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Note on a paper of Samuel concerning asymptotic properties of ideals

By

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Previously Samuel [4] defined an equivalence relation between ideals of a Noetherian ring as follows:

Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in a Noetherian ring $\mathfrak{o}$ having the same radical. Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are not nilpotent. For every natural number n, define the integers $v_{\mathfrak{b}}(\mathfrak{a}, n)$ and $w_{\mathfrak{b}}(\mathfrak{a}, n)$ such that[1)]

$$(1) \quad \mathfrak{a}^n \subseteq \mathfrak{b}^{v_{\mathfrak{b}}(\mathfrak{a}, n)}, \quad \mathfrak{a}^n \nsubseteq \mathfrak{b}^{v_{\mathfrak{b}}(\mathfrak{a}, n)+1}$$

$$(2) \quad \mathfrak{b}^{w_{\mathfrak{b}}(\mathfrak{a}, n)} \subseteq \mathfrak{a}^n, \quad \mathfrak{b}^{w_{\mathfrak{b}}(\mathfrak{a}, n)-1} \nsubseteq \mathfrak{a}^n.$$

Then $\mathfrak{a}$ and $\mathfrak{b}$ are said to be *equivalent* if $\lim(v_{\mathfrak{b}}(\mathfrak{a}, n)/n) = \lim(w_{\mathfrak{b}}(\mathfrak{a}, n)/n) = 1$[2)]. He showed that this defines actually an equivalence relation and that the operations of multiplication and addition are compatible with the equivalence relation.

Concerning this equivalence relation, Muhly [1] proved that if $\mathfrak{o}$ is a Noetherian integral domain, then this equivalence relation is characterized by integral dependence. Namely, we define the integral dependence as follows: An element a is *integral* over an ideal $\mathfrak{a}$ if there are elements $c_1, c_2, \cdots, c_n$ such that (i) $c_i \in \mathfrak{a}^i$ and (ii) $a^n + c_1 a^{n-1} + c_2 a^{n-2} + \cdots + c_n = 0$; an ideal $\mathfrak{b}$ is integally dependent on $\mathfrak{a}$ if every element of $\mathfrak{b}$ is integral over $\mathfrak{a}$. Then Muhly obtained the result: Two non-zero ideals $\mathfrak{a}$ and $\mathfrak{b}$ in a Noetherian integral domain are equivalent to each other if and only if $\mathfrak{a}$ and $\mathfrak{b}$ are integrally dependent on each other.

We shall prove at first that *the equivalence relation is characterized by integral dependence* without assuming that the ring is an integral domain (a generalization of the Muhly's result).

The second problem. Samuel [4] proved the following "Cancellation law":

If a, b and b' are equivalence classes of ideals having the same radical (in a Noetherian ring), then $ab=ab'$ implies $b=b'$.

Secondly we shall generalize this law, namely,

Cancellation law: Let a, b and b' are equivalence classes of ideals in a Noetherian ring. Then $ab=ab'$ implies $b=b'$ if the following condition is satisfied: If a minimal prime divisor $\mathfrak{p}$ of zero contains the radical of a, then $\mathfrak{p}$ contains the radicals of b and b'.

Here, the *radical* of an equivalence class is the radical of a member of the class (which is obviously determined uniquely).

The third problem. Samuel [4] asked following 4 questions:

(1) Are the limits $l_{\mathfrak{b}}(\mathfrak{a})=\lim v_{\mathfrak{b}}(\mathfrak{a}, n)/n$ and $L_{\mathfrak{b}}(\mathfrak{a})=\lim w_{\mathfrak{b}}(\mathfrak{a}, n)/n$ always rational numbers?

(2) Are the deviations $v_{\mathfrak{b}}(\mathfrak{a}, n)-l_{\mathfrak{b}}(\mathfrak{a})n$ and $L_{\mathfrak{b}}(\mathfrak{a})n-w_{\mathfrak{b}}(\mathfrak{a}, n)$ bounded?

(3) Let $\mathfrak{r}$ be a semi-prime ideal in a Noetherian ring A and let $\mathfrak{J}_{\mathfrak{r}}(A)$ be the equivalence classes of ideals which have $\mathfrak{r}$ as the radical. Then $\mathfrak{J}_{\mathfrak{r}}(A)$ can be imbedded in a lattice ordered group H. Does $\mathfrak{J}_{\mathfrak{r}}(A)$ contain all elements of H which are smaller than an element?

(4) Is the operation of intersection of ideals compatible with the equivalence relation?

We shall give here *affirmative answers of* (1) *and* (2) *and counter examples against* (3) *and* (4).

Furthermore we shall give some remarks concerning the non-Noetherian case and form ideals in local case.

§ 1. Integral dependence

From now on, we shall denote by $\mathfrak{o}$ a Noetherian ring, by $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ all of the minimal prime divisors of zero in $\mathfrak{o}$. If $x, y_1, \cdots, y_n$ are elements of $\mathfrak{o}$ and if x is not nilpotent, $\mathfrak{o}[y_1/x, \cdots, y_n/x]$ will denote the following ring: Let S be the set of powere of x. Then $0 \in S$ therefore we can consider the ring of quotients of $\mathfrak{o}$ with respect to S. Let ϕ_x be the natural homomorphism from $\mathfrak{o}$ into $\mathfrak{o}_S$. Then $\mathfrak{o}[y_1/x, \cdots, y_n/x]=\phi_x(\mathfrak{o})[\phi_x(y_1)/\phi_x(x), \cdots, \phi_x(y_n)/\phi_x(x)]$. Observe that the kernel of ϕ_x coincides with $0 : x^m\mathfrak{o}$ for sufficiently large m and is contained in every $\mathfrak{p}_i$ such that $x \in \mathfrak{p}_i$.

