$-image of $F(Y)$ vanish at $P'$. Put $\overline{\mathfrak{P}}=\mathfrak{P}\cap\mathfrak{v}[\eta]$, the ideal of $P'$ in $\mathfrak{v}[\eta]$. It is easily to be seen that the ring consisting of all homogeneous elements of degree 0 in the ring of quotients $\mathfrak{H}_{\mathfrak{P}}$ is equal to that ring in $\mathfrak{v}[\eta]_{\overline{\mathfrak{P}}}$, which is nothing but the local ring $\mathfrak{o}$ of $C$ at $P'$ over $\mathfrak{v}$. Further we see that, if we put $M=\mathfrak{v}[\eta]-\overline{\mathfrak{P}}$, the ring consisting of all homogeneous elements of degree 0 in $\mathfrak{v}[\zeta]_M$ is the $\tilde{\mathfrak{o}}$ derived from $\mathfrak{o}$ as in Lemma 14, and in the same way that ring in $\mathfrak{v}[\omega]_M$ is that $\hat{\mathfrak{o}}$. Since the curves $\widetilde{C}$ and $\widehat{C}$ are rational over the field $k$ the Chow-points of them are also rational over $k$ and the $\widetilde{C}$ and $\widehat{C}$ have uniquely determined specializations over the ground place $\phi$, which will be denoted by $\widetilde{C}'$ (in $\boldsymbol{P}^S$) and by $\widehat{C}'$ (in $\boldsymbol{P}^T$) respectively. Then the rings $\tilde{\mathfrak{o}}'$ and $\hat{\mathfrak{o}}'$ are the semi-local rings of the points on $\widetilde{C}'$ and those on $\widehat{C}'$, respectively, which correspond to the $P'$ of $C'$ (the correspondence is determined by the inclusion: $\mathfrak{v}[\eta]\subseteqq\mathfrak{v}[\omega]\subseteqq\mathfrak{v}[\zeta]$).

Now we shall observe the above obtained simultaneous specializations: $(\widehat{C},\ C)\rightarrow(\widehat{C}',\ C')$ over the $\phi$, and prove the Theorem 4. Applying the Therem which we have shown in [5], we have $\chi(\widehat{C},\ m)=\chi(\widehat{C}',\ m)$, for at every point $P'$ of $C'$ the local ring of $C$ at $P'$ over $\mathfrak{v}$ is obtained as a ring of quotients of the $\hat{\mathfrak{o}}$ derived from the local ring of $C$ at the corresponding point $P'$ of $C'$ over $\mathfrak{v}$ and for all such $\mathfrak{o}$ $I\mathfrak{o}$ is an unmixed ideal. But $\chi(\widehat{C},\ m)=dm+1-p_a(\widehat{C})$ and $\chi(\widehat{C}',\ m)=dm+1-p_a(\widehat{C}')$ whese $d$ is the common degree of $\widehat{C}$ and $\widehat{C}'$ in $\boldsymbol{P}^T$; therefore we have $p_a(\widehat{C})=p_a(\widehat{C}')$. Obviously $k[\eta]=k[\omega]$, hence $\widehat{C}$ and $C$ are biregularly equivalent to each other over the field $k$ and we have $p_a(C)=p_a(\widehat{C})=p_a(\widehat{C}')$. On the other hand; for each point $P'$ of $C'$, if we denote by $P_1'$, $P_2'$, $\cdots$, and $P_\mu'$ the corresponding points of $\widehat{C}'$ to $P'$, in the same way as in the proof of Theorem 1 we have $\delta(P':C')-\sum_{i=1}^{\mu}\delta(P':\widehat{C}')=\dim(\hat{\mathfrak{o}}':\mathfrak{o}')=\delta(P':C/C')$. Thus we have $p_a(C')-p_a(C)=p_a(C')-p_a(\widehat{C}')=\sum_{P'\in C'}\delta(P':C/C')$ applying the Genus Formula to $\widehat{C}'$ and to $C'$. We have completed the proof of Theorem 4.

Now we are going to prove the final result in this paper. In the following we shall assume that a non-singular curve can be derived from $C$ by normalization with reference to the field $k$; if not, we may replace $k$ by its finitely algebraic extension and $\phi$ by an arbitrary extension in it, which is also a ground place and cause the same specialization $C \to C'$. We take the $\widetilde{C}'$ constructed as above with respect to such a ground place.