We shall denote by ϕ_i the natural homomorphism from $\mathfrak{o}$ onto $\mathfrak{o}/\mathfrak{p}_i$ for each $i=1, \cdots, r$ and by L_i the field of quotients of $\mathfrak{o}/\mathfrak{p}_i$.

If $\mathfrak{f}$ is a subring of L_i which contains $\mathfrak{o}/\mathfrak{p}_i$ and if $\mathfrak{b}$ is an ideal of $\mathfrak{f}$, we shall denote by $\mathfrak{b}\cap\mathfrak{o}$ the ideal $\phi_i^{-1}(\phi_i(\mathfrak{o})\cap\mathfrak{b})$. We shall say that an ideal $\mathfrak{q}$ is a valuation ideal of $\mathfrak{o}$ if there exist one i, a valuation ring $\mathfrak{v}$ of L_i which contains $\mathfrak{o}/\mathfrak{p}_i$ and an ideal $\mathfrak{q}'$ of $\mathfrak{v}$ such that $\mathfrak{q}=\mathfrak{q}'\cap\mathfrak{o}$. When $\mathfrak{a}$ is an ideal of $\mathfrak{o}$, the intersection of all valuation ideals of $\mathfrak{o}$ containing $\mathfrak{a}$ will be called *the derived complete ideal of* $\mathfrak{a}$. If the derived complete ideal of $\mathfrak{a}$ coincides with $\mathfrak{a}$, then we shall say that $\mathfrak{a}$ is a *complete ideal.*

THEOREM 1. *An ideal* $\mathfrak{b}$ *of* $\mathfrak{o}$ *is integrally dependent on an ideal* $\mathfrak{a}$ *if and only if* $\mathfrak{b}$ *is contained in the derived complete ideal* $\mathfrak{a}'$ *of* $\mathfrak{a}$.

PROOF. Assume that $\mathfrak{b}$ is integrally dependent on $\mathfrak{a}$. Let b be an element of $\mathfrak{b}$. Then there are elements $c_i\in\mathfrak{a}^i$ such that $b^n+c_1b^{n-1}+\cdots+c_n=0$. For each $i=1, \cdots, r$, set $b_i=\phi_i(b)$. Then b_i is integrally dependent on $\phi_i(\mathfrak{a})$ because $\phi_i(c_j)\in\phi_i(\mathfrak{a})^j$. Therefore, for every valuation v of L_i whose valuation ring contains $\mathfrak{o}/\mathfrak{p}_i$, $v(b_i)\geq v(\phi_i(\mathfrak{a}))$, which proves that b is in $\mathfrak{a}'$, hence $\mathfrak{b}\subseteq\mathfrak{a}'$. Conversely, let b be an element of $\mathfrak{a}'$ (and we have only to show that b is integral over $\mathfrak{a}$). (i) If b is nilpotent, then b is integral over 0, hence over $\mathfrak{a}$. (ii) Now we assume that b is not nilpotent. Let $a_1, \cdots, a_n$ be a base of $\mathfrak{a}$ and consider the ring $\mathfrak{o}[a_1/b, \cdots, a_n/b]$. Assume for a moment that there exists a prime ideal $\mathfrak{P}$ of the ring containing $\phi_b(a_1)/\phi_b(b), \cdots, \phi_b(a_n)\phi_b(b)$. Let i be such that $\mathfrak{p}_i$ contains the kernel of ϕ_b and such that $\phi_b(\mathfrak{p}_i)$ is contained in $\mathfrak{P}$. Then there exists a prime ideal $\mathfrak{P}'$ in $(\mathfrak{o}/\mathfrak{p}_i)[\phi_i(a_1)/\phi_i(b), \cdots, \phi_i(a_n)/\phi_i(b)]$ containing $\phi_i(a_1)/\phi_i(b), \cdots, \phi_i(a_n)/\phi_i(b)$. Therefore there exists a valuation v of L_i whose valuation ring contains $\mathfrak{o}/\mathfrak{p}_i$ and such that $v(\phi_i(a_j)/\phi_i(b))>0$ for every j, which shows that $v(\phi_i(b))<v(\phi_i(\mathfrak{a}))$ and is a contradiction. Therefore $\phi_b(a_1)/\phi_b(b), \cdots, \phi_b(a_n)/\phi_b(b)$ generate the unit ideal in the ring $\mathfrak{o}[a_1/b, \cdots, a_n/b]$, that is, there exists a polynomial f with coefficients in $\phi_b(\mathfrak{o})$ such that $f(\phi_b(a_1)/\phi_b(b), \cdots, \phi_b(a_n)/\phi_b(b))=1$ and that the constant term of f is zero. Let d be the degree of f. Then we have a relation of the form:

$$\phi_b(b)^d=\phi_b(c_1)\phi_b(b)^{d-1}+\cdots+\phi_b(c_d)\quad(c_i\in\mathfrak{a}^i).$$

Since the kernel of ϕ_b coincides with $0:b^m\mathfrak{o}$ for a sufficiently large m, we have

$$b^{d+m}=c_1b^{d+m-1}+\cdots+c_db^m.$$

Since $c_i \in \mathfrak{a}^i$, we have proved that b is integral over $\mathfrak{a}$. Thus Theorem 1 is proved completely.

COROLLARY. *Let* $\mathfrak{n}$ *be the radical of* $\mathfrak{o}$. *Then an ideal* $\mathfrak{b}$ *is integrally dependent on another ideal* $\mathfrak{a}$ *if and only if* $\mathfrak{b}+\mathfrak{n}/\mathfrak{n}$ *is integrally dependent on* $\mathfrak{a}+\mathfrak{n}/\mathfrak{n}$.

REMARK. It is obvious that the derived complete ideal of an ideal $\mathfrak{a}$ is contained in the radical of $\mathfrak{a}$. Therefore if an ideal $\mathfrak{b}$ is integrally dependent on $\mathfrak{a}$, then $\mathfrak{b}$ is contained in the radical of $\mathfrak{a}$.

§ 2. Samuel's equivalence relation

Though Samuel [3] defined the equivalence relation only for non-nilpotent ideals under an additional condition on radicals, we shall generalize the difinition according to note 2) at the end of the presente paper.

THEOREM 2. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of* $\mathfrak{o}$. *Then* (1) *if there exists a sequence* $\{m_n\ (n=1, 2, \cdots)\}$ *of natural numbers such that* (i) $\lim m_n/n=1$ *and* (ii) $\mathfrak{b}^{m_n} \subseteq \mathfrak{a}^n$, *then* $\mathfrak{b}$ *is integrally dependent on* $\mathfrak{a}$, *and conversely,* (2) *if* $\mathfrak{b}$ *is integrally dependent on* $\mathfrak{a}$, *then there exists an integer* c *such that* $\mathfrak{b}^{n+c} \subseteq \mathfrak{a}^n$ *for every* $n=1, 2, \cdots$.

PROOF. (1) Let v be an arbitrary valuation of rank 1 in L_i whose valuation ring contains $\mathfrak{o}/\mathfrak{p}_i$ (i being also arbitrary). Then $m_n v(\phi_i(\mathfrak{b})) \geq n v(\phi_i(\mathfrak{a}))$. Therefore $v(\phi_i(\mathfrak{b})) \geq v(\phi_i(\mathfrak{a}))$, which shows that $\mathfrak{b}$ is contained in the derived complete ideal of $\mathfrak{a}$[3]. It follows that $\mathfrak{b}$ is integrally dependent on $\mathfrak{a}$.

(2) Since $\mathfrak{b}$ has a finite base, we have only to show the existence of c in the case where $\mathfrak{b}$ is generated by one element b. Since b is integrally dependent on $\mathfrak{a}$, there exists a relation $b^{c+1} + a_1 b^c + \cdots + a_c = 0$ with $a_i \in \mathfrak{a}^i$. Therefore $b^{c+1} \in \sum_0^c b^i \mathfrak{a}^{c+1-i} = \mathfrak{a}(\sum_0^c b^i \mathfrak{a}^{c-i})$. Then $b^{c+2} \in \mathfrak{a}(\sum_0^c b^{i+1}\mathfrak{a}^{c-i}) \subseteq \mathfrak{a}^2(\sum_0^c b^i \mathfrak{a}^{c-i})$ and so on. Thus we have $b^{n+c} \in \mathfrak{a}^n(\sum_0^c b^i \mathfrak{a}^{c-i}) \subseteq \mathfrak{a}^n$. Therefore our c is the required element.

COROLLARY. *Two ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *of* $\mathfrak{o}$ *are equivalent to each other if and only if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are integrally dependent on each other* (*or, equivalently,* $\mathfrak{a}+\mathfrak{b}$ *is integrally dependent on both* $\mathfrak{a}$ *and* $\mathfrak{b}$).

§ 3. The cancellation law

THEOREM 3. *Let* $\mathfrak{a}$, $\mathfrak{b}$ *and* $\mathfrak{c}$ *be ideals of* $\mathfrak{o}$. *Assume that*

$\mathfrak{ac}=\mathfrak{bc}$. *Then* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are equivalent to each other if the following condition is satisfied: If a minimal prime divisor* $\mathfrak{p}$ *of zero contains* $\mathfrak{c}$, *then* $\mathfrak{p}$ *contains* $\mathfrak{a}$ *and* $\mathfrak{b}$.

PROOF. Let $c_1, \cdots, c_n$ be a base of $\mathfrak{c}$. For each $b \in \mathfrak{b}$, there are $a_{ij} \in \mathfrak{a}$ such that $c_i b = \sum a_{ij} c_j$. Let d be the determinant $|\delta_{ij} b - a_{ij}|$. Then $dc_i = 0$ for every i, hence $d\mathfrak{c}=0$. By the condition, we have $d\mathfrak{b}$ in nilpotent, hence $d^m b^m = 0$ for a natural number m, which shows that b is integral over $\mathfrak{a}$. Thus $\mathfrak{b}$ is integrally dependent on $\mathfrak{a}$. Similarly, $\mathfrak{a}$ is integrally dependent on $\mathfrak{b}$ and therefore $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent to each other.

THEOREM 4. *If an ideal* $\mathfrak{b}$ *of* $\mathfrak{o}$ *contains another ideal* $\mathfrak{a}$ *of* $\mathfrak{o}$ *and if* $\mathfrak{b}$ *is integrally dependent on* $\mathfrak{a}$, *then there exists a natural number* t *such that* $\mathfrak{b}^n = \mathfrak{a}\mathfrak{b}^{n-1}$ *for* $n \geq t$.