*Theorem 5. Suppose $C$ is absolutely irreducible, we have*

$$\pi(C) - \pi(C') = \sum_{\widetilde{P}' \in \widetilde{C}'} \delta(\widetilde{P}' : \widetilde{C}') - (r-1)$$

*where $r$ is the number of the absolutely irreducible components of $C'$. Moreoverwe have $\sum_{\widetilde{P}' \in \widetilde{C}'} \delta(\widetilde{P}' : \widetilde{C}') \geqq (r-1)$, i.e., $\pi(C) \geqq \pi(C')$.*

(Proof) Applying the Theorem in [5] to the $\widetilde{C}$ and $\widetilde{C}'$, we have $p_a(\widetilde{C}) = p_a(\widetilde{C}')$. By assumption the above obtained $\widetilde{C}$ is a non-singular curve derived from $C$, and $p_a(\widetilde{C}) = \pi(C)$ by definition. Therefore the first assertion follows directly from the Genus Formula applied to the $C'$. Let $\widetilde{C}_1'$, $\widetilde{C}_2'$, and $\widetilde{C}_r'$ be the absolutely irreducible components of $\widetilde{C}'$. By Proposition 4 we have $\sum_{\widetilde{P}' \in \widetilde{C}'} \delta(\widetilde{P}' : \widetilde{C}') \geqq \deg(\bigwedge_{i=1}^{r} \widetilde{C}_i')$; therefore if we prove the inequality: $\deg(\bigwedge_{i=1}^{r} \widetilde{C}_i') \geqq r-1$, we shall conclude the second assertion. By virtue of the Principle of Degeneration,[22] however, the curve $\widetilde{C}'$ is connected and the inequality is proved elementarily, refering to Proposition 1 and Definition 1.

---

**Notes**

1) We shall say so instead of "absolutely non-singular."

2) We have only to prove that we have the same $i(P: C_1C_2)$ by replacing $k$ by its arbitrary extension $k'$. Let $R'$ be the local ring of $P^N$ at $P$ over $k'$. Since $C_i$'s are rational over $k$, $A_iR'$ is the ideal of $C_i$ in $R'$. And since $P$ is rational over $k$, for the maximal ideal $M$ of $R$ $Mk'R$ is maximal ideal ideal in $k'R$ and $R' = (k'R)_{(Mk'R)}$. By these we see that $R'/(A_1, A_2) = (R/(A_1, A_2)R) \otimes_k k'$. The independency follows from Lemmas 1 & 3.

3) It means the quotient ring of $R$ by the ideal of $C$.

4) cf. [9], in this paper "simple" means "absolutely simple."

5) cf. [11].

6) cf. [9].

7) cf. [9] or [12].

8) cf. [3].

9) cf. [4] or [12].

10) cf. [9].

11) cf. [10].

12) Here the $x_i$'s may contain zero-divisor (but no nilpotent elements). The specialization is defined in the same way as usual. On the other hand the precise meanings of $y_i/y_0=x_i$ is as follows: $x_i$ is the ratio of the images of $y_i$ and $y_0$ in the ring of quotients of $k[y]$ with respect to the set of all powers of $y_0$.

13) This is well-known fact when $k[x]$ is integrity domain. But the general case (so far as free from nilpotent elements) can be reduced easily to the case using the decomposition of unity.

14) cf. [3] or [11].

15) This is true for any graded $k[Y]$-module having a finite number of generators. Refer to the theorem of syzygies.

16) As in the note 13), we may use the decomposition of unity in the total ring of quotients of $k[y]$ and reduce it to the irreducible case.

17) cf. [6].

18) Or, see [1].

19) cf. [1].

20) cf. [6].

21) cf. [13].

22) cf. [7].

## REFERENCES


[1] M. Nagata, Basic theorems on general commutative rings, Memoirs Kyoto University, ser. A, Vol. 29, No. 1 (1955) pp. 59-77.

[2] M. Nagata, On the derived normal rings of Noetherian integral domains, Memoirs Kyoto University, ser. A, Vol. 29 No. 3, (1955) pp. 293-303.

[3] M. Nagata, The Theory of Multiplicity in General Local Rings, Proceedings of the International Symposium on Algebraic Number Theory, Tokyo-Nikko, (1955).

[4] M. Nagata, A general theory of algebraic geometry over Dedekind domains, I., American Journal of Mathmatics, Vol. 78, No. 1 (1956).

[5] H. Hironaka, On the invariance of the Hilbert characteristic function under specialization of projective varieties, forthcoming.

[6] O. Zariski, Some results in the arithmetic theory of algebraic varieties, American Journal of Mathematics, Vol. 61, (1939) pp. 249-294.

[7] O. Zariski, Theory and applications of holomorphic functions on algebraic varieties over arbitrary ground fields, Memoirs of the American Mathmatical Society, No. 5 (1951).

[8] O. Zariski, Eoundations of general theory of birational correspondences, Trancactions of American Mathematical Society, Vol. 53 (1943), pp. 490-542.

[9] O. Zariski, The concept of a simple point of an abstract algebraic variety, Transactions of the Americal Mathmatical Society, Vol. 61 (1947), pp. 1-52.

[10] D. G. Northcott, A note on the genus formula for plane curves, Journal of the London Mathematical Society, Vol. 30, (1955) pp. 376-382.

[11] P. Samuel, La notion de multiplicite en algebre et en geometrie algebrique, Journal de Mathematique Pures et Appliquees, (IX), Vol. 30 (1951), pp. 159-274.

[12] P. Samuel, Algebre locale, Memorial des Scence Mathmatiques, No. 123 (1953).

[13] M. Rosenlicht, Equivalence Relations on Algebraic Curves, Annals of Mathmatics, Vol. 56, No. 1, (1952).