PROOF. By the same way as in the proof of (2) in Theorem 2, we have $\mathfrak{b}^n \subseteq \sum_0^{n-1} \mathfrak{a}^{n-1}\mathfrak{b}^i$ for sufficiently large n. Since $\mathfrak{b}$ contains $\mathfrak{a}$, we have $\mathfrak{b}^n \subseteq \mathfrak{a}\mathfrak{b}^{n-1} \subseteq \mathfrak{b}^n$ and $\mathfrak{b}^n = \mathfrak{a}\mathfrak{b}^{n-1}$.

COROLLARY. *Two ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *of* $\mathfrak{o}$ *are equivalent to each other if and only if there exists an ideal* $\mathfrak{c}$ *of* $\mathfrak{o}$ *such that* (i) $\mathfrak{ac}=\mathfrak{bc}$ *and* (ii) $\mathfrak{c}$ *contains a power of* $\mathfrak{a}+\mathfrak{b}$.

PROOF. If there exists such a $\mathfrak{c}$, then Theorem 3 shows that $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent to each other. Conversely, assume that $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent to each other. Then for a sufficiently large t, $(\mathfrak{a}+\mathfrak{b})^t = \mathfrak{a}(\mathfrak{a}+\mathfrak{b})^{t-1} = \mathfrak{b}(\mathfrak{a}+\mathfrak{b})^{t-1}$ and $\mathfrak{c} = (\mathfrak{a}+\mathfrak{b})^{t-1}$ is the required ideal.

Now we come to the cancellation law:

THEOREM 5 (CANCELLATION LAW). *Let* a, b *and* b' *be equivalence classes of ideals in* $\mathfrak{o}$. *Then* $ab=ab'$ *implies* $b=b'$ *if the following condition is satisfied: If a minimal prime divisor* $\mathfrak{p}$ *of zero contains the radical of* a, *then* $\mathfrak{p}$ *contains the radicals of* b *and* b'.

PROOF. Let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{b}'$ be members of a, b and b' respectively. Then $\mathfrak{ab}$ is equivalent to $\mathfrak{ab}'$. Therefore there exists an ideal $\mathfrak{c}$ of $\mathfrak{o}$ which contains a power of $\mathfrak{a}(\mathfrak{b}+\mathfrak{b}')$ and such that $\mathfrak{abc}=\mathfrak{ab}'\mathfrak{c}$ by the colollary to Theorem 4. Assume that a minimal prime divisor $\mathfrak{p}$ of zero contains $\mathfrak{ac}$. Then $\mathfrak{p}$ contains $\mathfrak{a}(\mathfrak{b}+\mathfrak{b}')$. If $\mathfrak{p}$ contains $\mathfrak{a}$, then by the condition, $\mathfrak{p}$ contains $\mathfrak{b}+\mathfrak{b}'$. Therefore $\mathfrak{p}$ contains always $\mathfrak{b}+\mathfrak{b}'$. Therefore by Theorem 3 we have $\mathfrak{b}$ and $\mathfrak{b}'$ are equivalent to each other, which shows that $b=b'$.

REMARK. Observe that the condition in Theorem 5 is satisfied in each of the following cases and that the last case is nothing but the cancellation law due to Samuel [3]:

(1) A member of a is not contained in any minimal primed divisor of zero.

(2) The radical of a contains those of b and b'.

(3) a, b and b' have the same radical (this is a special case of (2)).

§ 4. Rationality of the limits $l_{\mathfrak{b}}(\mathfrak{a})$ and $L_{\mathfrak{b}}(\mathfrak{a})$

Let $\mathfrak{a}$ be a non-nilpotent ideal of $\mathfrak{o}$ and we renumber the $\mathfrak{p}_i$'s so that $\mathfrak{a} \in \mathfrak{p}_i$ if and only if $i \leq t$. Let $a_1, \cdots, a_s$ be a base of $\mathfrak{a}$. Set $a_{ij} = \phi_{ij}(a_j)$ (for $i=1, \cdots, t$; $j=1, \cdots, s$) and $\mathfrak{o}_{ij} = \phi_i(\mathfrak{o})[a_{i1}/a_{ij}, \cdots, a_{is}/a_{ij}]$ (for (i, j) such that $a_{ij} \neq 0$). Let $\mathfrak{o}_{ij}^*$ be the derived normal ring of $\mathfrak{o}_{ij}$. We set $\mathfrak{a}' = (\cap_{i,j} a_{ij}^n \mathfrak{o}_{ij}^* \cap \mathfrak{o}) \cap (\cap_{i>t} \mathfrak{p}_i)$. Then

THEOREM 6. *The ideal $\mathfrak{a}'$ is the derived complete ideal of $\mathfrak{a}^n$.*

Proof. Since $\mathfrak{o}_{ij}$ is a Noetherian integral domain, $\mathfrak{o}_{ij}^*$ is a Krull ring (see Nagata [3]). Therefore $\mathfrak{a}'$ is a complete ideal. If $i > t$, $\phi_i(\mathfrak{a}') = 0 = \phi_i(\mathfrak{a})$. For an arbitrary $i \leq t$, let v be an arbitrary valuation of L_i whose valuation ring $\mathfrak{v}$ contains $\mathfrak{o}/\mathfrak{p}_i$. Then $\mathfrak{v}$ contains at least one $\mathfrak{o}_{ij}$, hence $\mathfrak{o}_{ij}^*$. Then $v(\phi_i(\mathfrak{a}')) \geq v(a_{ij}^n) = v(\phi_i(\mathfrak{a}^n))$. Therefore $\mathfrak{a}'$ is contained in the derived complete ideal of $\mathfrak{a}^n$ and therefore $\mathfrak{a}'$ is the derived complete ideal of $\mathfrak{a}^n$.

THEOREM 7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be non-nilpotent ideals of $\mathfrak{o}$ which have the same radical. Then the limits $l_{\mathfrak{b}}(\mathfrak{a})$ and $L_{\mathfrak{b}}(\mathfrak{a})$ are rational numbers, provided that they are well defined.*[4)]

PROOF. Since $L_{\mathfrak{a}}(\mathfrak{b}) l_{\mathfrak{b}}(\mathfrak{a}) = 1$ (see Samuel [4]), we have only to prove that $L_{\mathfrak{a}}(\mathfrak{b})$ and $L_{\mathfrak{b}}(\mathfrak{a})$ are rational numbers. By the symmetry, we have only to show that $L_{\mathfrak{b}}(\mathfrak{a})$ is a rational number. We shall denote by $\mathfrak{a}_n$ and $\mathfrak{b}_n$ the derived complete ideals of $\mathfrak{a}^n$ and $\mathfrak{b}^n$ respectively. Let $m(n)$ be such that $\mathfrak{b}_{m(n)} \subseteq \mathfrak{a}_n$ and that $\mathfrak{b}_{m(n)-1} \not\subseteq \mathfrak{a}_n$. We shall use the same notations as a_i, a_{ij} and $\mathfrak{o}_{ij}^*$ as in Theorem 6 (applied to our $\mathfrak{a}$) and let $\mathfrak{p}_{ijk}^*$ $(k=1, \cdots, u(i, j))$ be all of the minimal prime divisors of $a_{ij}\mathfrak{o}_{ij}^*$ (for a_{ij} such that $a_{ij}\mathfrak{o}_{ij}^* \neq \mathfrak{o}_{ij}^*$) and let v_{ijk} be the normalized valuation defined by the valuation ring $(\mathfrak{o}_{ijk}^*)_{\mathfrak{p}^*_{ijk}}$. Let e be the maximum of $v_{ijk}(\phi_i(\mathfrak{a}))/v_{ijk}(\phi_i(\mathfrak{b}))$. Then Theorem 6 shows that our $m(n)$ is characterized by

$$m(n)/n \geq e > (m(n)-1)/n\,.$$

Therefore $\lim m(n)/n = e$ and e is obviously a rational number. Now we have only to show that $e = L_{\mathfrak{b}}(\mathfrak{a})$. If w/n (w and n being natural numbers) is not less than e, then $\mathfrak{b}^w$ is integrally dependent on $\mathfrak{a}^n$ by Theorem 1 and by our observation. Therefore $w/n \geq L_{\mathfrak{b}}(\mathfrak{a})$ by Theorem 2. If $w/n < e$, then $\mathfrak{b}^{wu}$ cannot be integrally dependent on $\mathfrak{a}^{nu}$ for any natural number u and $w/n < L_{\mathfrak{b}}(\mathfrak{a})$. Therefore $e = L_{\mathfrak{b}}(\mathfrak{a})$ and the proof is completed.

§ 5. The deviations $v_{\mathfrak{b}}(\mathfrak{a}, n) - l_{\mathfrak{b}}(\mathfrak{a})n$ and $L_{\mathfrak{b}}(\mathfrak{a})n - w_{\mathfrak{b}}(\mathfrak{a}, n)$.

THEOREM 8. *With the same* $\mathfrak{a}$, *and* $\mathfrak{b}$ *as in Theorem 7, the deviations* $v_{\mathfrak{b}}(\mathfrak{a}, n) - l_{\mathfrak{b}}(\mathfrak{a})n$ *and* $L_{\mathfrak{b}}(\mathfrak{a})n - w_{\mathfrak{b}}(\mathfrak{a}, n)$ *are bounded.*[5]

PROOF. By Theorem 7, $l_{\mathfrak{b}}(\mathfrak{a})$ is a rational number. Let f and g be natural numbers such that $l_{\mathfrak{b}}(\mathfrak{a}) = f/g$. By Theorem 2 $\mathfrak{a}^g$ is integrally dependent on $\mathfrak{b}^f$ and there exists an integer c such that $\mathfrak{a}^{ng+c} \subseteq \mathfrak{b}^{fn}$ for every $n = 1, 2, \cdots$. Therefore $\mathfrak{a}^{ng+g+c} \subseteq \mathfrak{b}^{fn+d}$ for $d \leq n$, which proves that $|v_{\mathfrak{b}}(\mathfrak{a}, n) - l_{\mathfrak{b}}(\mathfrak{a})n|$ is not greater than $g+c$, which completes the proof for $v_{\mathfrak{b}}(\mathfrak{a}, n) - l_{\mathfrak{b}}(\mathfrak{a})n$. The proof for $L_{\mathfrak{b}}(\mathfrak{a})n - w_{\mathfrak{b}}(\mathfrak{a}, n)$ can be done quite similarly.

§ 6. Counter examples against the 3-rd and the 4-th problems of Samuel [4]

Let k be a field, x and y algebraically independent elements over k and let $A = k[x, y]$, $\mathfrak{m} = xA + yA$.

(I) *The 3-rd problem:* We consider the equivalence classes of $\mathfrak{m}$-primary ideals in A; the set of the classes is denoted by $\mathfrak{J}_{\mathfrak{m}}(A)$. Consider a lattice ordered group H in which $\mathfrak{J}_{\mathfrak{m}}(A)$ is imbedded naturally (see Samuel [4]). Assume that there exists an element $c \in \mathfrak{J}_{\mathfrak{m}}(A)$ such that every element of H which is smaller than c belongs to $\mathfrak{J}_{\mathfrak{m}}(A)$. Since every primary ideal belonging to $\mathfrak{m}$ contains a power of $\mathfrak{m}$, we may assume that $c = m^n$, where m is the class of $\mathfrak{m}$.

Set $\mathfrak{a} = x^{2n}A + \mathfrak{m}^{2n+1}$. Then

LEMMA 1. $\mathfrak{a}$ *is a valuation ideal of* A, *hence* $\mathfrak{a}$ *is complete.*

PROOF. Set $f = x^{2n} + y^{2n+1}$. Then $f \in \mathfrak{a}$ and fA is a prime ideal. Let v' be a valuation of the field of quotients of A/fA such that $v'(y \bmod . fA) = 1$ and the valuation ring $\mathfrak{v}$ be the composite of the valuation ring A_{fA} and the valuation ring of v'. We shall show that $\mathfrak{a} = x^{2n}\mathfrak{v} \cap A$. Set $\bar{x} = x \bmod . fA$, $\bar{y} = y \bmod . fA$. Then $v'(\bar{x}) = 1 + 1/2n$ because $v'(\bar{y}) = 1$ and $\bar{x}^{2n} + \bar{y}^{2n+1} = 0$. Therefore $v'(\mathfrak{a}/fA) = 2n+1 = v'(\bar{x}^{2n})$. Thus we have $\mathfrak{a} \subseteq x^{2n}\mathfrak{v} \cap A$. Conversely, since different monomials in $\bar{x}$, $\bar{y}$ of degree less than $2n+1$ have different values under v', we see easily the converse inclusion. Therefore $\mathfrak{a}=x^{2n}\mathfrak{v}\cap A$.

Now, let a be the class of $\mathfrak{a}$. Since a is smaller than m^{2n}, am^{-n} is an element of H which is smaller than m^n. By our assumption, there exists an ideal $\mathfrak{b}$ whose class is am^{-n}, i. e., if b is the class of $\mathfrak{b}$, then $bm^n=a$.

Then $\mathfrak{b}\mathfrak{m}^n$ is equivalent to $\mathfrak{a}$. Since $\mathfrak{a}$ is a complete ideal, $\mathfrak{b}\mathfrak{m}^n$ is contained in $\mathfrak{a}$ and $\mathfrak{b}\subseteq\mathfrak{a}:\mathfrak{m}^n=\mathfrak{m}^{n+1}$. Therefore bm^n is larger than a, which is a contradiction. Thus we have proved that our $\mathfrak{J}_\mathfrak{m}(A)$ is a counter example against the 3-rd problem of Samuel [4].

REMARK. Let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ be ideals of $\mathfrak{o}$. Assume that (i) $\mathfrak{a}$ is complete, (ii) $\mathfrak{b}\mathfrak{c}$ is equivalent to $\mathfrak{a}$. Then $\mathfrak{a}$ is equivalent to $(\mathfrak{a}:\mathfrak{c})\mathfrak{c}$. Assume furthermore that iii) if a minimal prime divisor $\mathfrak{p}$ of zero contains $\mathfrak{c}$ then $\mathfrak{p}$ contains $\mathfrak{b}$ and $\mathfrak{a}:\mathfrak{c}$. Then $\mathfrak{b}$ is equivalent to $\mathfrak{a}:\mathfrak{c}$.

PROOF. Since $\mathfrak{a}$ is complete, we have $\mathfrak{b}\mathfrak{c}\subseteq\mathfrak{a}$ and $\mathfrak{b}\subseteq\mathfrak{a}:\mathfrak{c}$. Therefore $\mathfrak{b}\mathfrak{c}\subseteq(\mathfrak{a}:\mathfrak{c})\mathfrak{c}\subseteq\mathfrak{a}$. Therefore $(\mathfrak{a}:\mathfrak{c})\mathfrak{c}$ is equivalent to $\mathfrak{a}$ and $\mathfrak{b}\mathfrak{c}$, because $\mathfrak{b}\mathfrak{c}$ is equivalent to $\mathfrak{a}$. By the cancellation law we see also the last assertion.

(II) *The 4-th problem:*

LEMMA 2. *If an ideal $\mathfrak{a}$ of (a Noetherian ring) $\mathfrak{o}$ is generated by elements $a_1, \cdots, a_r$, then for every $n=1, 2, \cdots$, the ideal $\mathfrak{a}^n$ is equivalent to the ideal $\mathfrak{a}_n$ generated by $a_1^n, \cdots, a_r^n$.*

PROOF. For every valuation v of L_i, whose valuation ring contains $\mathfrak{o}/\mathfrak{p}_i$, $v(\phi_i(\mathfrak{a}^n))=v(\phi_i(\mathfrak{a}_n))$ and the assertion is proved.

Applying Lemma 2 to our $\mathfrak{m}$, we see that $\mathfrak{c}=x^2A+y^2A$ and $\mathfrak{d}=x^2A+(x+y)^2A$ are equivalent to $\mathfrak{m}^2$. But $\mathfrak{c}\cap\mathfrak{d}$ is contained in $x^2A+\mathfrak{m}^3$ which is not equivalent to $\mathfrak{m}^2$. Thus the operation of intersection of ideals is not compatible with the equivalence relation.

§ 7. Some remarks on non-Noetherian case

Let $\mathfrak{f}$ be a ring (with identity) which may not Noetherian. Let $\mathfrak{n}$ be the radical of $\mathfrak{f}$. Then —a generalization of the corollary to Theorem 1:

THEOREM 9. *An element b of $\mathfrak{f}$ is integral over an ideal $\mathfrak{a}$ of $\mathfrak{f}$ if and only if b mod. $\mathfrak{n}$ is integral over $\mathfrak{a}$ mod. $\mathfrak{n}$.*

PROOF. Only if part is obvious. If b mod. $\mathfrak{n}$ is integrally dependent on $\mathfrak{a}$ mod. $\mathfrak{n}$, there exists a relation

$$b^n + a_1 b^{n-1} + \cdots + a_n = c \in \mathfrak{n} \quad (a_i \in \mathfrak{a}^i).$$

Since c is nilpotent, we see that b is integral over $\mathfrak{a}$.

THEOREM 10. *Theorem 1 can be generalized to the non-Noetherian case under the assumption that there exists only a finite number of minimal prime divisor of zero.*

PROOF. By Theorem 9, we can reduce to the case where $\mathfrak{o}$ has no nilpotent elements. If $\mathfrak{o}$ is an integral domain, then the same proof is applied (the number of the a_i's may infinite). Then the following lemma proves our assertion:

LEMMA 3. *Let* $\mathfrak{n}_1, \cdots, \mathfrak{n}_r$ *be ideals of* $\mathfrak{f}$ *such that* $\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_r = 0$. *Let* σ_i *be the natural homomorphism from* $\mathfrak{f}$ *onto* $\mathfrak{f}/\mathfrak{n}_i$. *Let b be an element of* $\mathfrak{f}$ *and let* $\mathfrak{a}$ *be an ideal of* $\mathfrak{f}$. *Then b is integral over* $\mathfrak{a}$ *if and only if* $\sigma_i(b)$ *is integral over* $\sigma_i(\mathfrak{a})$ *for every i.*

PROOF. The only if part is obvious. From that $\sigma_i(b)$ is integral over $\sigma_i(\mathfrak{a})$, it follows the existence of relation of the form

$$b^n + c_1 b^{n-1} + \cdots + c_n \in \mathfrak{n}_i (c_i \in \mathfrak{a}^i).$$

Making the product of these monic polynomials in b, we see the integral dependence of b on $\mathfrak{a}$.

An analogy of the proof of Lemma 3 proves

LEMMA 4. *Assume that there exist only a finite number of minimal prime divisor of zero in* $\mathfrak{f}$. *Then an element b of the total quotient ring of* $\mathfrak{f}$ *is integral over* $\mathfrak{f}$ *if and only if* $v(\sigma(b)) \geq 0$ *for every v and σ, where σ is the natural homomorphism from* $\mathfrak{f}$ *onto* $\mathfrak{f}/\mathfrak{p}$ *with a minimal prime divisor* $\mathfrak{p}$ *of zero and v is a valuation of the field of quotients of* $\sigma(\mathfrak{f})$ *whose valuation ring contains* $\sigma(\mathfrak{f})$.

Furthermore, by the same proof as there,

THEOREM 11. *Theorem 2,* (2) *and Theorem 4 can be generalized to the non-Noetherian case if* $\mathfrak{b}/\mathfrak{a}$ *is generated by a finite number of elements.*

On the other hand, *Lemma 2 can be generalized to the non-Noetherian case* by the following proof ($\mathfrak{a}$ may have no finite base):

We have only to prove that $\mathfrak{a}^n$ is integrally dependent on $\mathfrak{a}_n$ which is quite easy because if w is a monomial of degree n in a base of $\mathfrak{a}$, then w^n is in $\mathfrak{a}_n^n$.

As an application of Lemma 4, we shall prove the following

THEOREM 12. *Let* $a_1, \cdots, a_r$, *be elements of* $\mathfrak{f}$ *which are not zero-divisors. Set* $\mathfrak{f}_i=\mathfrak{f}[a_1/a_i, \cdots, a_r/a_i]$ *and* $\mathfrak{d}=\cap_i \mathfrak{f}_i$. *Then* $\mathfrak{d}$ *is integral over* $\mathfrak{f}$.

PROOF. Let b be an element of $\mathfrak{d}$. Then there exists a natural number n such that $ba_i^n \in \mathfrak{a}^n$ for every i, where $\mathfrak{a}$ is the ideal generated by $a_1, \cdots, a_r$. Since there exists a finitely generated subring $\mathfrak{f}'$ of $\mathfrak{f}$ such that $b \in \cap \mathfrak{f}'[a_1/a_i, \cdots, a_r/a_i]$, $(a_i \in \mathfrak{f}')$, we may assume that $\mathfrak{f}$ is Noetherian. Then Lemma 4 can be applied and we see that b is integral over $\mathfrak{f}$.

§ 8. A remark on form ideals

Let P be a (Noetherian) local ring and let $\mathfrak{a}$ be an ideal of P. In the form ring F of P, there corresponds the form ideal $\bar{\mathfrak{a}}$ to $\mathfrak{a}$. If an element b of P is integral over $\mathfrak{a}$, then the corresponding form $\bar{b}$ to b is integral over $\bar{\mathfrak{a}}$, as is easily seen by the definition of integral dependence. Therefore

THEOREM. 13. *The form ideal of the derived complete ideal of* $\mathfrak{a}$ *is contained in the derived complete ideal of* $\bar{\mathfrak{a}}$. *In particular, if* $\bar{\mathfrak{a}}$ *is complete, then* $\mathfrak{a}$ *is also complete.*

But, even when $\mathfrak{a}$ is complete, $\bar{\mathfrak{a}}$ may not be complete. We shall construct such an example under additional conditions that (i) P is a regular local ring and (ii) $\mathfrak{a}$ is a primary ideal belonging to the maximal ideal.

EXAMPLE. Let x, y, z be algebraically independent elements over a field K and set $P=K[x, y, z]_{(x,y,z)}$. Let $\mathfrak{q}$ be the ideal of P generated by x^2+y^3, z^2, y^4. Then $\mathfrak{q}$ is a primary ideal belonging to the maximal ideal $\mathfrak{m}=(x, y, z)$. Let $\mathfrak{a}$ be the derived complete ideal of $\mathfrak{q}$. Then the form ideal of $\mathfrak{a}$ is not complete.

PROOF. We have only to show that the form ideal $\bar{\mathfrak{a}}$ of $\mathfrak{a}$ does not contain xz. Let $\mathfrak{v}_1$ be the valuation ring $P_{(x^2+y^3)}$ and let L' be the residue class field of $\mathfrak{v}_1$. y and z are algebraically independent over K in L'. Therefore there exists a valuation v' of L' such that $v'(f(y, z))$ = minimum of the values of terms of $f(y, z)$ for preasigned values of y and z, where $f(y, z)$ is an arbitrary element of $K[y, z]$. We choose v' so that $v'(y)=2$ and $v'(x)=4$. Then $v'(x)=3$. Let v be the composite of a valuation defined by $\mathfrak{v}_1$ with v'. Then $v(\mathfrak{q})=8$, whence $v(\mathfrak{a})=8$. We shall show that if $f \in \mathfrak{m}$ has xz as its leading form, then $v(f)=6$ or 7. $v(xz)=7$,

$v(x^2\mathfrak{m})=8$, $v(xz\mathfrak{m})=9$, $v(xy^2)=7$, $v(y^3)=6$, $v(y^2z)=8$, $v(z^2\mathfrak{m})=10$, $v(z\mathfrak{m}^2)=8$, $v(\mathfrak{m}^4)=8$. Therefore, if the coefficient of y^3 in f is different from zero, then $v(f)=6$ and we assume that the coefficient of y^3 is zero. Then we may assume that $f=xz+cxy^2$ $(c\in P)$, because we have only to know that $v(f)=7$. Then $f=x(z+cy^2)$ and by our choice of v, $v(f)=v(x)+v(z+cy^2)=3+4=7$, which completes the proof.

REMARK. The ideal $\mathfrak{b}$ of P generated by x^2+y^3 and z^2 is a valuation ideal, hence is a complete ideal, whose form ideal is not complete.

PROOF. Let $\mathfrak{v}_1$ be as before and let $\mathfrak{v}''$ be the valuation ring $P_{(x^2+y^2,z)}/(x^2+y^3)$. Let $\mathfrak{v}^*$ be the composite of $\mathfrak{v}_1$ with $\mathfrak{v}''$. Then $\mathfrak{b}=z^2\mathfrak{v}^*\cap P$, because $z^2\mathfrak{v}^*\cap P$ is a primary ideal containing x^2+y^3 and because $(x^2+y^3, z)/(x^2+y^3)$ is a principal prime ideal. The form ideal of $\mathfrak{b}$ is obviously generated by x^2 and z^2, which does not contain xz, whence it is not complete.

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Notes

1) These integers may not be defined (for example, let $\mathfrak{a}=\mathfrak{b}$ be idempotent) and therefore Samuel [4] assumed furthermore that the intersection of the powers of the radical of $\mathfrak{a}$ is zero. But we can treat similarly if these integers are well defined.

2) We shall generalize the definition of equivalence as follows (including the case where $v_{\mathfrak{b}}(\mathfrak{a}, n)$ and $w_{\mathfrak{b}}(\mathfrak{a}, n)$ are not defined): $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent to each other if there are integers $v(n)$ and $w(n)$ for $n=1, 2, 3, \cdots$ such that $\mathfrak{b}^{w(n)}\subseteq\mathfrak{a}^n\subseteq\mathfrak{a}^{v(n)}$ and such that $\lim v(n)/n=\lim w(n)/n=1$.

Theorem 2 below shows that this definition covers the definition of Samuel [4], and the operations of multiplication and addition are compatible with this equivalence relation.

3) We use here Theorem 6 below (the special case where $n=1$).

4) The assumption that $\mathfrak{a}$ and $\mathfrak{b}$ have the same radical is not essential, if we treat one of $l_{\mathfrak{b}}(\mathfrak{a})$ and $L_{\mathfrak{b}}(\mathfrak{a})$.

5) Cf. Note 4).